Quivers and Three-Dimensional Lie Algebras

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Abstract

We study a family of three-dimensional Lie algebras L_μ that depend on a continuous parameter μ. We introduce certain quivers, which we denote by $Q_{m,n}$ $(m, n \in \mathbb{Z})$ and $Q_{\infty\times\infty}$, and prove that idempotented versions of the enveloping algebras of the Lie algebras L_μ are isomorphic to the path algebras of these quivers modulo certain ideals in the case that μ is rational and non-rational, respectively. We then show how the representation theory of the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$ can be related to the representation theory of quivers of affine type A, and use this relationship to study representations of the Lie algebras L_μ. In particular, though it is known that the Lie algebras L_μ are of wild representation type, we show that if we impose certain restrictions on weight decompositions, we obtain full subcategories of the category of representations of L_μ that are of finite or tame representation type.

Acknowledgements

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Chapter 1

Introduction

In the late 20th century, the mathematician Pierre Gabriel discovered a beautiful relationship between the root systems of Lie algebras and the representations of *quivers* [5], which are directed graphs. This result fuelled further research of the connection between Lie theory and quivers, as the correspondence between the two allows for a more intuitive and geometric means of studying Lie algebras. In the present work, we study the representation theory of a particular collection of Lie algebras by first relating them to certain quivers.

When working over the complex numbers, one can completely classify all Lie algebras of dimension three, up to isomorphism. The possibilities are (see for example [3, Chapter 3]):

1. the three-dimensional abelian Lie algebra,
2. the direct sum of the unique nonabelian two-dimensional Lie algebra with the one-dimensional Lie algebra,
3. the Heisenberg algebra,
4. the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$,

5. the Lie algebra with basis $\{x, y, z\}$ and commutation relations $[x, y] = y$, $[x, z] = y + z$, $[y, z] = 0$.

6. the Lie algebras L_μ, $\mu \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ (see Section 3.2).

The representation theory of the first four Lie algebras in the above list is well understood, while the representation theory of the last two is less so. In particular, for generic μ, very little is known about the representation theory of the Lie algebras L_μ, and so in order to obtain a better understanding of the representation theory of three-dimensional Lie algebras over the complex numbers, we first require a better understanding of this particular family. Due to a result of Makedonskyi [10, Theorem 3], it is known that the Lie algebras L_μ are of wild representation type. However, in the current paper, we will exploit the relationship between quivers and Lie algebras to draw several conclusions about the representations of these Lie algebras.

The *Euclidean group* is the group of isometries of $\mathbb{R}^2$ having determinant 1, and the *Euclidean algebra* is the complexification of its Lie algebra. The Euclidean group is one of the oldest and most studied examples of a group: it was studied implicitly even before the notion of a group was formalized, and it has applications not only throughout mathematics but in quantum mechanics, relativity, and other areas of physics as well. In [12, Theorem 4.1], it is shown that the category of representations of the Euclidean algebra admitting weight space decompositions is equivalent to the category of representations of the preprojective algebras of quivers of type A_∞. In the current paper, we show that the category of representations of $L_{\frac{m}{n}}$ ($m, n \in \mathbb{Z}$, $n \neq 0$) admitting weight space decompositions can be embedded inside the category of representations of the preprojective algebra of the affine quiver of type $A^{(1)}_{m+n}$, where by convention $A^{(1)}_0$ denotes the quiver of type A_∞. It can be shown that the Euclidean algebra is isomorphic to the Lie algebra L_{-1}, and so if $\mu = -1$ then $m = -n$ so that this agrees with what is presented in [12]. Thus the current work can be thought of as a generalization of some of the results of that paper. Analogous to the rational case, we also show how the representations with weight space decompositions of L_μ, $\mu \in \mathbb{C} \setminus \mathbb{Q}$, form a subcategory of the category of representations of the preprojective algebra of the quiver of type A_∞.

We begin by defining the *modified enveloping algebras* of the Lie algebras L_μ, denoted $\widetilde{U}_\mu$, and we note that category of representations of $\widetilde{U}_\mu$ is equivalent to the category of representations of L_μ admitting weight space decompositions. We then introduce certain quivers, denoted $Q_{m,n}$ $(m, n \in \mathbb{Z})$ and $Q_{\infty\times\infty}$, and show that the algebras $\widetilde{U}_\mu$ are isomorphic to the path algebras of $Q_{m,n}$ and $Q_{\infty\times\infty}$ modulo certain ideals in the case that μ is rational and non-rational, respectively (see Proposition 4.1.2). We use the theory of quiver morphisms to relate the representation theory of the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$ to the representation theory of affine quivers of type A, which is well understood. In the case that $\mu \in \mathbb{Q}$, the main result is the following (Theorems 4.2.3 and 4.2.4):

Theorem. *Let $m, n \in \mathbb{Z}$, $\gcd(m, n) = 1$, and let $a, b \in \mathbb{Z}$ be such that $0 \le b - a < m$. Let $\mathcal{C}^{m,n}_{a,b}$ denote the full subcategory of $\widetilde{U}_\mu$-Mod consisting of modules V such that $V_k = 0$ whenever $k < a$ or $k > b$, where V_k denotes the k^{th} weight space of V. Then $\mathcal{C}^{m,n}_{a,b}$ is of finite representation type when $n \neq 1$, and $\mathcal{C}^{m,1}_{a,b}$ is of tame representation type.*

When $\mu \in \mathbb{C} \setminus \mathbb{Q}$, we introduce a $\mathbb{Z}$ action on $\widetilde{U}_\mu$-Mod and our main result is the following (Corollary 4.2.11):

Theorem. *Let A be a finite subset of $\mathbb{Z}$ with the property that A does not contain any five consecutive integers. Then there are a finite number of $\mathbb{Z}$-orbits of isomorphism classes of indecomposable $\widetilde{U}_\mu$-modules V such that $V_{ij} = 0$ whenever $i - j \notin A$, where V_{ij} denotes the (i, j) weight space of V.*

The organization of the paper is as follows. In Chapter 2 we recall some basic notions from the theory of quivers and their representations, and we develop the theory of morphisms between quivers. In Chapter 3 we introduce the family L_μ of 3-dimensional Lie algebras that will be studied in the rest of the paper. We then describe the (modified) universal enveloping algebras $\widetilde{U}_{m,n}$ and $\widetilde{U}_\mu$ associated with the Lie algebras L_μ. Finally, in Chapter 4, we establish a relationship between the representation theory of the Lie algebras L_μ and the representation theory of the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$, and use some of the results of Chapter 2 to study these quivers.

In Appendix A we cover some of the basics of category theory which we use throughout. We also include a brief discussion of the theory of Kan extensions, the use of which simplifies much of the discussion in Section 2.3, though is not strictly necessary. In Appendix B, we review some elementary definitions and results in the theory of affine varieties, which are used in Section 2.5.

Notation. Throughout this work, all vector spaces and linear maps will be over $\mathbb{C}$. Given a $\mathbb{C}$-algebra A, we will take the term *module over* A to mean *left* module over A. The category of (left) modules over A is denoted A-Mod. By the usual abuse of language, we will use the terms module and representation interchangeably. Below is a table of important notation we will use throughout.

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$Q_{m,n}$, $Q_{\infty\times\infty}$	Page 5	$\mathbb{C}Q$, $\mathcal{P}(Q)$	Page 6
$\widehat{Q}_s$, Q_∞	Page 10	φ^*, φ_*, $\varphi_!$	Definition 2.3.7
$\varphi^*_{R'}$, $\varphi^{R'}_*$, $\varphi^{R'}_!$	Page 22	E^Q_α, E^R_α, $\Lambda^{Q^*}_\alpha$, $\mathcal{N}(E^R_\alpha)$	Page 24
$U_{m,n}$, $\widetilde{U}_{m,n}$	Page 34	U_μ, $\widetilde{U}_\mu$	Page 36

Chapter 2

Quivers

This chapter introduces the theory of quivers and their representations. We put special emphasis on quiver morphisms and certain functors associated with them.

2.1 Quivers and the Path Algebra

A *quiver* Q is a 4-tuple (X, A, t, h), where X and A are sets, and t and h are functions from A to X. The sets X and A are called the vertex and arrow sets respectively. If $\rho \in A$, we call $t(\rho)$ the *tail* of ρ, and $h(\rho)$ the *head*. We can think of an element $\rho \in A$ as an arrow from the vertex $t(\rho)$ to the vertex $h(\rho)$. We will often denote a quiver simply by $Q = (X, A)$, or even more simply by a picture, leaving the maps t and h implied.

Example 2.1.1 (The quiver $Q_{m,n}$)**.** Let $m, n \in \mathbb{Z}$ be nonzero integers and consider the quiver $Q_{m,n} = (\mathbb{Z}, A^{m,n})$ where $A^{m,n} = \{\rho_j^k \mid k \in \mathbb{Z},\ j = 1, 2\}$ is the arrow set. We define a map $\sigma : \{1, 2\} \to \{m, n\}$ such that $\sigma(1) = m$ and $\sigma(2) = n$. Then ρ_j^k is an arrow whose tail, $t(\rho_j^k)$, is the vertex k and whose head, $h(\rho_j^k)$, is the vertex $k + \sigma(j)$. If $\gcd(m, n) \neq 1$, then the quiver $Q_{m,n}$ decomposes into a disjoint union of quivers of the form $Q_{m',n'}$, where $\gcd(m', n') = 1$. Thus we may assume when dealing with the quivers $Q_{m,n}$ that $\gcd(m, n) = 1$.

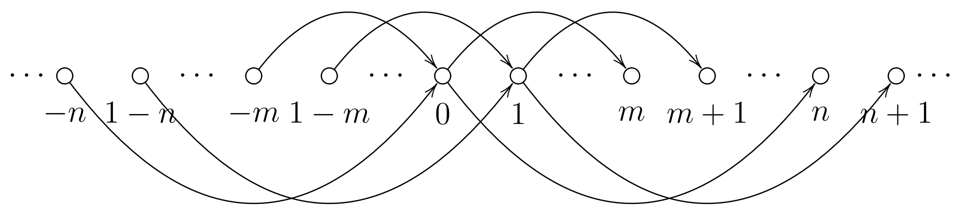


Figure 2.1: The quiver $Q_{m,n}$

Example 2.1.2 (The quiver $Q_{\infty\times\infty}$)**.** We now consider the quiver $Q_{\infty\times\infty} = (\mathbb{Z} \times \mathbb{Z}, A_{\infty\times\infty})$ where the set of arrows is $A_{\infty\times\infty} = \{\rho_d^k \mid d \in \{1, 2\}, k \in \mathbb{Z} \times \mathbb{Z}\}$. We define the map $\theta : \{1, 2\} \to \{(1, 0), (0, 1)\}$ by $\theta(1) = (1, 0)$ and $\theta(2) = (0, 1)$. Then ρ_d^k is the arrow whose tail, $t(\rho_d^k)$, is the vertex $k = (i, j)$ and whose head, $h(\rho_d^k)$, is the vertex $(i, j) + \theta(d)$.

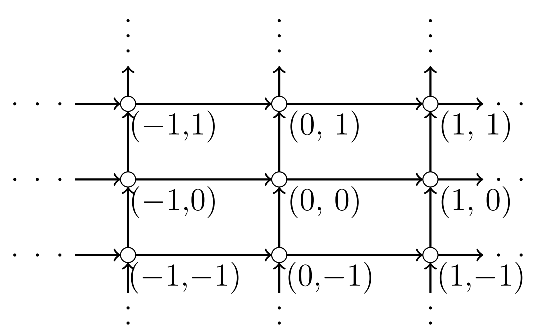


Figure 2.2: The quiver $Q_{\infty\times\infty}$

A *path* in a quiver Q is a sequence $\tau = \rho_n\rho_{n-1}\cdots\rho_1$ of arrows such that $h(\rho_i) = t(\rho_{i+1})$ for each $1 \le i \le n-1$. We define $t(\tau) = t(\rho_1)$ and $h(\tau) = h(\rho_n)$.

Definition 2.1.3 (Path algebra)**.** *Let Q be a quiver. The* path algebra *of Q is the $\mathbb{C}$-algebra whose underlying vector space has for basis the set of paths in Q and with* *multiplication given by:*

$$\tau_2 \cdot \tau_1 = \begin{cases} \tau_2\tau_1, & \text{if } h(\tau_1) = t(\tau_2), \\ 0, & \text{otherwise,} \end{cases}$$

where $\tau_2\tau_1$ denotes the concatenation of the paths τ_1 and τ_2. We denote the path algebra of Q by $\mathbb{C}Q$.

For any vertex $x \in X$ we let e_x denote the trivial path starting and ending at x and with multiplication given by $e_x\tau = \delta_{h(\tau)x}\tau$ and $\tau e_x = \delta_{t(\tau)x}\tau$ for any path τ.

For example, any nontrivial path in the quiver $Q_{m,n}$ of Example 2.1.1 above is of the form $\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_2}^{k+\sigma(j_1)}\rho_{j_1}^{k}$ with $k \in \mathbb{Z}$, $s \in \mathbb{N}$ and $j_i \in \{1, 2\}$ for $i = 1, \ldots, s$. Elements of the path algebra $\mathbb{C}Q_{m,n}$ are linear combinations of these paths and the trivial paths at each vertex. Similarly, elements of the form $\rho_{d_s}^{k+\theta(d_1)+\cdots+\theta(d_{s-1})} \cdots \rho_{d_2}^{k+\theta(d_1)}\rho_{d_1}^{k}$ along with the trivial paths at each vertex constitute a basis for the path algebra $\mathbb{C}Q_{\infty\times\infty}$, where $k \in \mathbb{Z} \times \mathbb{Z}$, $s \in \mathbb{N}$, and $d_n \in \{1, 2\}$ for $n = 1, ..., s$.

For every arrow $\rho \in A$ in a quiver $Q = (X, A)$, define an arrow $\bar{\rho}$ by $t(\bar{\rho}) = h(\rho)$ and $h(\bar{\rho}) = t(\rho)$, and let $\bar{A}$ denote the set of all such $\bar{\rho}$. Then we call $\overleftrightarrow{Q} = (X, A \cup \bar{A})$ the *double quiver* of Q.

Definition 2.1.4 (Preprojective algebra)**.** *Let Q be a quiver, $\overleftrightarrow{Q}$ its double quiver, and let I be the two-sided ideal of $\mathbb{C}\overleftrightarrow{Q}$ generated by elements of the form*

$$\sum_{\rho \in A, h(\rho)=i} \rho\bar{\rho} - \sum_{\rho \in A, t(\rho)=i} \bar{\rho}\rho. \tag{2.1.1}$$

Then the algebra $\mathbb{C}\overleftrightarrow{Q}/I$ is called the preprojective algebra *of Q, and is denoted $\mathcal{P}(Q)$. The relations* (2.1.1) *are called the* Gelfand-Ponomarev relations *in $\overleftrightarrow{Q}$.*

Now that we have introduced the concept of paths of a quiver $Q = (X, A)$, we can think of Q as being a small category as follows:

(i) the objects of Q are given by $\mathrm{Ob}(Q) = X$, and

(ii) for any $B, B' \in \mathrm{Ob}(Q)$, $\mathrm{Hom}_Q(B, B')$ is the set of all paths τ in Q with $t(\tau) = B$ and $h(\tau) = B'$.

It is not difficult to see that this defines a category. Indeed, composition is given by concatenation of paths and for any $B \in \mathrm{Ob}(Q)$ the identity morphism on B is given by the trivial path e_B. We will often pass between this categorical definition of a quiver and our original definition.

2.2 Quiver Representations

Definition 2.2.1 (Quiver Representation)**.** *Let Q be a quiver. A* representation *of Q is a covariant functor $V\colon Q \to \mathrm{Vect}_{\mathbb{C}}$.*

More explicitly, a representation of a quiver $Q = (X, A)$ is a collection of vector spaces $\{V(x) \mid x \in X\}$ along with a collection of linear maps $\{V(\rho)\colon V(t(\rho)) \to V(h(\rho)) \mid \rho \in A\}$. For any path $\tau = \rho_n \cdots \rho_1$ in Q we set $V(\tau) = V(\rho_n) \cdots V(\rho_1)$. Let V, W be two representations of a quiver $Q = (X, A)$. A *morphism* from V to W, $\sigma \in \mathrm{Hom}_{\mathrm{Rep}(Q)}(V, W)$, is a natural transformation $\sigma\colon V \to W$. A morphism σ is specified by a collection of linear maps $\{\sigma(x)\colon V(x) \to W(x) \mid x \in X)\}$ such that for every $\rho \in A$ the following diagram commutes:

$$\begin{array}{ccc} W(t(\rho)) & \xrightarrow{W(\rho)} & W(h(\rho)) \\ {\scriptstyle \sigma(t(\rho))}\uparrow & & \uparrow{\scriptstyle \sigma(h(\rho))} \\ V(t(\rho)) & \xrightarrow[V(\rho)]{} & V(h(\rho)). \end{array} \tag{2.2.1}$$

Thus we may consider the category $\mathrm{Rep}(Q)$ having as objects representations of Q, and with morphisms as described above. The category of representations of the quiver $Q = (X, A)$ is equivalent to the category of representations of the path algebra $\mathbb{C}Q$ (see for example [1, Theorem 1.6]).

Remark 2.2.2. When viewed as a category, the morphisms (paths) in a quiver Q are generated by the paths of length 1 (arrows). Thus the action of a representation V on any path in Q is uniquely determined by the action of V on the arrows of Q.

Definition 2.2.3 (Representation Type)**.** *Let Q be a quiver and let $\mathcal{C}$ be a subcategory of* $\mathrm{Rep}(Q)$*. We say $\mathcal{C}$ is of* finite type *if there are only finitely many isomorphism classes of indecomposable representations in $\mathcal{C}$. We say $\mathcal{C}$ is of* tame type *if for every $n \in \mathbb{N}$ all but finitely many isomorphism classes of indecomposable representations of dimension n in $\mathcal{C}$ occur in a finite number of families that are parametrized by a single, complex parameter. We say $\mathcal{C}$ is of* wild type *if it is neither of finite type nor tame type.*

In [5], Gabriel was able to prove that the notion of representation type of $\mathrm{Rep}(Q)$ is closely related to the study of Dynkin diagrams. This beautiful result was one of the main motivating factors for studying the representation theory of quivers and its relationship to Lie theory.

Theorem 2.2.4 (Gabriel's Theorem)**.** *Let Q be a quiver.*

1. $\mathrm{Rep}(Q)$ *is of finite type if and only if the underlying graph of Q is a union of Dynkin diagrams of type A, D, or E.*

2. $\mathrm{Rep}(Q)$ *is of tame type if and only if the underlying graph of Q is a union of Dynkin diagrams of type A, D, or E and of extended Dynkin diagrams of type $\hat{A}, \hat{D}$, or $\hat{E}$ (with at least one extended Dynkin diagram).*

Let $Q = (X, A)$ be some quiver. If $V, U \in \mathrm{Rep}(Q)$ and $\varphi \in \mathrm{Hom}_{\mathrm{Rep}(Q)}(V, U)$, then the kernel of φ, $\ker \varphi$, is the representation of Q with vector spaces given by $(\ker \varphi)(x) = \ker(\varphi(x))$ for every $x \in X$, and maps given by the restriction of the maps in V to these subspaces. The cokernel of φ is the representation of Q given by $(\operatorname{coker} \varphi)(x) = U(x)/\operatorname{im} \varphi(x)$ and with maps induced on these spaces by the maps of U.

2.3 Quiver Morphisms

Definition 2.3.1. *Let* $Q = (X, A, t, h)$ *and* $Q' = (X', A', t, h)$ *be two quivers. Then a* quiver morphism $\varphi \in \mathrm{Hom}(Q, Q')$ *consists of a pair of maps* $\varphi_x \colon X \to X'$ *and* $\varphi_a \colon A \to A'$ *such that* $\varphi_x t = t\varphi_a$ *and* $\varphi_a h = h\varphi_x$*.*

Remark 2.3.2. Throughout this chapter we have moved back and forth between thinking of quivers as directed graphs and thinking of quivers as the free category generated by the directed graph. While this can often be a useful identification, we should mention that the natural functor Quiv $\to$ Cat, where Quiv denotes the category of directed graphs, is a faithful embedding that is not full. That is, when thinking of two quivers Q, Q' as small categories, there can exist functors between Q and Q' that are not quiver morphisms.

Example 2.3.3. For all $s \in \mathbb{N}$, define the quiver $\widehat{Q}_s = (\mathbb{Z}/s\mathbb{Z}, \rho_i, \bar{\rho}_i)$, where $t(\rho_i) = h(\bar{\rho}_{i+1}) = i$ and $h(\rho_i) = t(\bar{\rho}_{i+1}) = i + 1$. Let $m, n \in \mathbb{Z}$ with $\gcd(m, n) = 1$. Then we have $\gcd(m, m + n) = 1$ and so for all $k \in \mathbb{Z}$ there is a unique integer $0 \le j_k < m + n$ such that $k \equiv j_k m \mod (m + n)$. Consider the morphism $g_{m,n} \in \mathrm{Hom}(Q_{m,n}, \widehat{Q}_{m+n})$, where $Q_{m,n}$ denotes the quiver of Example 2.1.1, given by $(g_{m,n})_x(k) = j_k$, $(g_{m,n})_a(\rho_1^k) = \rho_{j_k}$, and $(g_{m,n})_a(\rho_2^k) = \bar{\rho}_{j_k}$ for all $k \in \mathbb{Z}$. It is not difficult to see that this does indeed define a quiver morphism. The morphism $g_{2,1}$ is pictured below, where the coloured vertices and arrows of $Q_{2,1}$ are mapped under g to the vertices and arrows of the same colour. We will usually just write g, and let it be understood that the morphism g depends on m and n.

Example 2.3.4. Let Q_∞ denote the quiver $(\mathbb{Z}, \rho_i, \bar{\rho}_i)$, where $t(\rho_i) = i = h(\bar{\rho}_{i+1})$, and $h(\rho_i) = i + 1 = t(\bar{\rho}_{i+1})$. If $Q_{\infty\times\infty}$ denotes the quiver from Example 2.1.2, consider the morphism $f \in \mathrm{Hom}(Q_{\infty\times\infty}, Q_\infty)$ given by $f_x(i, j) = i - j$, $f_a(\rho_1^{(i,j)}) = \rho_{i-j}$, and $f_a(\rho_2^{(i,j)}) = \bar{\rho}_{i-j}$ for all $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. Once again it is easy to check that f is indeed a quiver morphism. The morphism f is displayed in the figure below, where

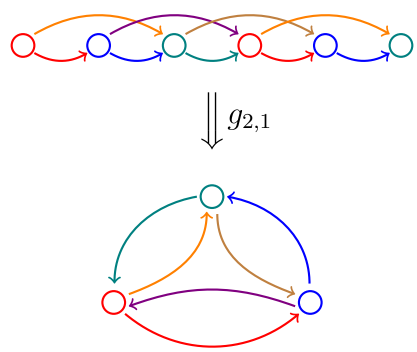


Figure 2.3: The quiver morphism $g_{2,1}\colon Q_{2,1} \to \widehat{Q}_3$

the coloured vertices and arrows of $Q_{\infty\times\infty}$ are mapped under f to the vertices and arrows of the same colour.

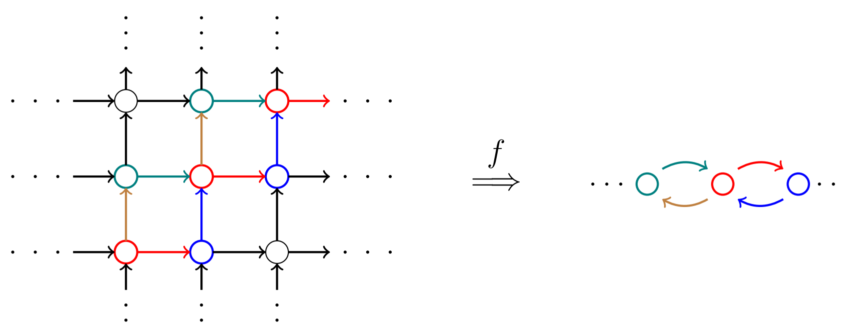


Figure 2.4: The quiver morphism $f\colon Q_{\infty\times\infty} \to Q_\infty$

For the most part, we will be interested in quiver morphisms satisfying a certain property, known as *covering morphisms*, which we introduce now.

Definition 2.3.5 (Covering Morphism)**.** *Let $Q = (X, A)$ and $Q' = (X', A')$ be quivers and let $\varphi \in \mathrm{Hom}(Q, Q')$. Then φ will be said to be a* covering morphism *if for every $x \in X$ and every path τ' in Q' such that $h(\tau') = \varphi(x)$ there exists a unique path τ in Q such that $h(\tau) = x$ and $\varphi(\tau) = \tau'$.*

For example, the morphisms g and f of Examples 2.3.3 and 2.3.4 are covering morphisms. On the other hand, the following example provides a quiver morphism which is not a covering morphism.

Example 2.3.6. Consider the quiver morphism φ pictured below. This morphism is not a covering morphism, since the path of length one ending at $\varphi(3)$ has two preimages under φ.

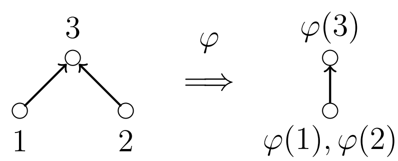


Figure 2.5: A quiver morphism that is not a covering morphism

Given a quiver morphism $\varphi\colon Q \to Q'$, there are three important functors one can use to study the relationship between the representations of the two quivers.

Definition 2.3.7. *Let $Q = (X, A)$ and $Q' = (X', A')$ be two quivers and let $\varphi \in \mathrm{Hom}(Q, Q')$.*

(i) The restriction functor *of φ is the functor $\varphi^*\colon \mathrm{Rep}(Q') \to \mathrm{Rep}(Q)$ defined for all $V \in \mathrm{Rep}(Q')$ by*

(a) For any $x \in X$, $\varphi^*(V)(x) = V(\varphi(x))$.

(b) For any $\rho \in A$, $\varphi^*(V)(\rho) = V(\varphi(\rho))$.

If $V, U \in \mathrm{Rep}(Q)$ and $f \in \mathrm{Hom}_Q(V, U)$ then the morphism $\varphi^*(f)$ is defined by $\varphi^*(f)(x) = f(\varphi(x))$ for every $x \in X$.

(ii) If φ is a covering morphism, then the right extension functor *of φ is the functor $\varphi_*\colon \mathrm{Rep}(Q) \to \mathrm{Rep}(Q')$ defined for all $V \in \mathrm{Rep}(Q)$ by:*

(a) *For any* $x' \in X'$, $\varphi_*(V)(x') = \prod_{x \in \varphi^{-1}(x')} V(x)$, *where by convention we take the empty product to be the zero vector space.*

(b) *For any* $\rho' \in A'$, $\varphi_*(V)(\rho') = \prod_{\rho \in \varphi^{-1}(\rho')} V(\rho)$.

If $V, U \in \mathrm{Rep}(Q')$ *and* $f \in \mathrm{Hom}_{Q'}(V, U)$ *then the morphism* $\varphi_*(f)$ *is defined by* $\varphi_*(f)(x') = \prod_{x \in \varphi^{-1}(x')} f(x)$.

(iii) *If* φ *is a covering morphism, then the* left extension functor *of* φ *is the functor* $\varphi_! \colon \mathrm{Rep}(Q) \to \mathrm{Rep}(Q')$ *defined for all* $V \in \mathrm{Rep}(Q)$ *by:*

(a) *For any* $x' \in X'$, $\varphi_!(V)(x') = \bigoplus_{x \in \varphi^{-1}(x')} V(x)$, *where by convention we take the empty coproduct to be the zero vector space.*

(b) *For any* $\rho' \in A'$, $\varphi_!(V)(\rho') = \bigoplus_{\rho \in \varphi^{-1}(\rho')} V(\rho)$.

If $V, U \in \mathrm{Rep}(Q')$ *and* $f \in \mathrm{Hom}_{Q'}(V, U)$ *then the morphism* $\varphi_!(f)$ *is defined by* $\varphi_!(f)(x') = \bigoplus_{x \in \varphi^{-1}(x')} f(x)$.

Remark 2.3.8. If we view a representation of Q' as a functor $V \colon Q' \to \mathrm{Vect}_{\mathbb{C}}$ then the restriction functor φ^* is described by the map $V \mapsto V \circ \varphi$. The (left and right) extension functors have similar categorical definitions in terms of what are known as (left and right) Kan extensions, and these definitions allow one to discuss the extension functors even in the case where φ is not a covering morphism. More precisely, in the language of Section A.3, the category $\mathrm{Rep}(Q)$ is written as the functor category $\mathrm{Vect}_{\mathbb{C}}^{Q}$, and then the restriction functor φ^* is written $\mathrm{Vect}_{\mathbb{C}}^{\varphi} \colon \mathrm{Vect}_{\mathbb{C}}^{Q'} \to \mathrm{Vect}_{\mathbb{C}}^{Q}$. Since Q is small and $\mathrm{Vect}_{\mathbb{C}}$ is complete, then for any functor $V \colon Q \to \mathrm{Vect}_{\mathbb{C}}$ (that is, for any representation $V \in \mathrm{Rep}(Q)$) there exists a right Kan extension of V along φ by Theorem A.3.2, which we denote by $\mathrm{Ran}_{\varphi} V \in \mathrm{Vect}_{\mathbb{C}}^{Q'}$. Moreover, for any vertex x in Q', we have

$$\mathrm{Ran}_{\varphi} V(x) = \varprojlim((x \downarrow \varphi) \xrightarrow{Q^x} Q \xrightarrow{V} \mathrm{Vect}_{\mathbb{C}}).$$

The comma category $x \downarrow \varphi$ is given by pairs $(y, \rho\colon x \to y)$ of vertices y in Q and arrows $\rho\colon x \to \varphi(y)$ in Q'. The functor Q^x takes pairs (y, ρ) to y. Thus $\operatorname{Ran}_\varphi V(x)$ can be described as taking the limit of the vector spaces $V(y)$ over all vertices y in Q and all arrows $\rho\colon x \to \varphi(y)$ in Q'. In order to generalize Definition 2.3.7 above, we set φ_* to be the functor $V \mapsto \operatorname{Ran}_\varphi V$. In the case that φ is a covering morphism, for any y in Q and any arrow $\rho\colon x \to \varphi(y)$ in Q', there is precisely one arrow mapped to ρ by φ and we retrieve the definition of φ_* given above. Similarly, one can generalize the definition of $\varphi_!$ in terms of left Kan extensions. Since in this work we deal only with covering morphisms, we will work with Definition 2.3.7 directly in order to keep the technical prerequisites to a minimum. However, we would like to point out that using these generalized definitions of the left and right extension functors, one can apply Theorem A.3.4 to deduce that φ_* is right adjoint to φ^*, and $\varphi_!$ is left adjoint to φ^*. This is a fact that was proven directly in [2, Theorem 4.1] for the case when φ is a covering morphism (we include the proof below in Theorem 2.3.9).

In Section 4.2 we will use the above functors to study the representation theory of the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$ of Examples 2.1.1 and 2.1.2. It turns out that the functors in Definition 2.3.7 are closely related.

Theorem 2.3.9. *[2, Theorem 4.1] Let Q, Q' be two quivers and let $\varphi \in \operatorname{Hom}(Q, Q')$ be a covering morphism. Then φ^* is left adjoint to φ_* and right adjoint to $\varphi_!$.*

Proof: In [2, Theorem 4.1] it is shown explicitly that φ^* is left adjoint to φ_*. The proof that φ^* is right adjoint to $\varphi_!$ is completely dual to the argument presented there. We include the proof here for completenes.

For any $V \in \operatorname{Rep}(Q), U \in \operatorname{Rep}(Q')$ we need to exhibit an isomorphism

$$\operatorname{Hom}_{\operatorname{Rep}(Q')}(\varphi_!(V), U) \cong \operatorname{Hom}_{\operatorname{Rep}(Q)}(V, \varphi^*(U))$$

that is natural in the arguments V and U. First, we note that for any $x' \in X'$ we have

$$\operatorname{Hom}_{\mathbb{C}}(\varphi_!(V)(x'), U(x')) \cong \bigoplus_{x:x'=\varphi(x)} \operatorname{Hom}_{\mathbb{C}}(V(x), U(x'))$$
$$\cong \operatorname{Hom}_{\mathbb{C}}(V(x), \varphi^*(U)(x)). \tag{2.3.1}$$

Now for any $\sigma \in \operatorname{Hom}_{\operatorname{Rep}(Q')}(\varphi_!(V), U)$, define $\bar{\sigma} \in \operatorname{Hom}_{\operatorname{Rep}(Q)}(V, \varphi^*(U))$ by $\bar{\sigma}(x_0) = \sigma(x')i_{x_0}$ for every $x_0 \in X$ with $\varphi(x_0) = x'$. Here i_{x_0} denotes the inclusion map $i_{x_0} \colon V(x_0) \to \bigoplus_{x\in\varphi^{-1}(x')} V(x) = \varphi_!(V)(x')$. Then the assignment $\sigma \mapsto \bar{\sigma}$ defines a morphism $\operatorname{Hom}_{\operatorname{Rep}(Q')}(\varphi_!(V), U) \to \operatorname{Hom}_{\operatorname{Rep}(Q)}(V, \varphi^*(U))$. Indeed, let $x_0, y_0 \in X$ and $\rho_0 \colon x_0 \to y_0$. Then if $x' = \varphi(x_0)$, $y' = \varphi(y_0)$, and $\rho' = \varphi(\rho_0)$, the diagram

$$\begin{array}{ccc} \bigoplus_{x\in\varphi^{-1}(x')} V(x) & \xrightarrow{\varphi_!(V)(\rho')} & \bigoplus_{y\in\varphi^{-1}(y')} V(y) \\ \downarrow{\scriptstyle \sigma(x')} & & \downarrow{\scriptstyle \sigma(y')} \\ \varphi^*(U)(x_0) & \xrightarrow[\varphi^*(U)(\rho_0)]{} & \varphi^*(U)(y_0) \end{array} \tag{2.3.2}$$

is commutative since σ is a morphism. But by the definition of $\varphi_!$, the diagram

$$\begin{array}{ccc} V(x_0) & \xrightarrow{V(\rho_0)} & V(y_0) \\ \downarrow{\scriptstyle i_{x_0}} & & \downarrow{\scriptstyle i_{y_0}} \\ \bigoplus_{x\in\varphi^{-1}(x')} V(x) & \xrightarrow[\varphi_!(V)(\rho')]{} & \bigoplus_{y\in\varphi^{-1}(y')} V(y) \end{array} \tag{2.3.3}$$

is also commutative. Placing diagram (2.3.3) on top of diagram (2.3.2) shows that $\bar{\sigma} \in \operatorname{Hom}_{\operatorname{Rep}(Q)}(V, \varphi^*(U))$.

It is not difficult to see that if we fix either V or U then the assignment $\sigma \mapsto \bar{\sigma}$ defines a natural transformation. Suppose we fix V. Then for any $\beta \in \operatorname{Hom}_{\operatorname{Rep}(Q')}(U_1, U_2)$ and any $\sigma \in \operatorname{Hom}_{\operatorname{Rep}(Q')}(\varphi_!(V), U_1)$ we have

$$\overline{\beta\sigma}(x_0) = \beta\sigma(\varphi(x_0))i_{x_0} = \beta(\varphi(x_0))\sigma(\varphi(x_0))i_{x_0}.$$

On the other hand,

$$(\varphi^*(\beta)\bar{\sigma})(x_0) = \varphi^*(\beta)(x_0)\bar{\sigma}(x_0) = \beta(\varphi(x_0))\sigma(\varphi(x_0))i_{x_0},$$

and so the assignment is natural in the argument U, as claimed. One shows similarly that the assignment is natural in V.

If is left to show that for any representations V and U, the map $\sigma \mapsto \bar{\sigma}$ defined above is an isomorphism. To do so, we construct its inverse. For any $\tau \in \mathrm{Hom}_{\mathrm{Rep}(Q)}(V, \varphi^*(U))$, define $\bar{\tau} \in \mathrm{Hom}_{\mathrm{Rep}(Q')}(\varphi_!(V), U)$ by $\bar{\tau}(x') = \bigoplus_{x \in \varphi^{-1}(x')} \tau(x)$ for every $x' \in X'$, where we have considered $\bar{\tau}$ as an element of the coproduct (2.3.1) above. Clearly the assignments $\sigma \to \bar{\sigma}$ and $\tau \to \bar{\tau}$ are mutual inverses, so if we can show that $\bar{\tau}$ defines a morphism of representations, then we are done. Let $\rho' \colon x' \to y'$ be an arrow in Q'. For every $y \in X$ such that $\varphi(y) = y'$, let ρ_y be the unique arrow in Q with $\varphi(\rho_y) = \rho'$ and $h(\rho_y) = y$. Then any morphism $\tau \colon V \to \varphi^*(U)$ gives a commutative diagram

$$\begin{array}{ccc} V(t(\rho_y)) & \xrightarrow{V(\rho_y)} & V(y) \\ {\scriptstyle \tau(t(\rho_y))}\downarrow & & \downarrow{\scriptstyle \tau(y)} \\ U(x') & \xrightarrow[U(\rho')]{} & U(y') \end{array}$$

for each such y. Taking the direct sum gives the following commutative diagram

$$\begin{array}{ccc} \bigoplus_{x \in \varphi^{-1}(x')} V(x) & \xrightarrow{\bigoplus V(\rho_y)} & \bigoplus_{y \in \varphi^{-1}(y')} V(y) \\ {\scriptstyle \bar{\tau}(x')}\downarrow & & \downarrow{\scriptstyle \bar{\tau}(y')} \\ U(x') & \xrightarrow[U(\rho')]{} & U(y'), \end{array}$$

which shows that $\bar{\tau}$ is a morphism of representations. ∎

Remark 2.3.10. It is clear from Definition 2.3.7 that if we restrict the functors φ_* and $\varphi_!$ to the subcategory of $\mathrm{Rep}(Q)$ consisting of finite dimensional representations, then φ_* and $\varphi_!$ are naturally isomorphic. In that case, φ^* and $\varphi_!$ are both left and right adjoint to each other when φ is a covering morphism. Such a pair is often called a *biadjoint* pair.

The following corollary will prove useful.

Corollary 2.3.11. *The functors φ^*, φ_*, and $\varphi_!$ are additive. Moreover, φ^* is exact, φ_* is left exact, and $\varphi_!$ is right exact.*

Proof: This follows immediately from Theorem 2.3.9 as all adjoint functors are additive, and left (resp. right) adjoint functors are right (resp. left) exact (see Section A.2). ∎

We will now focus on other properties of these functors.

Lemma 2.3.12. *Let Q, Q' be quivers and let $\varphi \in \mathrm{Hom}(Q, Q')$ be a covering morphism. Then both φ_* and $\varphi_!$ are faithful.*

Proof: Let $V, U \in \mathrm{Rep}(Q)$ and consider the map $(\varphi_!)_{VU} \colon \mathrm{Hom}_{\mathrm{Rep}(Q)}(V, U) \to \mathrm{Hom}_{\mathrm{Rep}(Q')}(\varphi_!(V), \varphi_!(U))$. We wish to show that this map is injective. Since it is a group homomorphism, it is enough to consider the preimage of the morphism $0 \colon \varphi_!(V) \to \varphi_!(U)$. This morphism is defined by the linear maps $0(x') \colon \varphi_!(V)(x') \to \varphi_!(U)(x')$ for every $x' \in X'$. So if $\varphi_!(\sigma) = 0$, then $\coprod_{x \in \varphi^{-1}(x')} \sigma(x) = 0$ for every x', and it follows that $\sigma(x) = 0$ for every $x \in X$. Hence $\sigma = 0$. To show that φ_* is faithful is similar. ∎

Remark 2.3.13. We have seen that if we use the generalized definitions of the left and right extension functors in terms of Kan extensions, Theorem 2.3.9 is still true. However, in this generalized setting, Lemma 2.3.12 does not hold.

Lemma 2.3.14. *Let Q, Q' be quivers and let $\varphi \in \mathrm{Hom}(Q, Q')$ be a covering morphism. Then both φ_* and $\varphi_!$ are exact.*

Proof: By Corollary 2.3.11, we need only show that φ_* is right exact and $\varphi_!$ is left exact. Since the categories $\mathrm{Rep}(Q)$ and $\mathrm{Rep}(Q')$ are abelian, it is enough to show that φ_* preserves cokernels and $\varphi_!$ preserves kernels.

Let $V, U \in \operatorname{Rep}(Q)$ and $\sigma \in \operatorname{Hom}_{\operatorname{Rep}(Q)}(V, U)$. The vector spaces of the representation $\varphi_*(\operatorname{coker}\sigma)$ are defined by $\varphi_*(\operatorname{coker}\sigma)(x') = \prod_{x\in\varphi^{-1}(x')} U(x)/\operatorname{im}\sigma(x)$. However, the map $\varphi_*(\sigma)(x')\colon \varphi_*(V)(x') \to \varphi_*(U)(x')$ is given by $\prod_{x\in\varphi^{-1}(x')}\sigma(x)$. Hence

$$\begin{aligned}\operatorname{coker}(\varphi_*(\sigma))(x') &= \varphi_*(U)(x')/\operatorname{im}\varphi_*(\sigma)(x') \\ &= \prod_{x\in\varphi^{-1}(x')} U(x)/\prod_{x\in\varphi^{-1}(x')}\operatorname{im}\sigma(x) \\ &\cong \prod_{x\in\varphi^{-1}(x')} U(x)/\operatorname{im}\sigma(x) \\ &= \varphi_*(\operatorname{coker}\sigma)(x').\end{aligned}$$

It is now clear that for any arrow $\rho' \in A'$ the maps $\varphi_*(\operatorname{coker}\sigma)(\rho')$ and $\operatorname{coker}(\varphi_*(\sigma))(\rho')$ will be equal, and hence φ_* preserves cokernels.

The representation $\varphi_!(\ker\sigma) \in \operatorname{Rep}(Q)$ has vector spaces given by

$$\varphi_!(\ker\sigma)(x) = \bigoplus_{x\in\varphi^{-1}(x')}(\ker\sigma)(x) = \bigoplus_{x\in\varphi^{-1}(x')}\ker(\sigma(x))$$

for all $x \in X$. On the other hand, the morphism $\varphi_!(\sigma) \in \operatorname{Hom}_{\operatorname{Rep}(Q)}(\varphi_!(V), \varphi_!(U))$ is defined by $\varphi_!(\sigma)(x') = \oplus_{x\in\varphi^{-1}(x')}\sigma(x)$ and hence

$$\ker(\varphi_!(\sigma))(x') = \ker\left(\bigoplus_{x\in\varphi^{-1}(x')}\sigma(x)\right) = \bigoplus_{x\in\varphi^{-1}(x')}\ker(\sigma(x)).$$

Thus $\varphi_!(\ker\sigma)(x') = \ker(\varphi_!(\sigma))(x')$ for all $x' \in X'$. Again, it is easy to see that for any $\rho' \in A'$ the maps $\varphi_!(\ker\sigma)(\rho')$ and $\ker(\varphi_!(\sigma))(\rho')$ will also coincide, and so $\varphi_!$ preserves kernels. ∎

2.4 Quivers with Relations

Definition 2.4.1 (Relation)**.** *A* relation *in a quiver Q is an element $r \in \mathbb{C}Q$. More explicitly, a relation in Q is an expression of the form $\sum_{j=1}^{k} a_j\tau_j$, where for every*

$j \in \{1, \dots, k\}$ we have $a_j \in \mathbb{C}$ and τ_j is a path in Q.

We say that a representation $V \in \mathrm{Rep}(Q)$ *satisfies the relation* $\sum_{j=1}^{k} a_j \tau_j$ if $\sum_{j=1}^{k} a_j V(\tau_j) = 0$. If R is a set of relations, we denote by $\mathrm{Rep}(Q, R)$ the category of representations of Q which satisfy the relations R. If I is the two-sided ideal in $\mathbb{C}Q$ generated by R, then there is an equivalence of categories $\mathrm{Rep}(\mathbb{C}Q/I) \cong \mathrm{Rep}(Q, R)$ (see for example [1, Theorem 1.6]).

Remark 2.4.2. If A is an associative algebra then there is a natural equivalence of categories $\mathrm{Rep}(A) \cong A$-Mod. Thus the equivalences described above imply that the categories $\mathrm{Rep}(Q)$ and $\mathrm{Rep}(Q, R)$ are abelian. For more information on categories of representations of quivers, see [1].

Any quiver morphism $\varphi\colon Q \to Q'$ induces an algebra homomorphism $\varphi\colon \mathbb{C}Q \to \mathbb{C}Q'$, and so for any set of relations R' in Q' we can consider the preimage

$$\varphi^{-1}(R') = \{\sum_{j=1}^{k} a_j \tau_j \mid \sum_{j=1}^{k} a_j \varphi(\tau_j) \in R'\}.$$

We will now show that if φ is a covering morphism and R' is a set of relations in Q' then the functors φ^*,φ_*, and $\varphi_!$ restrict naturally to the subcategories of representations satisfying the relations R' and $\varphi^{-1}(R')$. First we recall that if $\varphi\colon Q \to Q'$ is a covering morphism and τ' is a path in Q', then for any $y \in X$ with $\varphi(y) = h(\tau')$ there is a unique path τ in Q ending at y with $\varphi(\tau) = \tau'$. We will denote this unique path τ by $\varphi^{-1}(\tau')_y$.

Lemma 2.4.3. *Let $\varphi\colon Q \to Q'$ be a covering morphism. Then for any path τ' in Q' and any representation $V \in \mathrm{Rep}(Q)$ we have*

$$\varphi_!(V)(\tau') = \bigoplus_{y : h(\tau') = \varphi(y)} V(\varphi^{-1}(\tau')_y), \text{ and} \tag{2.4.1}$$

$$\varphi_*(V)(\tau') = \prod_{y : h(\tau') = \varphi(y)} V(\varphi^{-1}(\tau')_y). \tag{2.4.2}$$

Proof: We begin by noting that since φ is a covering morphism, for any path τ' in Q' there is a bijection between paths τ in Q with $\varphi(\tau) = \tau'$ and vertices $x \in X$ with $\varphi(x) = h(\tau')$ given by $x \leftrightarrow \varphi^{-1}(\tau')_x$. Let τ' be a path in Q'. We will proceed by induction on the length of the path, which we denote by n. If $n = 0$, then the path τ' is a trivial path at some vertex y' in Q'. Then for any vertex $y \in \varphi^{-1}(y')$, $\varphi^{-1}(\tau')_y$ is the trivial path at y. Hence we have

$$\varphi_!(V)(\tau') = \mathrm{id}_{\varphi_!(V)(y')} = \bigoplus_{y \in \varphi^{-1}(y')} \mathrm{id}_{V(y)} = \bigoplus_{y : h(\rho') = \varphi(y)} V(\varphi^{-1}(\rho')_y).$$

Now let $n \geq 1$ and write $\tau' = \alpha'\rho'$, where α' is a path in Q' of length $n-1$ and $\rho' \in A'$. Then if we assume the result is true for the path α', we have

$$\varphi_!(V)(\alpha')\varphi_!(V)(\rho') = \bigoplus_{y : h(\alpha') = \varphi(y)} V(\varphi^{-1}(\alpha')_y) \bigoplus_{x : h(\rho') = \varphi(x)} V(\varphi^{-1}(\rho')_x). \tag{2.4.3}$$

Since φ is a covering morphism, $|\{\alpha \mid \varphi(\alpha) = \alpha'\}| = |\{\tau \mid \varphi(\tau) = \tau'\}|$ as both sets are in bijection with the set of vertices in X that get mapped to $h(\alpha') = h(\tau')$. Moreover, for any $y \in X$ with $\varphi(y) = h(\tau')$, if we let $x = t(\varphi^{-1}(\alpha')_y)$ then we have $\varphi^{-1}(\tau')_y = \varphi^{-1}(\alpha')_y \varphi^{-1}(\rho')_x$. It follows that each $\varphi^{-1}(\tau')_y$ shows up exactly once in the decomposition (2.4.3) above. On the other hand, if $x \in X$ is a vertex such that the arrow $\varphi^{-1}(\rho')_x$ cannot be extended to a path which maps to τ' under φ, then in the decomposition (2.4.3) $V(x)$ is mapped to 0. We can therefore ignore such components, and we conclude that

$$\varphi_!(\tau') = \bigoplus_{y : h(\tau') = \varphi(y)} V(\varphi^{-1}(\tau')_y),$$

from which the result follows by induction. To show that

$$\varphi_*(\tau') = \prod_{y : h(\tau') = \varphi(y)} V(\varphi^{-1}(\tau')_y)$$

is similar. ∎

Proposition 2.4.4. *Let Q, Q' be quivers, R' a set of relations in Q', and $\varphi\colon Q \to Q'$ a covering morphism.*

(i) If $V \in \mathrm{Rep}(Q', R')$, then $\varphi^*(V) \in \mathrm{Rep}(Q, \varphi^{-1}(R'))$.

(ii) If $V \in \mathrm{Rep}(Q, \varphi^{-1}(R'))$ then $\varphi_!(V) \in \mathrm{Rep}(Q', R')$ and $\varphi_*(V) \in \mathrm{Rep}(Q', R')$.

Proof:

(i) Let $\sum_{j=1}^{k} a_j \tau'_j \in R'$. Then since $V \in \mathrm{Rep}(Q', R')$, we have $\sum_{j=1}^{k} a_j V(\tau'_j) = 0$. Hence for any $y \in X$ with $\varphi(y) = h(\tau'_j)$ we have

$$\sum_{j=1}^{k} a_j \varphi^*(V)(\varphi^{-1}(\tau'_j)_y) = \sum_{j=1}^{k} a_j V(\tau'_j) = 0.$$

It follows that $\varphi^*(V) \in \mathrm{Rep}(Q, \varphi^{-1}(R'))$.

(ii) Once again, suppose $\sum_{j=1}^{k} a_j \tau'_j \in R'$. Let $V \in \mathrm{Rep}(Q, \varphi^{-1}(R'))$ and let $h(\tau'_j) = y'$ for every j. Then using Lemma 2.4.3 we get:

$$\begin{aligned}\sum_{j=1}^{k} a_j \varphi_!(V)(\tau'_j) &= \sum_{j=1}^{k} a_j \bigoplus_{y:y'=\varphi(y)} V(\varphi^{-1}(\tau'_j)_y) \\ &= \bigoplus_{y:y'=\varphi(y)} \sum_{j=1}^{k} a_j V(\varphi^{-1}(\tau'_j)_y) \\ &= 0.\end{aligned}$$

It follows that $\varphi_!(V) \in \mathrm{Rep}(Q', R')$. To show that $\varphi_*(V) \in \mathrm{Rep}(Q', R')$ is similar.

∎

We will denote by $\varphi^*_{R'}$ the restriction of the functor φ^* to the subcategory $\mathrm{Rep}(Q', R')$. Similarly, we denote by $\varphi_!^{R'}$ and $\varphi_*^{R'}$ the restrictions of the functors

$\varphi_!$ and φ_* to the subcategory $\operatorname{Rep}(Q, \varphi^{-1}(R'))$. By the results of Section 2.3, for any set of relations R' in Q' the functors $\varphi^*_{R'}$, $\varphi_!^{R'}$, and $\varphi_*^{R'}$ are all additive, exact, and both $\varphi_!^{R'}$ and $\varphi_*^{R'}$ are faithful. Moreover, Theorem 2.3.9 implies that $\varphi^*_{R'}$ is left adjoint to $\varphi_*^{R'}$ and right adjoint to $\varphi_!^{R'}$.

2.5 Graded Dimension

In [8], Lusztig introduced certain varieties associated to quivers, now called Lusztig's nilpotent quiver varieties. These varieties were used to introduce a canonical basis of the lower half of the universal enveloping algebra associated to a Kac-Moody Lie algebra. These quiver varieties have now become an important tool in the study of Lie algebras and their representations, and so a natural question to ask is how the results of this chapter translate into the language of these varieties. While a thorough investigation of this relationship is outside the scope of the current work, we provide a starting point for such questions here. More precisely, we show that the restriction functor and the left and right extension functors associated to a quiver morphism induce a morphism between the associated quiver varieties, and that the image of these morphisms are subvarieties of their codomains. It would be fruitful to better understand the images of these maps, and this provides a potential avenue for further study. For example, it would be useful to know when the images of these maps are isomorphic to irreducible components of a Lusztig quiver variety, as these components correspond to the elements of the canonical basis alluded to earlier. We begin with some definitions which lead to the description of the Lusztig quiver varieties, and then we examine how the functors described in Section 2.4 behave with respect to this construction.

Definition 2.5.1 (Graded Dimension)**.** *Let $Q = (X, A)$ and let $V \in \operatorname{Rep}(Q)$ be such that $\dim V(x) < \infty$ for all $x \in X$. The* graded dimension *of V is the function*

$\alpha\colon X \to \mathbb{N}$ *defined by* $\alpha(x) = \dim V(x)$*. If* α *is the graded dimension of* V*, we write* $\dim V = \alpha$*. We say* α *is* of finite type *if* $\sum_{x\in X} \alpha(x) < \infty$*, and we say* V *is* finite dimensional *if* $\dim V$ *is of finite type.*

We will denote the set of all functions $\alpha\colon \mathbb{N} \to X$ by $\mathbb{N}^X$. Let $\alpha \in \mathbb{N}^X$ and let $V \in \operatorname{Rep}(Q)$ be such that $\dim V = \alpha$. Then for any $x \in X$, we have $V(x) \cong \mathbb{C}^{\alpha(x)}$, and so by fixing bases for each $V(x)$ we can identify V with an element of the space

$$E_\alpha^Q := \bigoplus_{\rho\in A} \operatorname{Hom}(\mathbb{C}^{\alpha(t(\rho))}, \mathbb{C}^{\alpha(h(\rho))}).$$

We call E_α^Q the *representation space of dimension* α of Q. Of course, E_α^Q is isomorphic to the space $\bigoplus_{\rho\in A} \operatorname{Mat}_{\alpha(t(\rho))\times\alpha(h(\rho))}(\mathbb{C})$, where $\operatorname{Mat}_{n\times m}(\mathbb{C})$ denotes the space of $n\times m$ matrices with complex entries.

For any $\alpha \in \mathbb{N}^X$, we define the group

$$\operatorname{GL}(\alpha) := \prod_{x\in X} \operatorname{GL}_{\alpha(x)}(\mathbb{C}),$$

where $\operatorname{GL}_n(\mathbb{C})$ denotes the group of automorphisms of $\mathbb{C}^n$. Then $\operatorname{GL}(\alpha)$ acts on the space E_α^Q via $(g\cdot T)_\rho = g_{h(\rho)} T_\rho g_{t(\rho)}^{-1}$ for all $\rho \in A, T \in E_\alpha^Q$. This action is illustrated by the commutativity of the following diagram

$$\begin{array}{ccc} \mathbb{C}^{\alpha(t(\rho))} & \xrightarrow{T_\rho} & \mathbb{C}^{\alpha(h(\rho))} \\ {\scriptstyle g_{t(\rho)}}\downarrow & & \downarrow{\scriptstyle g_{h(\rho)}} \\ \mathbb{C}^{\alpha(t(\rho))} & \xrightarrow[(g\cdot T)_\rho]{} & \mathbb{C}^{\alpha(h(\rho))}. \end{array}$$

Clearly, two elements $T, S \in E_\alpha^Q$ are isomorphic when thought of as representations of Q if and only if they lie in the same $\operatorname{GL}(\alpha)$-orbit.

Now suppose $\varphi : Q \to Q'$ is a quiver morphism, and let V be a representation of Q' having graded dimension α. Then for any $x \in X$ we have

$$\dim \varphi^*(V)(x) = \dim V(\varphi(x)) = \alpha(\varphi(x)),$$

and hence $\dim \varphi^*(V) = \alpha\varphi$. In other words, φ^* induces a map $E_\alpha^{Q'} \to E_{\alpha\varphi}^{Q}$ which is obviously linear.

For any $\alpha \in \mathbb{N}^X$ of finite type, define $\varphi_!(\alpha) \in \mathbb{N}^{X'}$ by $\varphi_!(\alpha)(x') = \sum_{x\in\varphi^{-1}(x')} \alpha(x)$. Then if $T \in E_\alpha^Q$, we have $\dim \varphi_!(T)(x') = \varphi_!(\alpha)(x')$, and hence $\varphi_!$ induces a map $E_\alpha^Q \to E_{\varphi_!(\alpha)}^{Q'}$.

Due to the results of Section 2.4, this discussion extends to the case of quivers with relations. That is, if R is a set of relations in a quiver Q, then we can identify representations $V \in \operatorname{Rep}(Q, R)$ with elements of the space

$$E_\alpha^R := \{T \in E_\alpha^Q \mid T \text{ satisfies the relations } R\}.$$

Then if R' is a set of relations in Q', $\varphi_{R'}^*$ gives a linear map $E_\alpha^{R'} \to E_{\alpha\varphi}^{\varphi^{-1}(R')}$, and $\varphi_!^{R'}$ gives a linear map $E_\alpha^{\varphi^{-1}(R')} \to E_{\varphi_!(\alpha)}^{R'}$.

We now examine a specific example of this. We start with a quiver morphism $\varphi\colon Q \to Q'$, and suppose $Q' = \overleftrightarrow{Q^*}$ for some quiver Q^*. That is, Q' is the double quiver of some other quiver. We will consider certain subsets of the spaces $E_\alpha^{R'}$.

Definition 2.5.2 (Nilpotent Representation)**.** *A representation V of a quiver $Q = (X, A)$ is said to be* nilpotent *if there exists some $N \in \mathbb{N}$ such that for any path of the form $\rho_N \cdots \rho_1$ in Q, where $\rho_i \in A$ for all i, we have $V(\rho_N \cdots \rho_1) = 0$.*

Now let R' be the set of Gelfand-Ponomarev relations (2.1.1) in Q'. We will study the sets

$$\Lambda_\alpha^{Q^*} := \{T \in E_\alpha^{R'} \mid T \text{ is nilpotent}\}.$$

These objects were introduced by Lusztig [8], and are known as *Lusztig's (nilpotent) quiver varieties*. Later on we will show that if φ is a covering morphism, then $\varphi_!(T)$ is nilpotent if T is nilpotent. Hence $\varphi_!$ gives a map from the set of nilpotent elements of $E_\alpha^{\varphi^{-1}(R')}$, which we denote by $\mathcal{N}(E_\alpha^{\varphi^{-1}(R')})$, into the Lusztig quiver variety $\Lambda_{\varphi_!(\alpha)}^{Q^*}$.

Let $n = \dim E_\alpha^Q$. If we fix a basis of E_α^Q, then we can identify E_α^Q with n-dimensional affine space $\mathbb{A}^n$ using coordinates with respect to this basis. We will usually consider elements of E_α^Q to be matrices and choose as our basis the set $\{e_{ij}^\rho \mid 1 \le i \le \alpha(t(\rho)), 1 \le j \le \alpha(h(\rho))\}_{\rho \in A}$, where e_{ij}^ρ denotes the $\alpha(t(\rho)) \times \alpha(h(\rho))$ matrix whose (i,j)-entry is equal to 1, and all other entries are 0. Under this identification, the condition that a representation $T \in E_\alpha^Q$ satisfies a given relation in Q becomes a system of polynomial equations in the affine coordinates of T. Thus $E_\alpha^{\varphi^{-1}(R')}$ is a Zariski-closed subset of $\mathbb{A}^n$. In fact, $\mathcal{N}(E_\alpha^{\varphi^{-1}(R')})$ is also a Zariski-closed subset of $\mathbb{A}^n$ as it is an intersection of closed sets.

Note that if α is of finite type, so is $\varphi_!(\alpha)$, and moreover $n = \dim E_\alpha^Q \le \dim E_{\varphi_!(\alpha)}^{Q'} =: m$. Thus we have:

$$\begin{aligned}
E_{\varphi_!(\alpha)}^{Q'} &= \bigoplus_{\rho' \in A'} \mathrm{Hom}(\mathbb{C}^{\varphi_!(\alpha)(t(\rho'))}, \mathbb{C}^{\varphi_!(\alpha)(h(\rho'))}) \\
&= \bigoplus_{\rho' \in A'} \mathrm{Hom}\left(\bigoplus_{x : t(\rho') = \varphi(x)} \mathbb{C}^{\alpha(x)}, \bigoplus_{y : h(\rho') = \varphi(y)} \mathbb{C}^{\alpha(y)} \right) \\
&= \bigoplus_{\rho' \in A'} \bigoplus_{\substack{x : t(\rho') = \varphi(x) \\ y : h(\rho') = \varphi(y)}} \mathrm{Hom}(\mathbb{C}^{\alpha(x)}, \mathbb{C}^{\alpha(y)}) \\
&= \bigoplus_{\rho' \in A'} \bigoplus_{\rho : \rho' = \varphi(\rho)} \mathrm{Hom}(\mathbb{C}^{\alpha(t(\rho))}, \mathbb{C}^{\alpha(t(\rho))}) \bigoplus_{\substack{x : t(\rho') = \varphi(x) \\ y : h(\rho') = \varphi(y) \\ \nexists \rho : x \to y}} \mathrm{Hom}(\mathbb{C}^{\alpha(x)}, \mathbb{C}^{\alpha(y)}) \\
&= E_\alpha^Q \oplus \bigoplus_{\rho' \in A'} \bigoplus_{\substack{x : t(\rho') = \varphi(x) \\ y : h(\rho') = \varphi(y) \\ \nexists \rho : x \to y}} \mathrm{Hom}(\mathbb{C}^{\alpha(x)}, \mathbb{C}^{\alpha(y)}).
\end{aligned}$$

In particular, E_α^Q is naturally isomorphic to a subspace of $E_{\varphi_!(\alpha)}^{Q'}$. Hence we may complete the set $\{e_{ij}^\rho\}$ mentioned above to a basis of $E_{\varphi_!(\alpha)}^{Q'}$, and thereby identify $E_{\varphi_!(\alpha)}^{Q'}$ with $\mathbb{A}^m$. Under these identifications, the map $\varphi_!^{R'} : \mathcal{N}(E_\alpha^{\varphi^{-1}(R')}) \to \Lambda_{\varphi_!(\alpha)}^{Q^*}$ is the restriction of the canonical injection $\mathbb{A}^n \hookrightarrow \mathbb{A}^m$ to the variety $\mathcal{N}(E_{\varphi_!(\alpha)}^{Q'})$. This shows that $\varphi_!^{R'}$ is a morphism of affine varieties. In fact, we claim that $\varphi_!^{R'}(\mathcal{N}(E_\alpha^{\varphi^{-1}(R')}))$ is isomorphic to a subvariety of $\Lambda_{\varphi_!(\alpha)}^{Q^*}$. Indeed, consider the pullback of $\varphi_!^{R'}$ between the coordinate rings of $\mathcal{N}(E_\alpha^{\varphi^{-1}(R')})$ and $\Lambda_{\varphi_!(\alpha)}^{Q^*}$:

$$\mathbb{C}[\Lambda_{\varphi_!(\alpha)}^{Q^*}] \to \mathbb{C}[\mathcal{N}(E_\alpha^{\varphi^{-1}(R')})]$$
$$f \mapsto f \circ \varphi_!^{R'}.$$

If $f \in \mathbb{C}[\mathcal{N}(E_\alpha^{\varphi^{-1}(R')})]$, then fix a representative of the coset of f (under the identification (B.1)) and write $f = [g]$, where $g \in \mathbb{C}[x_1, \ldots, x_n]$. Let $g \mapsto g'$ under the canonical injection $\mathbb{C}[x_1, \ldots, x_n] \hookrightarrow \mathbb{C}[x_1, \ldots, x_m]$ and define $f' \in \mathbb{C}[\Lambda_{\varphi_!(\alpha)}^{Q^*}]$ by $f' = [g']$. Then

$$f' \circ \varphi_!^{R'} = [g' \circ \varphi_!^{R'}(x_1, \ldots, x_n)] = [g'(x_1, \ldots, x_n, 0, \ldots, 0] = [g(x_1, \ldots, x_n)] = f,$$

and therefore the pullback between the coordinate rings is surjective. It follows that the image of the morphism $\varphi_!^{R'}$ is isomorphic to a subvariety of $\Lambda_{\varphi_!(\alpha)}^{Q^*}$ by Lemma B.

We will now give a concise summary of how the functors φ^*, φ_*, and $\varphi_!$ act with respect to certain subcategories of $\mathrm{Rep}(Q)$ and $\mathrm{Rep}(Q')$.

First we consider the subcategories $\mathrm{Rep}(Q, \varphi^{-1}(R'))$ and $\mathrm{Rep}(Q', R')$ where R' is a set of relations in Q'. In Proposition 2.4.4, we showed that φ^* maps $\mathrm{Rep}(Q', R')$ to $\mathrm{Rep}(Q, \varphi^{-1}(R'))$ and that the functors φ_* and $\varphi_!$ map $\mathrm{Rep}(Q, \varphi^{-1}(R'))$ to $\mathrm{Rep}(Q', R')$. For any quiver Q, we will denote by $\mathrm{NRep}(Q)$ the subcategory of $\mathrm{Rep}(Q)$ consisting of nilpotent representations of Q. We claim that the functors induced by quiver morphisms on the categories of representations respect the property of being nilpotent.

Proposition 2.5.3. *(i) If $V \in \mathrm{NRep}(Q')$, then $\varphi^*(V) \in \mathrm{NRep}(Q)$.*

(ii) If $V \in \mathrm{NRep}(Q)$, then $\varphi_!(V), \varphi_*(V) \in \mathrm{NRep}(Q')$.

Proof:

(i) Let $V \in \mathrm{NRep}(Q')$ and suppose $\varphi^*(V)$ were not nilpotent. Then for any $m \in \mathbb{N}$, there is some path $\rho_1 \ldots \rho_m$ in Q with $\varphi^*(V)(\rho_1 \ldots \rho_m) \neq 0$. But then we have $0 \neq V(\rho'_1 \ldots \rho'_m)$, where $\rho'_i = \varphi(\rho_i)$ for each $i = 1, \ldots, m$, which contradicts the fact that V is nilpotent.

(ii) Let $V \in \mathrm{NRep}(Q)$. Let N be such that for any path τ of length N in Q, $V(\tau) = 0$. Then for any path τ' of length N in Q', for any vertex $x \in X$, the path $\varphi^{-1}(\tau)_x$ is also of length N. It follows from Lemma 2.4.3 that we have $\varphi_!(V)(\tau') = \bigoplus_{x:h(\tau')=\varphi(x)} V(\varphi^{-1}(\tau')_x) = 0$. The proof for φ_* is similar.

∎

We conclude that φ^* maps $\mathrm{NRep}(Q', R')$ to $\mathrm{NRep}(Q, \varphi^{-1}(R'))$ and that $\varphi_!$ and φ_* map $\mathrm{NRep}(Q, \varphi^{-1}(R'))$ to $\mathrm{NRep}(Q', R')$, where $\mathrm{NRep}(Q, R)$ denotes the category of nilpotent representations of Q satisfying the relations R.

Next we consider the subcategories $\mathrm{Rep}_{\mathrm{fd}}(Q)$ and $\mathrm{Rep}_{\mathrm{fd}}(Q')$ consisting of finite dimensional representations of Q and Q' respectively.

Lemma 2.5.4. *If $V \in \mathrm{Rep}_{\mathrm{fd}}(Q)$ then $\varphi_!(V), \varphi_* \in \mathrm{Rep}_{\mathrm{fd}}(Q')$.*

Proof: If $V \in \mathrm{Rep}_{\mathrm{fd}}(Q)$ and $\dim V = \alpha$, then the graded dimension of both $\varphi_!(V)$ and $\varphi_*(V)$ is given by $\varphi_!(\alpha)(x') = \sum_{x \in \varphi^{-1}(x')} \alpha(x)$ for all $x' \in X'$. The result follows. ∎

On the other hand, it is not difficult to see that φ^* does *not* map $\mathrm{Rep}_{\mathrm{fd}}(Q')$ to $\mathrm{Rep}_{\mathrm{fd}}(Q)$ in general. Indeed, suppose φ is a morphism having the property that there is some vertex $x' \in Q'$ such that there are infinitely many vertices $x \in X$ with $\varphi(x) = x'$. If V is a finite dimensional representation of Q' such that $\dim V(x') > 0$, then $\varphi^*(V)$ will not be finite dimensional. This motivates the definition of *locally finite* representations.

Definition 2.5.5 (Locally Finite Representation)**.** *For a quiver $Q = (X, A)$, we will say a representation $V \in \mathrm{Rep}(Q)$ is* locally finite *if $V(x)$ is a finite dimensional vector space for every $x \in X$. We denote by $\mathrm{Rep}_{\mathrm{lf}}(Q)$ the subcategory of $\mathrm{Rep}(Q)$ consisting of locally finite representations.*

Lemma 2.5.6. *If* $V \in \mathrm{Rep}_{\mathrm{lf}}(Q')$ *then* $\varphi^*(V) \in \mathrm{Rep}_{\mathrm{lf}}(Q)$.

Proof: If $\dim V(x') < \infty$ for every $x' \in X'$, then $\dim \varphi^*(V)(x) = \dim V(\varphi(x)) < \infty$ for every $x \in X$, and so φ^* preserves the property of being locally finite. ∎

The extension functors $\varphi_!$ and φ_*, however, do not necessarily preserve the property of being locally finite. Again, if φ maps infinitely many vertices of Q to a single vertex in Q', it is easy to construct a locally finite representation V of Q such that $\varphi_!(V)$ is not locally finite.

The discussion in the two preceeding paragraphs is summarized in the tables below. The entries in the tables represent the subcategory of Rep(Q) in which the image of representations having the properties specified by the row and column live.

φ^*	Satisfies Relations R'	Nilpotent	Both
Finite dimensional	$\mathrm{Rep}_{\mathrm{lf}}(Q, \varphi^{-1}(R'))$	$\mathrm{NRep}_{\mathrm{lf}}(Q)$	$\mathrm{NRep}_{\mathrm{lf}}(Q, \varphi^{-1}(R'))$
Locally finite	$\mathrm{Rep}_{\mathrm{lf}}(Q, \varphi^{-1}(R'))$	$\mathrm{NRep}_{\mathrm{lf}}(Q)$	$\mathrm{NRep}_{\mathrm{lf}}(Q, \varphi^{-1}(R'))$
Neither	$\mathrm{Rep}(Q, \varphi^{-1}(R'))$	$\mathrm{NRep}(Q)$	$\mathrm{NRep}(Q, \varphi^{-1}(R'))$

$\varphi_*, \varphi_!$	Satisfies Relations $\varphi^{-1}(R')$	Nilpotent	Both
Finite dimensional	$\mathrm{Rep}_{\mathrm{fd}}(Q, R')$	$\mathrm{NRep}_{\mathrm{fd}}(Q)$	$\mathrm{NRep}_{\mathrm{fd}}(Q, R')$
Locally finite	$\mathrm{Rep}(Q, R')$	$\mathrm{NRep}(Q)$	$\mathrm{NRep}(Q, R')$
Neither	$\mathrm{Rep}(Q, R')$	$\mathrm{NRep}(Q)$	$\mathrm{NRep}(Q, R')$

We remark that all of the properties discussed in Section 2.3 (adjointness, faithfulness, exactness) are unaffected by the restriction of a functor to a subcategory. Thus we may use the results of that section even when we are considering the action of $\varphi^*, \varphi_!$ or φ_* on one of the subcategories mentioned above.

Chapter 3

Lie Algebras

We begin this chapter with a brief review of some elementary notions of Lie algebras, and recall the classification of three-dimensional Lie algebras over $\mathbb{C}$. We then focus on a specific infinite family of non-isomorphic three-dimensional Lie algebras, and introduce modified versions of their universal enveloping algebras.

3.1 Elementary Notions

A *Lie algebra* over a field F is an F-vector space, L, together with a bilinear map, the *Lie bracket*,

$$L \times L \to L$$
$$(x, y) \mapsto [x, y]$$

satisfying the following properties:

$$[x, x] = 0 \qquad \forall x \in L,$$
$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \qquad \forall x, y, z \in L \qquad \text{(the Jacobi identity).}$$

A Lie algebra is said to be *abelian* if $L = Z(L)$, where

$$Z(L) = \{x \in L \mid [x, y] = 0 \quad \forall y \in L\}$$

is the centre of L. We define the *derived algebra* of a Lie algebra L as

$$L' = \mathrm{Span}\{[x, y] \mid x, y \in L\}.$$

Given two Lie algebras, L_1 and L_2, a map $\varphi : L_1 \to L_2$ is a *Lie algebra homomorphism* if $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for every $x, y \in L_1$. An important Lie algebra homomorphism is the *adjoint homomorphism*, $\mathrm{ad} : L \to \mathrm{End}(L)$ defined by $(\mathrm{ad}\, x)(y) := [x, y]$.

By a *representation* of a Lie algebra L, we shall mean a homomorphism $\varphi : L \to \mathfrak{gl}(V)$ for some vector space V. Here $\mathfrak{gl}(V)$ denotes the Lie algebra of endomorphisms of V with Lie bracket given by $[x, y] = xy - yx$ for all $x, y \in \mathfrak{gl}(V)$. If $\varphi : L \to \mathfrak{gl}(V)$ is a representation of a Lie algebra L, then V is an *L-module* with action given by $x \cdot v = \varphi(x)v$ for all $v \in V$. By the usual abuse of terminology, we will often use the terms representation and module interchangeably.

3.2 Lie Algebras of Low Dimension

In this section we study complex Lie algebras of dimension at most 3. We review the classification of such Lie algebras, and then study one particular family of 3-dimensional Lie algebras. This family depends on a continuous parameter $\mu \in \mathbb{C}$, and we denote its members by L_μ. The Lie algebras L_μ are not as well understood as the other 3-dimensional complex Lie algebras, and their representation theory is the subject of the current paper.

First, we note that every 1-dimensional Lie algebra is abelian. It is not difficult to see that two abelian Lie algebras are isomorphic if and only if they have the same dimension, and hence all 1-dimensional Lie algebras are isomorphic. In the 2-dimensional case, there is a unique (up to isomorphism) non-abelian Lie algebra, which has a basis $\{x, y\}$ such that its Lie bracket is described by $[x, y] = x$. See for example [3, Theorem 3.1] for a proof of this statement.

In dimension 3, all cases of non-abelian Lie algebras can be classified by relating L' to $Z(L)$. First we will consider the cases $\dim L' = 1$ and $\dim L' = 3$.

Lemma 3.2.1 ([3, Section 3.2]). *There are unique 3-dimensional Lie algebras having the following properties:*

(i) $\dim L' = 1$ *and* $L' \subseteq Z(L)$. *This Lie algebra is known as the* Heisenberg algebra.

(ii) $\dim L' = 1$ *but* $L' \nsubseteq Z(L)$. *This Lie algebra is the direct sum of the 2-dimensional non-abelian Lie algebra with the 1-dimensional Lie algebra.*

(iii) $L' = L$. *In this case,* $L = \mathfrak{sl}(2, \mathbb{C})$, *the Lie subalgebra of* $\mathfrak{gl}(2, \mathbb{C}) := \mathfrak{gl}(\mathbb{C}^2)$ *consisting of trace zero operators.*

We will now consider the case where $\dim L = 3$ and $\dim L' = 2$. Let $\{y, z\}$ be a basis of L' and let $x \in L \setminus L'$. We now have a basis of L, $\{x, y, z\}$. We need the following lemma.

Lemma 3.2.2 ([3, Lemma 3.3]). *Let L be a Lie algebra such that* $\dim L = 3$ *and* $\dim L' = 2$, *and let* $x \in L \setminus L'$. *Then:*

(i) L' *is abelian.*

(ii) The map $\operatorname{ad} x : L' \to L'$ *is an isomorphism.*

We can separate Lie algebras L having the properties $\dim L = 3$ and $\dim L' = 2$ into two cases. The first happens when there is some $\beta \in L \setminus L'$ such that $\operatorname{ad} \beta : L' \to L'$ is diagonalisable. The second case occurs if $\operatorname{ad} x : L' \to L'$ is not diagonalisable for any $x \in L \setminus L'$.

In the latter case, we get that the Jordan canonical form of the matrix of $\operatorname{ad} x$ must be a single 2×2 Jordan block. Since $\operatorname{ad} x$ is an isomorphism it must have an eigenvector with nonzero eigenvalue, and by proper scaling of x we may assume that this eigenvalue is 1. Thus the Jordan form of the matrix of $\operatorname{ad} x$ acting on L' is

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

which completely determines the Lie algebra L.

The case of interest for us will be the one where there is some $\beta \notin L'$ such that the map $\operatorname{ad}\beta : L' \to L'$ is diagonalisable. We will choose basis $\{\alpha_1, \alpha_2\}$ of L' consisting of eigenvectors of $\operatorname{ad}\beta$. Then the associated eigenvalues of α_1 and α_2 must be nonzero by part (ii) of Lemma 3.2.2.

Since α_1 and α_2 are eigenvectors of $\operatorname{ad}\beta$, $[\beta, \alpha_1] = \eta\alpha_1$ and $[\beta, \alpha_2] = \mu\alpha_2$ for some $\eta, \mu \in \mathbb{C}^*$. We have $[\eta^{-1}\beta, \alpha_1] = \alpha_1$, therefore, with proper scaling, we may assume that $\eta = 1$. With respect to the basis $\{\alpha_1, \alpha_2\}$, $\operatorname{ad}\beta : L' \to L'$ has matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & \mu \end{pmatrix}. \tag{3.2.1}$$

We will call this Lie algebra L_μ. Furthermore, by [3, Exercise 3.2] we can see that $L_\mu \cong L_\nu \Leftrightarrow \mu = \nu$ or $\mu = \nu^{-1}$. This is true for all $\mu \in \mathbb{C}^*$. First we will focus on the case where $\mu \in \mathbb{Q}^*$. The matrix of $\operatorname{ad} x$ is

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{n}{m} \end{pmatrix},$$

where $\mu = \frac{n}{m}$. With a simple change of basis we get

$$\begin{pmatrix} m & 0 \\ 0 & n \end{pmatrix}. \tag{3.2.2}$$

Therefore, L_μ has basis $\{\alpha_1, \beta, \alpha_2\}$ and commutation relations

$$[\beta, \alpha_1] = m\alpha_1, \qquad [\beta, \alpha_2] = n\alpha_2, \qquad [\alpha_1, \alpha_2] = 0. \tag{3.2.3}$$

Remark 3.2.3. A special case of this algebra occurs when $\mu = -1$ and we get L_{-1} which is the *Euclidean algebra*. See [12, Section 2] for a discussion of this Lie algebra in the same context as the current paper.

Remark 3.2.4. Since $\mu^{-1} = (\frac{n}{m})^{-1} = \frac{m}{n}$, we have $L_{\frac{n}{m}} \cong L_{\frac{m}{n}}$.

On the other hand, if $\mu \in \mathbb{C} \setminus \mathbb{Q}$ then by (3.2.1) the commutation relations are

$$[\beta, \alpha_1] = \alpha_1, \qquad [\beta, \alpha_2] = \mu\alpha_2, \qquad [\alpha_1, \alpha_2] = 0. \tag{3.2.4}$$

3.3 The Universal Enveloping Algebra of L_μ and its Representations

3.3.1 Universal Enveloping Algebras

If L is a Lie algebra, then the *universal enveloping algebra* of L is the pair (U, i), where U is a unital associative algebra and $i : L \to U$ is a map satisfying

$$i([x, y]) = i(x)i(y) - i(y)i(x) \quad \forall x, y \in L. \tag{3.3.1}$$

Moreover, the pair (U, i) is *universal* with respect to this property. That is, for any pair (U', i') satisfying (3.3.1) there exists a unique homomorphism $\varphi : U \to U'$ such that $\varphi i = i'$. Since we have defined the universal enveloping algebra by a universal property, it is unique up to isomorphism. Furthermore, this definition makes it clear that the category of representations of a Lie algebra L is equivalent to the category of representations of its universal enveloping algebra U. If we denote by T the tensor algebra of the Lie algebra L and by I_T the two sided ideal of T generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$, $x, y \in L$, then it can be shown that $U \cong T/I_T$. If $\{x_i \mid i \in I\}$ is a basis of L ordered by some indexing set I, then the set $\{x_{i_1} \cdots x_{i_n} \mid i_1 \le \cdots \le i_n\}$ forms a basis for the space T/I_T. This is known as the *Poincaré-Birkhoff-Witt* (PBW) Theorem. See for example [7, Section 17] for a proof of both the isomorphism $U \cong T/I_T$ and the PBW Theorem.

3.3.2 Rational Case

Let $\mu \in \mathbb{Q}^*$, $\mu = \frac{n}{m}$ with $\gcd(m, n) = 1$. Then for any indecomposable representation V of L_μ, the eigenvalues of the action of β on V will come from a set of the form $\gamma + \mathbb{Z}$ for some $\gamma \in \mathbb{C}$. We will write V_λ to represent the eigenspace of β with eigenvalue λ. This gives the following eigenspace decomposition:

$$V = \bigoplus_{k\in\mathbb{Z}} V_{\gamma+k}, \qquad V_\lambda = \{v \in V \mid \beta \cdot v = \lambda v\}.$$

Let $U_{m,n}$ be the universal enveloping algebra of L_μ and let U^0, U^1, U^2 be the subalgebras generated by $\beta, \alpha_1, \alpha_2$ respectively. Then, by the PBW Theorem, we have

$$U_{m,n} \cong U^1 \otimes U^0 \otimes U^2 \quad \text{(as vector spaces)}. \tag{3.3.2}$$

From (3.2.3) we obtain the following relations in the universal enveloping algebra:

$$\beta\alpha_1 - \alpha_1\beta = m\alpha_1, \qquad \beta\alpha_2 - \alpha_2\beta = n\alpha_2, \qquad \alpha_1\alpha_2 = \alpha_2\alpha_1. \tag{3.3.3}$$

Let $x \in V_\lambda$. Then $\beta x = \lambda x$ so we have:

$$\begin{aligned} \beta(\alpha_1 x) &= (\beta\alpha_1)x \\ &= (m\alpha_1 + \alpha_1\beta)x \qquad \text{by (3.3.3)} \\ &= (\lambda + m)(\alpha_1 x) \\ &\Rightarrow \alpha_1 V_\lambda \subseteq V_{m+\lambda}. \end{aligned}$$

Similarly, we find that $\alpha_2 V_\lambda \subseteq V_{n+\lambda}$.

Following [12, Section 2] we will consider the modified enveloping algebra $\widetilde{U}_{m,n}$ of L_μ by replacing U^0 with a sum of 1-dimensional algebras:

$$\widetilde{U}_{m,n} = U^1 \otimes \left(\bigoplus_{k\in\mathbb{Z}} \mathbb{C}a_k\right) \otimes U^2. \tag{3.3.4}$$

Multiplication in the modified enveloping algebra is given by

$$a_k a_\ell = \delta_{k\ell} a_k,$$

$$\alpha_1 a_k = a_{k+m}\alpha_1, \quad \alpha_2 a_k = a_{k+n}\alpha_2,$$
$$\alpha_1\alpha_2 a_k = \alpha_2\alpha_1 a_k, \tag{3.3.5}$$

where $k, \ell \in \mathbb{Z}$. We can think of a_k as the projection onto the k-th weight space $V_{\gamma+k}$. For any associative algebra A, we denote the category of representations of A by $\text{Rep}(A)$. If we denote by wt-rep$(U_{m,n})$ the category of representations of $U_{m,n}$ on which β acts semisimply with eigenvalues from the set $\gamma + \mathbb{Z}$, then we have the equivalence of categories wt-rep$(U_{m,n}) \cong \text{Rep}(\widetilde{U}_{m,n})$.

3.3.3 Non-Rational Case

Using the same notation as in Section 3.3.2, we have V a representation of L_μ and V_λ the eigenspace of β with eigenvalue λ. However, when $\mu \in \mathbb{C} \setminus \mathbb{Q}$, β will act on indecomposable representations of L_μ with eigenvalues of the form $\gamma + k$ for some $\gamma \in \mathbb{C}$, where $k \in \mathbb{Z} + \mathbb{Z}\mu$. Therefore, we have $V = \bigoplus_{k \in \mathbb{Z}+\mathbb{Z}\mu} V_{\gamma+k}$.

When $\mu \in \mathbb{C} \setminus \mathbb{Q}$, we get the same decomposition of the universal enveloping algebra as found in (3.3.2). However, the relations found in (3.3.3) become

$$\beta\alpha_1 - \alpha_1\beta = \alpha_1, \qquad \beta\alpha_2 - \alpha_2\beta = \mu\alpha_2, \qquad \alpha_1\alpha_2 = \alpha_2\alpha_1. \tag{3.3.6}$$

As in the rational case, we find that $\alpha_1 V_\lambda \subseteq V_{\lambda+1}$ and $\alpha_2 V_\lambda \subseteq V_{\lambda+\mu}$.

Again we consider the modified enveloping algebra, $\widetilde{U}_\mu$, in this case given by

$$\widetilde{U}_\mu = U^1 \otimes \left(\bigoplus_{k \in \mathbb{Z}+\mu\mathbb{Z}} \mathbb{C}a_k \right) \otimes U^2. \tag{3.3.7}$$

Multiplication in this modified enveloping algebra is given by

$$a_k a_\ell = \delta_{k\ell} a_k,$$
$$\alpha_1 a_k = a_{k+1}\alpha_1, \quad \alpha_2 a_k = a_{k+\mu}\alpha_2,$$
$$\alpha_1\alpha_2 a_k = \alpha_2\alpha_1 a_k, \tag{3.3.8}$$

where $k, \ell \in \mathbb{Z} + \mu\mathbb{Z}$. Since $\mu \in \mathbb{C} \setminus \mathbb{Q}$, we have $\mathbb{Z} + \mu\mathbb{Z} \cong \mathbb{Z} \times \mathbb{Z}$, so we can reindex the projections a_k by defining $a_{ij} = a_{i+j\mu}$. In this notation the modified enveloping algebra has the form

$$\widetilde{U}_\mu = U^1 \otimes \left(\bigoplus_{i,j\in\mathbb{Z}} \mathbb{C}a_{ij}\right) \otimes U^2. \tag{3.3.9}$$

The multiplication is given by:

$$a_{ij}a_{st} = \begin{cases} a_{ij}, & \text{if } i = s, j = t, \\ 0, & \text{otherwise,} \end{cases}$$

$$\alpha_1 a_{ij} = a_{(i+1)j}\alpha_1, \quad \alpha_2 a_{ij} = a_{i(j+1)}\alpha_2,$$

$$\alpha_1\alpha_2 a_{ij} = \alpha_2\alpha_1 a_{ij}. \tag{3.3.10}$$

Remark 3.3.1. The importance of this last point is that we have eliminated any dependence on μ. This shows that when $\mu \in \mathbb{C} \setminus \mathbb{Q}$ the modified enveloping algebras of all the Lie algebras L_μ are isomorphic. Thus from now on we will denote $\widetilde{U}_\mu$ simply by $\widetilde{U}$ when μ is not rational.

Again we have an equivalence of categories wt-rep$(U_\mu) \cong$ Rep$(\widetilde{U})$, where U_μ is the universal enveloping algebra of L_μ. Note that the categories of weight representations of L_μ are all equivalent whenever $\mu \in \mathbb{C} \setminus \mathbb{Q}$.

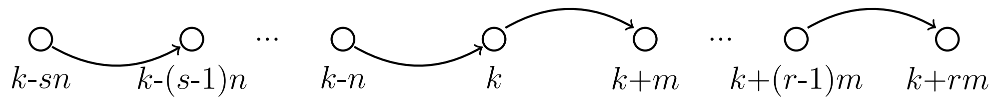
Chapter 4

The Quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$

In this chapter we relate the quivers $Q_{m,n}$ and $Q_{\infty\times\infty}$ to the modified enveloping algebras of the Lie algebras L_μ and use this relationship to study the representation theory of these Lie algebras.

4.1 Relation to the Lie Algebras L_μ

In this section we prove how the modified enveloping algebra $\widetilde{U}_{m,n}$ is related to the path algebra $\mathbb{C}Q_{m,n}$. We have seen in Section 3.3.2 that a basis of $\widetilde{U}_{m,n}$ is given by the elements $\alpha_1^r a_k \alpha_2^s$, and so in order to establish a connection between the two algebras, we need a way of associating a path in $Q_{m,n}$ to such an element. Recall that in $\widetilde{U}_{m,n}$, the operator a_k corresponds to projection onto the k^{th} weight space, α_1 moves the weight i to the weight $i+m$, and α_2 moves the weight i to the weight $i+n$. Since the vertices of $Q_{m,n}$ correspond to the weights in $\widetilde{U}_{m,n}$, the idea is to map $\alpha_1^r a_k \alpha_2^s$ to the path in $Q_{m,n}$ pictured below.



Of course, such paths do not form a basis of $\mathbb{C}Q_{m,n}$, and so this map will not be surjective. In order to obtain an isomorphism, we need to quotient by a certain ideal in $\mathbb{C}Q_{m,n}$. Loosely speaking, this quotient identifies the path above with any path with the same endpoints that travels along upper arrows r times and lower arrows s times. That is, the equivalence class is uniquely determined by r, s, and k, and hence will establish an isomorphism with $\widetilde{U}_{m,n}$. A similar strategy can be employed in the non-rational case, but since the proof runs analogously to the rational case, we do not include the details here.

4.1.1 Rational Case

Consider the linear map φ determined by:

$$\begin{gathered}\varphi : \mathbb{C}Q_{m,n} \to \tilde{U}, \\ \rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^{k} \mapsto \alpha_{j_s} \cdots \alpha_{j_1} a_k. \end{gathered} \tag{4.1.1}$$

Lemma 4.1.1. *The map φ is a homomorphism of algebras.*

Proof: Since φ is linear, it suffices to show that it commutes with the multiplication. Let

$$\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^{k}, \rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})} \cdots \rho_{i_1}^{l} \in \mathbb{C}Q_{m,n}.$$

Then

$$\begin{aligned}
&(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^{k}) \cdot (\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})} \cdots \rho_{i_1}^{l}) \\
&\quad = \begin{cases} \rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^{k} \rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})} \cdots \rho_{i_1}^{l} & \text{if } l+\sigma(i_1)+\cdots+\sigma(i_r)=k \\ 0 & \text{otherwise} \end{cases} \\
&\implies \varphi\left((\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^{k}) \cdot (\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})} \cdots \rho_{i_1}^{l})\right) \\
&\quad = \begin{cases} \alpha_{j_s} \cdots \alpha_{j_1} \alpha_{i_r} \cdots \alpha_{i_1} a_l & \text{if } l+\sigma(i_1)+\cdots+\sigma(i_r)=k, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\varphi\left(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}\right)\varphi\left(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l}\right)\\
&\quad=(\alpha_{j_s}\cdots\alpha_{j_1}a_k)(\alpha_{i_r}\cdots\alpha_{i_1}a_l)\\
&\quad=\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}a_{k-\sigma(i_r)}\alpha_{i_{r-1}}\cdots\alpha_{i_1}a_l\\
&\quad=\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\alpha_{i_{r-1}}a_{k-\sigma(i_r)-\sigma(i_{r-1})}\alpha_{i_{r-2}}\cdots\alpha_{i_1}a_l\\
&\quad\vdots\\
&\quad=\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\cdots\alpha_{i_1}a_{k-\sigma(i_r)-\cdots-\sigma(i_1)}a_l\\
&\quad=\begin{cases}\alpha_{j_s}\cdots\alpha_{j_1}\alpha_{i_r}\cdots\alpha_{i_1}a_l & \text{if } l=k-\sigma(i_1)-\cdots-\sigma(i_r),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

And hence

$$\begin{aligned}
&\varphi\left((\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k})\cdot(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l})\right)\\
&\quad=\varphi\left(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})}\cdots\rho_{j_1}^{k}\right)\varphi\left(\rho_{i_r}^{l+\sigma(i_1)+\cdots+\sigma(i_{r-1})}\cdots\rho_{i_1}^{l}\right).
\end{aligned}$$

∎

We now establish a relationship between the path algebra $\mathbb{C}Q_{m,n}$ and the modified enveloping algebra $\widetilde{U}_{m,n}$.

Proposition 4.1.2. *For any $m,n\in\mathbb{Z}^*$ there is an isomorphism of algebras $\widetilde{U}_{m,n}\cong\mathbb{C}Q_{m,n}/I^{m,n}$, where $I^{m,n}$ is the two-sided ideal generated by elements of the form $\rho_1^{k+n}\rho_2^k-\rho_2^{k+m}\rho_1^k$.*

Proof: We claim that $I^{m,n}\subseteq\ker\varphi$. Indeed:

$$\begin{aligned}
\varphi(\rho_1^{k+n}\rho_2^k-\rho_2^{k+m}\rho_1^k)&=\varphi(\rho_1^{k+n}\rho_2^k)-\varphi(\rho_2^{k+m}\rho_1^k)\\
&=\alpha_1\alpha_2a_k-\alpha_2\alpha_1a_k\\
&=\alpha_1\alpha_2a_k-\alpha_1\alpha_2a_k \qquad \text{(by (3.3.5))}\\
&=0
\end{aligned}$$

$$\implies \rho_1^{k+n}\rho_2^k - \rho_2^{k+m}\rho_1^k \in \ker\varphi \quad \forall k \in \mathbb{Z}.$$

Thus φ induces a morphism

$$\bar{\varphi} : \mathbb{C}Q_{m,n}/I^{m,n} \to \tilde{U},$$
$$x + I \mapsto \varphi(x) \qquad \forall x \in \mathbb{C}Q_{m,n}.$$

We will also consider the linear map determined by:

$$\psi : \widetilde{U} \to \mathbb{C}Q_{m,n}/I^{m,n}$$
$${\alpha_1}^r a_k {\alpha_2}^s \mapsto \rho_1^{k+(r-1)m} \cdots \rho_1^k \rho_2^{k-n} \cdots \rho_2^{k-sn} + I.$$

We will show that $\psi\bar{\varphi}$ and $\bar{\varphi}\psi$ are identity maps. Seeing as both maps are linear, it will suffice to show that this is the case for basis elements. Let $\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k + I \in \mathbb{C}Q/I^{m,n}$. Then

$$\begin{aligned}
(\psi\bar{\varphi})\left(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k + I\right) &= \psi\left(\bar{\varphi}(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k + I)\right) \\
&= \psi\left(\varphi(\rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k)\right) \\
&= \psi(\alpha_{j_s} \cdots \alpha_{j_1} a_k) \\
&= \psi(\alpha_1^r \alpha_2^t a_k), \quad \text{for some } r + t = s \\
&= \psi(\alpha_1^r a_{k+tn} \alpha_2^t) \\
&= \rho_1^{k+tn+(r-1)m} \cdots \rho_1^{k+tn} \rho_2^{k+(t-1)n} \cdots \rho_2^k + I \\
&= \rho_{j_s}^{k+\sigma(j_1)+\cdots+\sigma(j_{s-1})} \cdots \rho_{j_1}^k + I
\end{aligned}$$

$$\implies \psi\bar{\varphi} = \mathrm{id} : \mathbb{C}Q_{m,n}/I^{m,n} \to \mathbb{C}Q_{m,n}/I^{m,n}.$$

In the last line of the computation we used the relation $\rho_1^{k+n}\rho_2^k - \rho_2^{k+m}\rho_1^k \in I^{m,n}$ to reorder the terms into a new member of the same equivalence class.

Now let ${\alpha_1}^r a_k {\alpha_2}^s \in \widetilde{U}$. Then

$$(\bar{\varphi}\psi)({\alpha_1}^r a_k {\alpha_2}^s) = \bar{\varphi}\left(\psi({\alpha_1}^r a_k {\alpha_2}^s)\right)$$

$$
\begin{aligned}
&= \bar{\varphi}\left(\rho_1^{k+(r-1)m}\cdots\rho_1^k\rho_2^{k-n}\cdots\rho_2^{k-sn}+I\right)\\
&= \varphi\left(\rho_1^{k+(r-1)m}\cdots\rho_1^k\rho_2^{k-n}\cdots\rho_2^{k-sn}\right)\\
&= {\alpha_1}^r{\alpha_2}^s a_{k-sn}\\
&= {\alpha_1}^r a_k{\alpha_2}^s
\end{aligned}
$$

$$\implies \bar{\varphi}\psi = \mathrm{id} : \widetilde{U} \to \widetilde{U}.$$

Combining this result with Lemma 4.1.1, we see that $\bar{\varphi}$ is a bijective homomorphism of algebras, which completes the proof. ∎

Thus there is an equivalence of categories $\mathrm{Rep}(\widetilde{U}_{m,n}) \cong \mathrm{Rep}(\mathbb{C}Q_{m,n}/I^{m,n})$.

4.1.2 Non-Rational Case

Consider the linear map Ω defined by:

$$
\begin{aligned}
\Omega : \mathbb{C}Q_{\infty\times\infty} &\to \widetilde{U},\\
\rho_{d_s}^{k+\theta(d_1)+\cdots+\theta(d_{s-1})}\cdots\rho_{d_1}^k &\mapsto \alpha_{d_s}\cdots\alpha_{d_1}a_{ij},
\end{aligned}
\tag{4.1.2}
$$

where $k=(i,j)$.

Lemma 4.1.3. *The map Ω is a homomorphism of algebras.*

Proof: The proof is similar to that of Lemma 4.1.1. ∎

Proposition 4.1.4. *There is an isomorphism of algebras $\widetilde{U} \cong \mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty}$ where $I_{\infty\times\infty}$ is the two-sided ideal generated by elements of the form $\rho_1^{k+(0,1)}\rho_2^k - \rho_2^{k+(1,0)}\rho_1^k$, where $k \in \mathbb{Z}\times\mathbb{Z}$.*

Proof: The proof is similar to that of Proposition 4.1.2. ∎

So we have an equivalence of categories $\mathrm{Rep}(\widetilde{U}) \cong \mathrm{Rep}(\mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty})$.

4.2 Representation Theory

4.2.1 Rational Case

Let $\mu = \frac{n}{m}$, $\gcd(m, n) = 1$. We have the equivalences

$$\operatorname{Rep}(\widetilde{U}_{m,n}) \cong \operatorname{Rep}(\mathbb{C}Q_{m,n}/I^{m,n}) \cong \operatorname{Rep}(Q_{m,n}, R^{m,n}),$$

where $R^{m,n} = \{\rho_1^{k+n}\rho_2^k - \rho_2^{k+m}\rho_1^k \mid k \in \mathbb{Z}\}$. If we define $\widehat{R} = \{\bar{\rho}_{i+1}\rho_i - \rho_{i-1}\bar{\rho}_i \mid i \in \mathbb{Z}/s\mathbb{Z}\}$, then we can study the representations of $\widetilde{U}_{m,n}$ by relating the category $\operatorname{Rep}(Q_{m,n}, R^{m,n})$ to the category $\operatorname{Rep}(\widehat{Q}_{m+n}, \widehat{R})$, where $\widehat{Q}_s$ is the quiver defined in Example 2.3.3. This is an interesting connection, and moreover much is known about the category $\operatorname{Rep}(\widehat{Q}_{m+n}, \widehat{R})$, see [4] for example.

Let $V = (V_k, V(\rho_i^k)) \in \operatorname{Rep}(Q_{m,n})$ and let $j \in \mathbb{Z}/(m+n)\mathbb{Z}$. If $k \in \mathbb{Z}$, we will write $k \equiv j \mod (m+n)$ simply as $k \equiv j$. Then if $g \in \operatorname{Hom}(Q_{m,n}, \widehat{Q}_{m+n})$ is the morphism described in Example 2.3.3, the representation $g_!(V)$ has vector spaces given by

$$g_!(V)(j) = \bigoplus_{k \equiv jm} V_k.$$

The linear maps of the representation are given by $g_!(V)(\rho_j) = \bigoplus_{k\equiv jm} V(\rho_1^k)$ and $g_!(V)(\bar{\rho}_j) = \bigoplus_{k\equiv jm} V(\rho_2^k)$. Note that $g_!(V)(\rho_j)$ maps $g_!(V)(j)$ to $g_!(V)(j+1)$, and $g_!(V)(\bar{\rho}_j)$ maps $g_!(V)(j)$ to $g_!(V)(j-1)$. Moreover, if $V, U \in \operatorname{Rep}(Q_{m,n})$ and $\varphi \in \operatorname{Hom}_{Q_{m,n}}(V, U)$ then $g_!(\varphi) = \{g_!(\varphi)(j) \colon g_!(V)(j) \to g_!(U)(j)\}$, where $g_!(\varphi)(j) = \bigoplus_{k\equiv jm} \varphi_k$.

Let $V, U \in \operatorname{Rep}(Q_{m,n})$. Then $g_!(V) \ncong g_!(U) \Rightarrow V \ncong U$ since $g_!$ is a functor. Further, it follows from Corollary 2.3.11 that if $g_!(V)$ is indecomposable then V is indecomposable, since additive functors preserve finite coproducts, which in the categories $\operatorname{Rep}(Q_{m,n})$ and $\operatorname{Rep}(\widehat{Q}_{m+n})$ are finite direct sums.

Note that the preimage of the relations R_{m+n} in $\widehat{Q}_{m+n}$ under $g_!$ are exactly the relations $R_{m,n}$, that is, $g_!^{-1}(R_{m+n}) = R_{m,n}$. Thus, by Proposition 2.4.4, we can restrict the functor $g_!$ to the subcategory $\operatorname{Rep}(Q_{m,n}, R^{m,n})$ of $\operatorname{Rep}(Q_{m,n})$ to get a functor $g_!^{R_{m+n}} : \mathrm{Rep}(Q_{m,n}, R_{m,n}) \to \mathrm{Rep}(\widehat{Q}_{m+n}, \widehat{R}_{m+n})$. Further, all of the properties that were proven for $g_!$ still hold for the restricted functor $g_!^{R_{m+n}}$, and we can therefore relate the categories $\mathrm{Rep}(\widetilde{U}_{m,n})$ and $\mathrm{Rep}(\widehat{Q}_{m+n}, \widehat{R}_{m+n})$. It is natural to ask whether or not $g_!^{R_{m+n}}$ gives an equivalence of categories, and it turns out that this is true only when $\mu = -1$, in which case $g_!$ is the identity functor. When $\mu \neq -1$, the functor $g_!^{R_{m+n}}$ is neither full nor essentially surjective, as the following examples illustrate:

Example 4.2.1. Let $V \in \mathrm{Rep}(Q_{m,n}, R_{m,n})$ be the representation given by $V(0) = V(n+m) = \mathbb{C}$, and $V(i) = 0$ for all other $i \in \mathbb{Z}$. This representation is pictured in Figure 4.1, where any vector space or linear map not pictured is assumed to be zero. The endomorphism space of V is $\mathrm{Hom}_{Q_{m,n}}(V, V) \cong \mathrm{Hom}(\mathbb{C}, \mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C}, \mathbb{C}) \cong \mathbb{C}^2$.

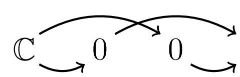


Figure 4.1: The functor $g_!^{R_{m+n}}$ is not full

The representation $g_!^{R_{m+n}}(V) \in \mathrm{Rep}(\widehat{Q}_{m+n}, \widehat{R}_{m+n})$ is the representation such that $g_!^{R_{m+n}}(V)_0 = \mathbb{C}^2$, and all other vector spaces are zero. Thus

$$\mathrm{Hom}_{\widehat{Q}_{m+n}}(g_!^{R_{m+n}}(V), g_!^{R_{m+n}}(V)) \cong \mathrm{Hom}(\mathbb{C}^2, \mathbb{C}^2) \cong \mathrm{Mat}_{2\times 2}(\mathbb{C}).$$

Since the dimension of the endomorphism space of $g_!^{R_{m+n}}(V)$ is greater than the dimension of the endomorphism space of V, the map $g_{!VV}^{R_{m+n}}$ is not surjective. Hence $g_!^{R_{m+n}}$ is not full.

Example 4.2.2. Let $U \in \mathrm{Rep}(Q_{m,n}, R_{m,n})$. Then if U is finite dimensional, there are only finitely many nonzero $U(k)$, and so there exists some $t \in \mathbb{Z}$ such that

$$U(\rho_1^{k+tm}\rho_1^{k+(t-1)m} \cdots \rho_1^k) = 0$$

for any $k \in \mathbb{Z}$. Then $\bigoplus_{k \equiv jm} U(\rho_1^{k+tm}\rho_1^{k+(t-1)m} \cdots \rho_1^k) = 0$, and so there is a path in the representation $g_!^{R_{m+n}}(U) \in \mathrm{Rep}(\widehat{Q}_{m+n}, \widehat{R}_{m+n})$ that acts by zero. Consider the representation $V \in \mathrm{Rep}(\widehat{Q}_{m+n}, \widehat{R}_{m+n})$ defined by $(V(i), V(\rho_i), V(\bar{\rho}_i)) = (\mathbb{C}, \lambda, 1)$ for all $i \in \mathbb{Z}/(m+n)\mathbb{Z}$, where $\lambda \in \mathbb{C}$. For $m = 2$ and $n = 1$ this representation is pictured in Figure 4.2. Suppose there were some representation $U \in \mathrm{Rep}(Q_{m,n}, R^{m,n})$

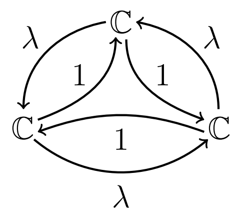


Figure 4.2: The functor $g_!^{R_{m+n}}$ is not essentially surjective

such that $g_!^{R_{m+n}}(U) \cong V$. Then since V is finite dimensional, U must also be finite dimensional. However, since there are no paths in the representation V that act by zero, this is a contradiction. Hence $g_!^{R_{m+n}}$ is not essentially surjective.

If $\widehat{I}$ denotes the two sided ideal of $\mathbb{C}\widehat{Q}_{m+n}$ generated by the relations $\widehat{R}$, then we have $\mathbb{C}\widehat{Q}_{m+n}/\widehat{I} = \mathcal{P}(\widehat{Q}^*_{m+n})$, where $\widehat{Q}^*_{m+n}$ is a subquiver of $\widehat{Q}_{m+n}$ obtained by omitting the arrows $\{\bar{\rho}_i \mid i \in \mathbb{Z}/(m+n)\mathbb{Z}\}$. The functor $g_!^{R_{m+n}}$ provides an embedding of the category $\mathrm{Rep}(\mathbb{C}^{m,n}/I^{m,n})$ in $\mathrm{Rep}(\mathcal{P}(\widehat{Q}^*_{m+n}))$.

Let $\alpha \in \mathbb{N}^{\mathbb{Z}}$ and consider the space $E_\alpha^{Q_{m,n}}$. Since the relations R_{m+n} are exactly the Gelfand-Ponomarev relations (2.1.1) in Q_{m+n}, the functor $g_!^{R_{m+n}}$ induces a morphism of affine varieties $\mathcal{N}(E_\alpha^{Q_{m,n}}) \to \Lambda_{g_!(\alpha)}^{Q^*_{m+n}}$ that identifies $\mathcal{N}(E_\alpha^{Q_{m,n}})$ with a subvariety of $\Lambda_{g_!(\alpha)}^{Q^*_{m+n}}$ (see Section 2.5).

While the category $\mathrm{Rep}(Q_{m,n}, R_{m,n})$ can be quite difficult to study in general, if we restrict our attention to representations which are supported on certain numbers of vertices, we can obtain certain classification results.

Theorem 4.2.3. *Let $\mu = m \in \mathbb{Z}$ and let $a, b \in \mathbb{Z}$ be such that $0 \leq b - a \leq m$. Let $\mathcal{C}_{a,b}$ denote the subcategory of $\mathrm{Rep}(Q_{m,1}, R_{m,1})$ consisting of representations V such that $V(x) = 0$ whenever $x < a$ or $x > b$. Then $\mathcal{C}_{a,b}$ is of tame representation type.*

Proof: Note that any representation $V \in \mathrm{Rep}(Q_{m,1})$ which is supported on at most $m+1$ consecutive vertices automatically satisfies the relations $R_{m,1}$ in a trivial way. Thus we may identify representations $V \in \mathcal{C}_{a,b}$ with representations of a quiver whose underlying graph is an extended Dynkin diagram of type $\hat{A}_m$. The result then follows from Theorem 2.2.4. ∎

Theorem 4.2.4. *Let $\mu = \frac{m}{n}$, $\gcd(m,n) = 1$, $n \neq 1$. Let $a, b \in \mathbb{Z}$ be such that $0 \leq b - a \leq m$. Then there are only finitely many isomorphism classes of indecomposable representations $V \in \mathrm{Rep}(Q_{m,n}, R_{m,n})$ such that $V(x) = 0$ whenever $x < a$ or $x > b$.*

Proof: As previously noted, any such representation V trivially satisfies the relations $R_{m,n}$. Thus we may identify V with a representation of a quiver whose underlying graph is a union of Dynkin diagrams of type A, and the result follows from Theorem 2.2.4. ∎

4.2.2 Non-Rational Case

When $\mu \in \mathbb{C} \setminus \mathbb{Q}$ we are interested in representations of the quiver $Q_{\infty\times\infty}$ introduced in Example 2.1.2. Then we have the equivalences $\mathrm{Rep}(\widetilde{U}) \cong \mathrm{Rep}(\mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty}) \cong \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$, where $R_{\infty\times\infty} = \{\rho_1^{k+(0,1)}\rho_2^k - \rho_2^{k+(1,0)}\rho_1^k \mid k \in \mathbb{Z}\times\mathbb{Z}\}$. In order to study the representations of $Q_{\infty\times\infty}$ we will relate the category $\mathrm{Rep}(Q_{\infty\times\infty})$ to the category $\mathrm{Rep}(Q_\infty)$. Here Q_∞ denotes the quiver $(\mathbb{Z}, \rho_i, \bar{\rho}_i)$, where $t(\rho_i) = i = h(\bar{\rho}_{i+1})$, and $h(\rho_i) = i + 1 = t(\bar{\rho}_{i+1})$. We will then be able to relate the category $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ to the category $\mathrm{Rep}(Q_\infty, R_\infty)$, where $R_\infty = \{\bar{\rho}_{i+1}\rho_i - \rho_{i-1}\bar{\rho}_i \mid i \in \mathbb{Z}\}$. We will obtain a relationship between these two categories which is similar to the relationship we studied in Section 4.2.1. The representation theory of the quiver Q_∞ subject to relations R_∞ is well known, see [12] for a summary of the results.

Remark 4.2.5. Given any representation $V \in \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ and any $i \in \mathbb{Z}$, we can consider the representation of the quiver of type A_∞ given by the i^{th} row of V, that is, the representation $V_i \in \mathrm{Rep}(A_\infty)$ with $V_i(j) = V(i,j)$. Then the fact that V satisfies the relations $R_{\infty\times\infty}$ implies that the collection $\{V(\rho_2^{ij}) \mid j \in \mathbb{Z}\}$ defines a morphism of A_∞ representations $V_i \to V_{i+1}$. Thus we may think of elements of $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ as chains of representations of the quiver of type A_∞.

Let V be a representation of the quiver $Q_{\infty\times\infty}$ and let $f \in \mathrm{Hom}(Q_{\infty\times\infty}, Q_\infty)$ be the morphism described in Example 2.3.4. Then the functor $f_!$ has vector spaces given by

$$f_!(V)(k) := \bigoplus\nolimits_{i-j=k} V(i,j).$$

The linear maps between these spaces are given by $f_!(V)(\rho_k) = \bigoplus_{i-j=k} V(\rho_1^{ij})$ and $f_!(V)(\bar{\rho}_k) = \bigoplus_{i-j=k} V(\rho_2^{ij})$. Note that $f_!(V)(\rho_k)$ maps $f_!(V)(k)$ to $f_!(V)(k+1)$ and $f_!(\bar{\rho}_k)$ maps $f_!(V)(k)$ to $f_!(V)(k-1)$. The functor $f_!$ acts on morphisms of $\mathrm{Rep}(Q_{\infty\times\infty})$ as follows: if $\varphi = \{\varphi(i,j)\}$ is a morphism between two representations V and U of $Q_{\infty\times\infty}$, where $\varphi(i,j)\colon V(i,j) \to U(i,j)$, then $f_!(\varphi) = \{f_!(\varphi)(k)\}$, where $f_!(\varphi)(k) = \bigoplus_{i-j=k} \varphi(i,j)$.

Once again Corollary 2.3.11 implies that $f_!$ is an additive functor. If two objects $f_!(V), f_!(U) \in \mathrm{Rep}(Q_\infty)$ are non-isomorphic, the objects $V, U \in \mathrm{Rep}(Q_{\infty\times\infty})$ must be non-isomorphic. Also, if an object $f_!(V) \in \mathrm{Rep}(Q_\infty)$ is indecomposable, then the object $V \in \mathrm{Rep}(Q_{\infty\times\infty})$ is also indecomposable, as in the rational case.

The relations R_∞ are the Gelfand-Ponomarev relations in Q_∞. Thus $f_!$ can be restricted to the subcategory $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ of $\mathrm{Rep}(Q_{\infty\times\infty})$ to yield a functor $f_!^{R_\infty} : \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty}) \to \mathrm{Rep}(Q_\infty, R_\infty)$, and this restricted functor shares the properties proven for $f_!$. However, the following examples illustrate that $f_!^{R_\infty}$ is neither full nor essentially surjective:

Example 4.2.6. Consider the representation $V \in \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ pictured in Figure 4.3, where all vector spaces not shown in the diagram are zero. The endo-

$$\begin{array}{ccc} 0 & \longrightarrow & \mathbb{C} \\ \uparrow & & \uparrow \\ \mathbb{C} & \longrightarrow & 0 \end{array}$$

Figure 4.3: The functor $f_!^{R_\infty}$ is not full

morphism space of V is given by $\mathrm{Hom}_{Q_{\infty\times\infty}}(V,V) \cong \mathrm{Hom}(\mathbb{C},\mathbb{C}) \oplus \mathrm{Hom}(\mathbb{C},\mathbb{C}) \cong \mathbb{C}^2$. The object $f_!^{R_\infty}(V)$ is the representation of Q_∞ such that $f_!^{R_\infty}(V)_0 = \mathbb{C}^2$, and all other vertices are 0. The endomorphism space of this representation, however, is $\mathrm{Hom}_{Q_\infty}(f_!^{R_\infty}(V), f_!^{R_\infty}(V)) \cong \mathrm{Hom}(\mathbb{C}^2,\mathbb{C}^2) \cong \mathrm{Mat}_{2\times 2}(\mathbb{C})$. Since the dimension of $\mathrm{Hom}_{Q_\infty}(f_!^{R_\infty}(V), f_!^{R_\infty}(V))$ is greater than the dimension of $\mathrm{Hom}_{Q_{\infty\times\infty}}(V,V)$, the induced functor $f_{!VV}^{R_\infty}$ is not surjective, and hence $f_!^{R_\infty}$ is not full.

Example 4.2.7. Next, consider the representation $V \in \mathrm{Rep}(Q_\infty, R_\infty)$ given by $(V(i), V(\rho_i), V(\bar{\rho}_i)) = (\mathbb{C}, \lambda, 1)$ for all $i \in \mathbb{Z}$, where $\lambda \in \mathbb{C}$ is nonzero. This representation is pictured in Figure 4.4. Suppose $V \cong f_!^{R_\infty}(U)$ for some $U \in \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$.

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} & \longrightarrow & \cdots \\ & & 1\uparrow & & 1\uparrow & & 1\uparrow & & \\ \cdots & \longrightarrow & \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} & \longrightarrow & \cdots \\ & & 1\uparrow & & 1\uparrow & & 1\uparrow & & \\ \cdots & \longrightarrow & \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Figure 4.4: The functor $f_!^{R_\infty}$ is not essentially surjective

Recall that the vertical maps $U(\rho_2^{ij})$ of the representation U correspond to the leftward maps $V(\bar{\rho}_i)$ of V. Since each $V(i)$ is one-dimensional, and each $V(i)$ maps to each $V(i-1)$ through the identity map, all nonzero $U(i,j)$ must lie along the same column. Relabelling if necessary, we may assume it's the first column. Then we must have $U(1,j) \cong V(j)$ and $U(\rho_2^{1j}) \cong 1$. A similar argument shows that all nonzero $U(i,j)$ must lie along the first *row*, with $U(i,1) \cong V(i)$ and $U(\rho_1^{i1}) \cong \lambda$. Clearly, no such U exists, and hence $f_!^{R_\infty}$ is not essentially surjective.

We will now consider an example which shows that the representation theory of the quiver $Q_{\infty\times\infty}$ is at least of tame type. To do this, we will show that there exists a family of pairwise nonisomorphic indecomposable representations of $Q_{\infty\times\infty}$ which depend upon a continuous parameter.

Example 4.2.8. For any $\lambda \in \mathbb{C}$, let $V_\lambda \in \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ denote the representation pictured in Figure 4.5, where all vector spaces and maps not displayed are assumed to be zero. We will assume that $V_\lambda(0,0) = \mathbb{C}^2$, and label all other

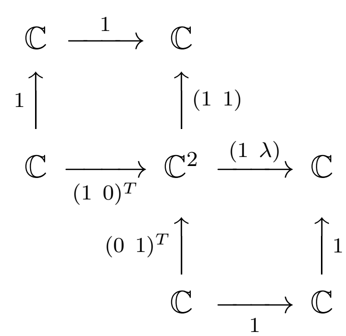


Figure 4.5: The category $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$ is at least tame

vertices accordingly (see Example 2.1.2). First we will show that V_λ is indecomposable for all $\lambda \in \mathbb{C}$. Suppose $V_\lambda = U \oplus W$. Then we may assume $U(-1,1) = \mathbb{C}$. Since $V_\lambda(\rho_1^{-1,1}) = V_\lambda(\rho_2^{-1,0}) = 1$, we must have $U(0,1) = U(-1,0) = \mathbb{C}$. We then have $(1\ \lambda)(1\ 0)^T(\mathbb{C}) =\subseteq U(0,1)$, and it follows that $U(0,1) = \mathbb{C}$. But then $U(0,-1) = U(1,-1) = \mathbb{C}$ since $V_\lambda(\rho_1^{0,-1}) = V_\lambda(\rho_2^{1,-1}) = 1$. Finally, we have $(1\ 0)^T(\mathbb{C}) \subseteq U(0,0)$ and $(0\ 1)^T(\mathbb{C}) \subseteq U(0,0)$, and we conclude that $U = V_\lambda$, so V_λ is indecomposable.

Now suppose $V_\lambda \cong V_\mu$. Then there exists an invertible 2×2 matrix A and nonzero complex numbers $z_1, z_2, z_3, z_4 \in \mathbb{C}$ such that the following equations hold:

$$(1\ 0)^T z_1 = A(1\ 0)^T,$$
$$(0\ 1)^T z_2 = A(0\ 1)^T,$$
$$z_3(1\ 1) = (1\ 1)A,$$
$$(1\ \mu)A = z_4(1\ \lambda).$$

The first two equations insist that A is a diagonal matrix. The third equation then implies that it is a scalar matrix, and then the fourth equation forces $\lambda = \mu$. Hence when $\lambda \neq \mu$, V_λ and V_μ are nonisomorphic. One can show in a similar manner that the images of these representations under the functor $f_!^{R_\infty}$ gives a family of indecomposable pairwise nonisomorphic representations in $\mathrm{Rep}(Q_\infty, R_\infty)$.

While we have seen that it is neither full nor essentially surjective, the functor $f_!^{R_\infty}$ can still be used to study the category $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$. First, we note that the group $\mathbb{Z}$ acts on the vertices and arrows of $Q_{\infty\times\infty}$ via

$$z \cdot (i,j) = (i+z, j+z), \quad z \cdot \rho_1^{ij} = \rho_1^{(i+z)(j+z)}, \quad z \cdot \rho_2^{ij} = \rho_2^{(i+z)(j+z)}.$$

Given any representation $V \in \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$, we denote by $V^{(z)}$ the representation obtained from V by twisting with the action of $\mathbb{Z}$. More precisely, the representation $V^{(z)}$ is defined by

$$V^{(z)}(i,j) = V(i+z, j+z), \quad V^{(z)}(\rho_1^{ij}) = V(\rho_1^{(i+z)(j+z)}), \quad V^{(z)}(\rho_2^{ij}) = V(\rho_2^{(i+z)(j+z)}).$$

Lemma 4.2.9. *Let $a, b \in \mathbb{Z}$ be integers such that $0 < b - a \leq 4$. Let $V \in \mathrm{Rep}_{\mathrm{fd}}(Q_\infty, R_\infty)$ be a finite dimensional representation such that $V(k) = 0$ whenever $k < a$ or $k > b$. Then V is isomorphic to $f_!^{R_\infty}(U)$ for some $U \in \mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty})$, which is unique up to translation $U \mapsto U^{(z)}$ by the group $\mathbb{Z}$.*

Proof: Since the functor $f_!^{R_\infty}$ is additive, we may assume that V is indecomposable. Any indecomposable representation $V \in \mathrm{Rep}_{\mathrm{fd}}(Q_\infty, R_\infty)$ such that $V(k) = 0$

whenever $k < a$ or $k > b$ is supported on at most 5 vertices, and hence may be thought of as a representation of the preprojective algebra of the quiver of type A_5. The lemma then follows from [6, Lemma 9.1], which states a similar result in the case of preprojective algebras of type A_n for $2 \leq n \leq 5$. ∎

The translation $V \mapsto V^{(z)}$ by $\mathbb{Z}$ on representations of $Q_{\infty\times\infty}$ induces an action of $\mathbb{Z}$ on the collection of isomorphism classes of representations of U_μ admitting a weight space decomposition via the equivalence $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty}) \cong \text{wt-rep}(U_\mu)$. We then have the following result:

Proposition 4.2.10. *For $a, b \in \mathbb{Z}$ with $0 \leq b - a \leq 3$, there are a finite number of $\mathbb{Z}$-orbits of isomorphism classes of indecomposable U_μ-modules V such that $V_{ij} = 0$ whenever $i - j < a$ or $i - j > b$.*

Proof: By [12, Theorem 4.3], there are a finite number of isomorphism classes of indecomposable modules $V \in \mathrm{Rep}(Q_\infty, R_\infty)$ such that $V(k) = 0$ for $k < a$ or $k > b$. The proposition then follows from the equivalence $\mathrm{Rep}(Q_{\infty\times\infty}, R_{\infty\times\infty}) \cong U_\mu\text{-Mod}$ and Lemma 4.2.9. ∎

Corollary 4.2.11. *Let A be a finite subset of $\mathbb{Z}$ with the property that A does not contain any five consecutive integers. Then there are a finite number of $\mathbb{Z}$-orbits of isomorphism classes of indecomposable U_μ-modules V such that $V_{ij} = 0$ whenever $i - j \notin A$.*

If I_∞ denotes the two sided ideal of $\mathbb{C}Q_\infty$ generated by the relations R_∞, then we have $\mathbb{C}Q_\infty/I_\infty = \mathcal{P}(Q^*_\infty)$, where Q^*_∞ is a subquiver of Q_∞ obtained by omitting the maps $\{\bar{\rho}_i \mid i \in \mathbb{Z}\}$. The functor $f_!^{R_\infty}$ provides an embedding of the category $\mathrm{Rep}(\mathbb{C}Q_{\infty\times\infty}/J)$ in $\mathrm{Rep}(\mathcal{P}(Q^*_\infty))$. The category $\mathrm{Rep}(\mathcal{P}(Q^*_\infty))$ is well understood, and it is known that every finite dimensional representation of $\mathcal{P}(Q^*_\infty)$ is nilpotent, see ([12]). Thus the functor $f_!^{R_\infty}$ embeds the finite dimensional representations of $\mathbb{C}Q_{\infty\times\infty}/I_{\infty\times\infty}$ inside the Lusztig quiver varieties defined in Section 2.5. More specifically, if $\alpha \in \mathbb{N}^{\mathbb{Z}\times\mathbb{Z}}$, then $f_!^{R_\infty}$ defines a morphism of quiver varieties $E_\alpha^{Q_{\infty\times\infty}} \to \Lambda_{f_!(\alpha)}^{Q_\infty^*}$ which identifies $E_\alpha^{Q_{\infty\times\infty}}$ with a subvariety of $\Lambda_{f_!(\alpha)}^{Q_\infty^*}$.

Appendix A

Category Theory

This chapter is intended to provide a terse review of some of the notions from category theory that we have used throughout this work. The reader interested in category theory can consult [9] for more information.

A.1 Categories and Functors

In this section we will review some of the basic definitions from category theory.

Definition A.1.1 (Category)**.** *A* category $\mathcal{C}$ *consists of:*

1. *a class of objects* $\mathrm{Ob}(\mathcal{C})$*,*

2. *for every* $X, Y \in \mathrm{Ob}(\mathcal{C})$ *a class of* morphisms *from* X *to* Y*,* $\mathrm{Hom}_{\mathcal{C}}(X, Y)$*, and*

3. *a binary operation (called a composition law)*

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \times \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$$

for all $X, Y, Z \in \mathrm{Ob}(\mathcal{C})$ *denoted by* $(\sigma, \tau) \mapsto \tau \circ \sigma$

such that the following axioms hold:

(i) *(*associativity*) if* $\sigma \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$*,* $\tau \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$*, and* $\rho \in \mathrm{Hom}_{\mathcal{C}}(Z,W)$ *then* $\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$*, and*

(ii) *(*identity*) for every* $X \in \mathrm{Ob}(\mathcal{C})$*, there exists* $\mathrm{id}_X \in \mathrm{Hom}_{\mathcal{C}}(X,X)$ *such that for every* $\sigma \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ *and every* $\tau \in \mathrm{Hom}_{\mathcal{C}}(X,Z)$ *we have* $\mathrm{id}_X \circ \sigma = \sigma$ *and* $\tau \circ \mathrm{id}_X = \tau$*.*

Let $\sigma \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$. Then σ is called a *monomorphism* if for every $Z \in \mathrm{Ob}(\mathcal{C})$ and every $\tau, \rho \in \mathrm{Hom}_{\mathcal{C}}(Z,X)$ we have $\sigma \circ \tau = \sigma \circ \rho$ implies $\tau = \rho$. The morphism σ is called an *epimorphism* if for every $Z \in \mathrm{Ob}(\mathcal{C})$ and every $\tau, \rho \in \mathrm{Hom}_{\mathcal{C}}(Y,Z)$ we have $\tau \circ \sigma = \rho \circ \sigma$ implies $\tau = \sigma$. The morphism σ is called an *isomorphism* if there exists $\tau \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$ such that $\sigma \circ \tau = \mathrm{id}_Y$ and $\tau \circ \sigma = \mathrm{id}_X$. In some categories there can exist morphisms that are monomorphisms and epimorphisms but are not isomorphisms.

If $\mathcal{C}$ is a category, we define its *opposite category*, $\mathcal{C}^{\mathrm{op}}$, to be the category with $\mathrm{Ob}(\mathcal{C}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{C})$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A,B) = \mathrm{Hom}_{\mathcal{C}}(B,A)$ for all $A, B \in \mathrm{Ob}(\mathcal{C})$. By a *covariant functor* between two categories, $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, we shall mean a mapping which associates to every object $A \in \mathcal{C}$ an object $\mathcal{F}(A) \in \mathcal{D}$ and associates to each morphism $\sigma \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ a morphism $\mathcal{F}(\sigma) \in \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B))$ such that these associations preserve identity morphisms and composition of morphisms. By a *contravariant functor*, $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, we shall mean a covariant functor $\mathcal{F}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$. A category is said to be *small* if $\mathrm{Ob}(\mathcal{C})$ is a set and $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ is also a set for every $X, Y \in \mathrm{Ob}(\mathcal{C})$. The collection of all small categories along with functors between them forms a category, which we denote Cat.

If $\mathcal{F}$ and $\mathcal{G}$ are covariant functors between the categories $\mathcal{C}$ and $\mathcal{D}$, then a *natural transformation* from $\mathcal{F}$ to $\mathcal{G}$, $\eta\colon \mathcal{F} \to \mathcal{G}$, is a collection of morphisms $\{\eta_A\colon \mathcal{F}(A) \to \mathcal{G}(A) \mid A \in \mathrm{Ob}(\mathcal{C})\}$ such that for every $\sigma \in \mathrm{Hom}_{\mathcal{C}}(A,B)$ the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{F}(A) & \xrightarrow{\mathcal{F}(\sigma)} & \mathcal{F}(B) \\
\eta_A \big\downarrow & & \big\downarrow \eta_B \\
\mathcal{G}(A) & \xrightarrow[\mathcal{G}(\sigma)]{} & \mathcal{G}(B)
\end{array}$$

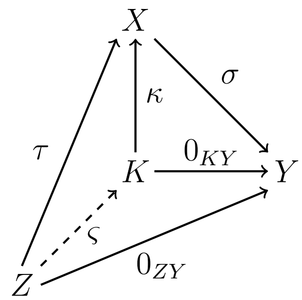
If η_A is an isomorphism for every $A \in \mathrm{Ob}(\mathcal{C})$ then we say η is a *natural isomorphism*.

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and denote by $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$ the endofunctors of $\mathcal{C}$ and $\mathcal{D}$ respectively that send every object and every morphism to themselves. An *equivalence of categories* between $\mathcal{C}$ and $\mathcal{D}$ consists of a covariant functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$, a covariant functor $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$, and two natural isomorphisms $\varepsilon\colon \mathcal{F}\mathcal{G} \to I_{\mathcal{D}}$ and $\eta\colon I_{\mathcal{C}} \to \mathcal{G}\mathcal{F}$. We say $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exists an equivalence of categories between them.

A.2 Abelian Categories and Adjunctions

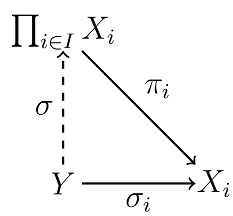
Throughout this work we will deal exclusively with abelian categories, which are well-behaved categories. In this section we will briefly review the definition of an abelian category as well as some basic results concerning functors in abelian categories. In order to define abelian categories we will first need to review some other categorical definitions.

An object O in a category $\mathcal{C}$ is called a *zero object* if for any $X \in \mathrm{Ob}(\mathcal{C})$ there is exactly one morphism $O \to X$ and for any $Y \in \mathrm{Ob}(\mathcal{C})$ there is exactly one morphism $Y \to O$. If $\mathcal{C}$ has a zero object O then for any two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$ we define the zero map from X to Y, $0_{XY}\colon X \to Y$, to be the (unique) map $X \to O \to Y$. For any morphism $\sigma\colon X \to Y$, the *kernel* of σ is a morphism $\kappa\colon K \to X$ such that $\sigma \circ \kappa = 0_{KY}$ and for any other morphism $\tau\colon Z \to X$ such that $\sigma \circ \tau = 0_{ZY}$ there exists a unique morphism $\varsigma\colon Z \to K$ such that $\kappa \circ \varsigma = \tau$. This definition can be visualized by the commutativity of the following diagram.



The kernel of σ is often written $\ker(\sigma) \to X$. The definition of a *cokernel* can be obtained by reversing the arrows in the definition of the kernel.

Let I be an index set, and let $X_i \in \mathrm{Ob}(\mathcal{C})$ for all $i \in I$. Then the *product* of the family of objects $\{X_i\}_{i\in I}$ is an object $\prod_{i\in I} X_i$ along with a collection of morphisms $\pi_i \colon \prod_{i\in I} X_i \to X_i$ such that for every object Y and every I-indexed family of morphisms $\sigma_i \colon Y \to X_i$ there exists a unique morphism $\sigma \colon Y \to \prod_{i\in I} X_i$ such that the following diagram commutes for all $i \in I$.



The definition of the *coproduct* of a family of objects can be obtained by reversing the arrows in the definition of the product. The coproduct of a family of objects is denoted $\coprod_{i\in I} X_i$.

We are now ready to define abelian categories.

Definition A.2.1 (Abelian Category)**.** *A category* $\mathcal{C}$ *is said to be* abelian *if it satisfies the following properties*

1. *there exists a zero object* $O \in \mathrm{Ob}(\mathcal{C})$*,*
2. *for any two objects* $X, Y \in \mathrm{Ob}(\mathcal{C})$*, the product* $X \prod Y$ *and the coproduct* $X \coprod Y$ *both exist,*

3. *for any morphism $\sigma\colon X \to Y$, the kernel of σ and the cokernel of σ both exist,*

4. *any monomorphism is the kernel of some morphism and any epimorphism is the cokernel of some morphism.*

Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories, and let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. Then for any $A, B \in \mathrm{Ob}(\mathcal{C})$, the functor $\mathcal{F}$ induces a map $\mathcal{F}_{AB}\colon \mathrm{Hom}_{\mathcal{C}}(A,B) \to \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), \mathcal{F}(B))$. If $\mathcal{F}_{AB}$ is a group homomorphism for all A, B, we say $\mathcal{F}$ is *additive*. If $\mathcal{F}_{AB}$ is injective (resp. surjective) for all A, B, we say $\mathcal{F}$ is *faithful* (resp. *full*). If $\mathcal{F}$ is additive, then it preserves finite coproducts; $\mathcal{F}(A \coprod B) \cong \mathcal{F}(A) \coprod \mathcal{F}(B)$, see for example [11, Corollary 5.88] for a proof of this statement. If for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ the sequence $0 \to \mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C)$ is exact, then we say the functor $\mathcal{F}$ is *left exact*. Similarly, if for any short exact sequence $0 \to A \to B \to C \to 0$ in $\mathcal{C}$ the sequence $\mathcal{F}(A) \to \mathcal{F}(B) \to \mathcal{F}(C) \to 0$ is exact, then we say $\mathcal{F}$ is *right exact*. One can show that a covariant functor is left exact if and only if it preserves kernels, that is, $\mathcal{F}(\ker\sigma) = \ker(\mathcal{F}(\sigma))$ for every morphism $\sigma \in \mathcal{C}$, and similarly a covariant functor is right exact if and only if it preserves cokernels, i.e $\mathcal{F}(\operatorname{coker}\sigma) = \operatorname{coker}(\mathcal{F}(\sigma))$ for any morphism σ (see for example [9, Section VIII.3]). Any functor which is both left exact and right exact is said to be *exact*.

If $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ and $\mathcal{G}\colon \mathcal{D} \to \mathcal{C}$ are two (covariant) functors then we can consider the functors $\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}-,-)$ and $\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{G}-)$ between the categories $\mathcal{C}^{\mathrm{op}} \times \mathcal{D}$ and Set. If there exists a natural isomorphism between these two functors, then we say $\mathcal{F}$ is *left adjoint* to $\mathcal{G}$, and $\mathcal{G}$ is *right adjoint* to $\mathcal{F}$. More explicitly, $\mathcal{F}$ is right adjoint to $\mathcal{G}$ if there exists a collection of isomorphisms

$$\{\Phi_{A,B}\colon \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) \to \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{G}(B)) \mid A \in \mathcal{C}, B \in \mathcal{D}\}$$

such that for any morphisms $\sigma \in \mathrm{Hom}_{\mathcal{C}}(A', A)$ and $\tau \in \mathrm{Hom}_{\mathcal{D}}(B, B')$ the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A), B) & \xrightarrow{\mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(\sigma),\tau)} & \mathrm{Hom}_{\mathcal{D}}(\mathcal{F}(A'), B') \\ \Phi_{A,B} \downarrow & & \downarrow \Phi_{A',B'} \\ \mathrm{Hom}_{\mathcal{C}}(A, \mathcal{G}(B)) & \xrightarrow[\mathrm{Hom}_{\mathcal{C}}(\sigma,\mathcal{G}(\tau))]{} & \mathrm{Hom}_{\mathcal{C}}(A', \mathcal{G}(B')). \end{array}$$

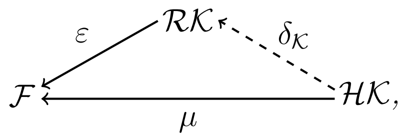
Here the horizontal arrows correspond to morphisms induced by composition with σ and τ. If a functor $\mathcal{F}$ has two right (or left) adjoints, $\mathcal{G}$ and $\mathcal{G}'$, then $\mathcal{G}$ and $\mathcal{G}'$ are naturally isomorphic. Adjoint functors have many important properties. For example, if $\mathcal{C}$ and $\mathcal{D}$ are abelian categories, then two adjoint functors $\mathcal{F}$ and $\mathcal{G}$ are necessarily additive and a left (resp. right) adjoint functor is right (resp. left) exact (see for example [9, Section V.5, Theorem 1]).

A.3 Kan Extensions

In this section we describe briefly the notion of left and right Kan extensions. Kan extensions are very powerful tools in category theory (indeed, [9, Section X.7] is titled "All Concepts Are Kan Extensions"), and their use in this work, while not imperative, would simplify the proof of Theorem 2.3.9. The interested reader is encouraged to consult [9, Chapter X].

Given two categories $\mathcal{A}$ and $\mathcal{C}$, we can consider the category $\mathcal{A}^{\mathcal{C}}$, whose objects are all functors $\mathcal{F}\colon \mathcal{C} \to \mathcal{A}$, and whose morphisms are natural transformations between such functors. Then for any category $\mathcal{B}$ and any covariant functor $\mathcal{K}\colon \mathcal{B} \to \mathcal{C}$, we can construct a functor $\mathcal{A}^{\mathcal{K}}\colon \mathcal{A}^{\mathcal{C}} \to \mathcal{A}^{\mathcal{B}}$ given by $\sigma \mapsto \sigma \circ \mathcal{K}$. The study of Kan extensions amounts to attempting to construct right and left adjoints to this functor.

Definition A.3.1 (Kan Extension)**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be categories, and let $\mathcal{K}\colon \mathcal{B} \to \mathcal{C}$ and $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ be covariant functors. Then the* right Kan extension *of $\mathcal{F}$ along $\mathcal{K}$ consists of a functor $\mathcal{R}\colon \mathcal{C} \to \mathcal{A}$ and a natural transformation $\varepsilon\colon \mathcal{R}\mathcal{K} \to \mathcal{F}$ such that* *for any functor* $\mathcal{H}\colon \mathcal{C} \to \mathcal{A}$ *and any natural transformation* $\mu\colon \mathcal{HK} \to \mathcal{F}$*, there exists a unique natural transformation* $\delta\colon \mathcal{H} \to \mathcal{R}$ *making the following diagram commute:*



where $\delta_{\mathcal{K}}$ *is the natural transformation such that*

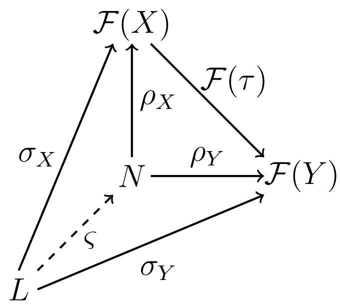
$$\delta_{\mathcal{K}}(X) = \delta(\mathcal{K}(X))\colon \mathcal{HK}(X) \to \mathcal{RK}(X)$$

for every $X \in \mathrm{Ob}(\mathcal{A})$*. We often write* $\mathrm{Ran}_{\mathcal{K}}\,\mathcal{F}$ *for the right Kan extension of* $\mathcal{F}$ *along* $\mathcal{K}$*, and* ε *is called the* counit *of the extension. The definition for a* left Kan extension *of* $\mathcal{F}$ *along* $\mathcal{K}$ *can be obtained by reversing all arrows in the above diagram, and we write* $\mathrm{Lan}_{\mathcal{K}}\mathcal{F}$ *for the left Kan extension of* $\mathcal{F}$ *along* $\mathcal{K}$*. In the left case we use the symbol* η *instead of* ε*, and* η *is called the* unit *of the extension.*

From the universality of this definition, if $\mathrm{Ran}_{\mathcal{K}}\,\mathcal{F}$ exists for every $\mathcal{F} \in \mathcal{A}^{\mathcal{B}}$, then the functor $\mathrm{Ran}_{\mathcal{K}}\colon \mathcal{A}^{\mathcal{B}} \to \mathcal{A}^{\mathcal{C}}$ given by $\mathcal{F} \mapsto \mathrm{Ran}_{\mathcal{K}}\,\mathcal{F}$ is a right adjoint to $\mathcal{A}^{\mathcal{K}}$. Similarly, if $\mathrm{Lan}_{\mathcal{K}}\,\mathcal{F}$ exists for every $\mathcal{F} \in \mathcal{A}^{\mathcal{B}}$, the functor $\mathcal{F} \mapsto \mathrm{Lan}_{\mathcal{K}}\,\mathcal{F}$ provides a left adjoint for $\mathcal{A}^{\mathcal{K}}$.

While this definition of Kan extensions in terms of their universal property is the most general, it is not always convenient for computing them explicitly, nor does it guarantee their existence. In order to introduce a more computationally useful characterization of Kan extensions, we first need to introduce the notions of limits and colimits.

Let $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor. A *cone* over $\mathcal{F}$ is an object $N \in \mathrm{Ob}(\mathcal{D})$ along with a family of morphisms $\sigma_X\colon N \to \mathcal{F}(X)$, $X \in \mathrm{Ob}(\mathcal{C})$, such that for every morphism $\tau\colon X \to Y$ in $\mathcal{C}$, we have $D(\tau) \circ \sigma_X = \sigma_Y$. The *limit* of $\mathcal{F}$, $\varprojlim \mathcal{F}$, is a cone $(N, \{\sigma_X\}_{X\in\mathcal{C}})$ over $\mathcal{F}$ such that for any other cone $(L, \{\rho_X\}_{X\in\mathcal{C}})$ over $\mathcal{F}$ there exists a unique morphism $\varsigma\colon L \to N$ such that $\sigma_X \circ \varsigma = \rho_X$ for all $X \in \mathcal{C}$. This can be visualized by the commutativity of the following diagram.



The definition of the *colimit* of $\mathcal{F}$, $\varinjlim \mathcal{F}$ can be obtained by dualizing the definition of the limit. If $\mathcal{D}$ is a category such that for any small category $\mathcal{C}$ and any functor $\mathcal{F}\colon \mathcal{C} \to \mathcal{D}$ the limit (resp. colimit) of $\mathcal{F}$ exists, then we say $\mathcal{D}$ is *complete* (resp. cocomplete).

In the case that $\mathcal{A}$ is a small category and $\mathcal{C}$ is a complete category, the following theorem gives a more explicit description of Kan extensions (see for example [9, Section X.3, Theorem 1]).

Theorem A.3.2. *Suppose $\mathcal{A}, \mathcal{B}$, and $\mathcal{C}$ are categories with $\mathcal{B}$ small and $\mathcal{A}$ complete. Let $\mathcal{K}\colon \mathcal{B} \to \mathcal{C}$ and $\mathcal{F}\colon \mathcal{B} \to \mathcal{A}$ be covariant functors. Then the right Kan extension of $\mathcal{F}$ along $\mathcal{K}$ exists and is given by*

$$\operatorname{Ran}_{\mathcal{K}} \mathcal{F}(X) = \varprojlim((X \downarrow \mathcal{K}) \xrightarrow{Q^X} \mathcal{B} \xrightarrow{\mathcal{F}} \mathcal{A}).$$

for all $X \in \mathcal{C}$.

In Theorem A.3.2 above, $X \downarrow \mathcal{K}$ denotes the *comma category* , which has as objects pairs (Y, σ) where $Y \in \operatorname{Ob}(B)$ and $\sigma \in \operatorname{Hom}_{\mathcal{C}}(X, \mathcal{K}(Y))$. A morphism in $X \downarrow \mathcal{K}(Y)$ between (Y, σ) and (Y', σ') is given by a morphism $\tau \in \operatorname{Hom}_{\mathcal{C}}(Y, Y')$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle\sigma}\downarrow & & \downarrow{\scriptstyle\sigma'} \\ \mathcal{K}(Y) & \xrightarrow[\mathcal{K}(\tau)]{} & \mathcal{K}(Y'). \end{array} \tag{A.3.1}$$

Along with every comma category there is a projection functor $Q^X : X \downarrow \mathcal{K} \to \mathcal{B}$ that takes (σ, Y) to Y. The limit in Theorem A.3.2 is the limit of the composition of $\mathcal{F}$ with this projection functor.

A similar result to Theorem A.3.2 holds for left Kan extensions, though we suppose that $\mathcal{C}$ is *cocomplete*, and then $\mathrm{Lan}_{\mathcal{K}} \mathcal{F}(X)$ is the colimit over the comma category $\mathcal{K} \downarrow X$ (a description of which can be obtained from the description of $X \downarrow \mathcal{K}$ by reversing all of the arrows).

The relationship between Kan extensions and limits of functors goes even deeper than Theorem A.3.2, as shown by the following result.

Theorem A.3.3. *[9, Theorem X.7.1] A functor $\mathcal{F} : \mathcal{B} \to \mathcal{A}$ has a limit if and only if it has a right Kan extension along the functor $\mathcal{K}_1 : \mathcal{B} \to *$ (where $*$ is the category with a single object and a single morphism), in which case $\varprojlim \mathcal{F}$ is the value of $\mathrm{Ran}_{\mathcal{K}_1} \mathcal{F}$ on the unique object of $*$.*

A similar theorem holds for left Kan extensions and colimits of functors.

One of the strengths of Theorem A.3.2 is that it guarantees the existence of Kan extensions under certain conditions. Thus, if one is interested in a particular functor that can be interpreted as being of the form $\mathcal{A}^{\mathcal{K}}$ for some suitable category $\mathcal{A}$ and some suitable functor $\mathcal{K}$, then we can guarantee the existence of adjoints to that functor. In fact, we can use Kan extensions to prove a deeper result concerning the existence of adjoints.

Theorem A.3.4. *[9, Theorem X.7.2] A functor $\mathcal{G} : \mathcal{A} \to \mathcal{C}$ has a left adjoint if and only if $\mathrm{Ran}_{\mathcal{G}}(I_{\mathcal{A}})$ exists with counit ε and $G \circ \mathrm{Ran}_{\mathcal{G}}(I_{\mathcal{A}}) = \mathrm{Ran}_{\mathcal{G}} \mathcal{G}$ with $\mathcal{G}\varepsilon$ as the counit. When this is the case, $\mathrm{Ran}_{\mathcal{G}}(I_{\mathcal{A}})$ is the left adjoint of $\mathcal{G}$.*

This theorem can be used to simplify the proof of Theorem 2.3.9.

Appendix B

Affine Varieties

Here we quickly review some of the theory of affine varieties. The material presented here will be very concise, for a more detailed exposition the reader can consult [13].

The main objects we will study in this section are the solution sets to systems of polynomials over $\mathbb{C}$. First we define *affine n-space* to be $\mathbb{A}^n := \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{C} \text{ for all } i\}$. We use the notation $\mathbb{A}^n$ rather than $\mathbb{C}^n$ to emphasize that we are not considering this set as being equipped with its natural vector space structure. If $f \in \mathbb{C}[x_1, \ldots, x_n]$, then a point $p \in \mathbb{A}^n$ will be called a *zero* of f is $f(p) = 0$. We then define

$$Z(f) = \{p \in \mathbb{A}^n \mid p \text{ is a zero of } f\}.$$

We can naturally extend this idea to any set $T \subseteq \mathbb{C}[x_1, \ldots, x_n]$:

$$Z(T) = \{p \in \mathbb{A}^n \mid p \text{ is a zero of } \textit{every } f \in T\}.$$

It follows immediately from these definitions that for any $T \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we have $Z(T) = Z(\langle T \rangle)$, where $\langle T \rangle$ denotes the ideal of $\mathbb{C}[x_1, \ldots, x_n]$ generated by T. Since $\mathbb{C}[x_1, \ldots, x_n]$ is a Noetherian ring (this is known as Hilbert's Basis Theorem), it follows that for any ideal $I \subseteq \mathbb{C}[x_1, \ldots, x_n]$, we have $Z(I) = Z(\{f_1, \ldots, f_m\})$, where $I = \langle f_1, \ldots, f_m \rangle$.

It is not difficult to see that the collection of all sets $Y \subseteq \mathbb{A}^n$ such that $Y = Z(T)$ for some $T \subseteq \mathbb{C}[x_1, \ldots, x_n]$ form the closed sets of a topology on $\mathbb{A}^n$. This topology is known as the *Zariski Topology*. This motivates the following definition:

Definition B.1 (Affine Variety)**.** *Let $Y \subseteq \mathbb{A}^n$ be such that there exists $T \subseteq \mathbb{C}[x_1, \ldots, x_n]$ with $Y = Z(T)$. Then Y is called an* affine variety.

Remark B.2. *Many authors use the term* algebraic set *for the objects defined above and reserve the term affine variety for algebraic sets which are irreducible with respect to the Zariski Topology.*

On the other hand, given a subset $Y \subseteq A^n$ we can consider the set

$$I(Y) := \{f \in \mathbb{C}[x_1, \ldots, x_n] \mid f(Y) = 0\}.$$

Clearly, $I(Y)$ is a radical ideal of $\mathbb{C}[x_1, \ldots, x_n]$ for any Y. This gives two functions between ideals I of $\mathbb{C}[x_1, \ldots, x_n]$ and subsets Y of affine space, namely $I \mapsto Z(I)$ and $Y \mapsto I(Y)$. It turns out that there exists a very important fact about these two functions known as *Hilbert's Nullstellensatz*:

Theorem B.3. *Let $\mathfrak{U}$ be an ideal of $\mathbb{C}[x_1, \ldots, x_n]$. Then $I(Z(\mathfrak{U})) = \sqrt{\mathfrak{U}}$, where $\sqrt{\mathfrak{U}}$ denotes the radical of $\mathfrak{U}$.*

Proof: See, for example, [13, Section 2.3]. ∎

Thus, if we restrict ourselves to radical ideals of $\mathbb{C}[x_1, \ldots, x_n]$, the maps $I \mapsto Z(I)$ and $Y \mapsto I(Y)$ are inverse to each other. This correspondence allows us to translate back and forth between affine varieties and ideals of $\mathbb{C}[x_1, \ldots, x_n]$.

Now that we have defined the objects we wish to study, it is natural to consider certain classes of functions on them. For any affine variety V, the restriction of any $f \in \mathbb{C}[x_1, \ldots, x_n]$ to V gives a function $f|_V : V \to \mathbb{C}$. The restrictions of all polynomials in $\mathbb{C}[x_1, \ldots, x_n]$ to V form a $\mathbb{C}$-algebra under pointwise addition and multiplication, which we denote by $\mathbb{C}[V]$. We call $\mathbb{C}[V]$ the *coordinate ring* of V. Clearly we have a natural surjection $\mathbb{C}[x_1, \ldots, x_n] \twoheadrightarrow \mathbb{C}[V]$ that has kernel $I(V)$, and hence for any affine variety V we have

$$\mathbb{C}[V] \cong \mathbb{C}[x_1, \ldots, x_n]/I(V), \tag{B.1}$$

which we will often view as an identification. We now turn our attention to morphisms between affine varieties. The simplest example of a morphism between two affine varieties is a map of the form

$$\begin{aligned} \mathbb{A}^n &\to \mathbb{A}^m \\ x &\mapsto (F_1(x), F_2(x), \ldots, F_m(x)), \end{aligned}$$

where F_i is a polynomial in the coordinates $x_1, \ldots, x_n$ for every i. Such a map is called a *polynomial map*. More generally, we make the following definition:

Definition B.4 (Morphism of Affine Varieties)**.** *Let $V \subseteq \mathbb{A}^n$ and $W \subseteq A^m$ be affine varieties. A map $F\colon V \to W$ is a* morphism of affine varieties *if it is the restriction to V of a polynomial map on the ambient affine spaces $\mathbb{A}^n \to \mathbb{A}^m$.*

A morphism of affine varieties is said to be an *isomorphism* if it has an inverse morphism. That is, a morphism is an isomorphism if it is bijective and its inverse map is a morphism of affine varieties. Given a morphism of affine varieties, $F\colon V \to W$, we get a naturally induced map between the coordinate rings $\mathbb{C}[W]$ and $\mathbb{C}[V]$ given by

$$\begin{aligned} \mathbb{C}[W] &\to \mathbb{C}[V], \\ g &\mapsto g \circ F. \end{aligned}$$

This induced map is called the *pullback* of F. It is easy to check that the pullback of F defines a $\mathbb{C}$-algebra homomorphism from $\mathbb{C}[W]$ to $\mathbb{C}[V]$. Thus for any affine variety V we get a finitely generated, reduced $\mathbb{C}$-algebra $\mathbb{C}[V]$, and for any morphism of affine varieties, we get a morphism of $\mathbb{C}$-algebras. In fact, this association defines an equivalence of categories.

Theorem B.5. *There is an equivalence of categories between the category of affine varieties and the category of finitely generated, reduced* $\mathbb{C}$*-algebras.*

Proof: See, for example, [13, Section 2.5]. ∎

Thus we may freely pass back and forth between the geometric concept of affine varieties and the algebraic concept of $\mathbb{C}$-algebras. The following lemma will prove useful.

Lemma B.6. *A morphism of affine varieties $F\colon V \to W$ defines an isomorphism between V and some algebraic subvariety of W if and only if the pullback of F is surjective.*

Proof: See, for example, [13, Section 2.5]. ∎

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