Asymptotics for the Sequential Empirical Process and Testing for Distributional Change for Stationary Linear Models

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Abstract

Detecting a change in the structure of a time series is a classical statistical problem. Here we consider a short memory causal linear process $X_i = \sum_{j=0}^{\infty} a_j \xi_{i-j}$, $i = 1, \cdots, n$, where the innovations ξ_i are independent and identically distributed and the coefficients a_j are summable. The goal is to detect the existence of an unobserved time at which there is a change in the marginal distribution of the X_i's. Our model allows us to simultaneously detect changes in the coefficients and changes in location and/or scale of the innovations. Under very simple moment and summability conditions, we investigate the asymptotic behaviour of the sequential empirical process based on the X_i's both with and without a change-point, and show that two proposed test statistics are consistent. In order to find appropriate critical values for the test statistics, we then prove the validity of the moving block bootstrap for the sequential empirical process under both the hypothesis and the alternative, again under simple conditions. Finally, the performance of the proposed test statistics is demonstrated through Monte Carlo simulations.

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Dedication

This work is dedicated to my "little ones" Yasmine Jannat, Lina and Lilya, to my lovely wife Aouatif, to my mother Nadia and to the loving memory of my grandparents.

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Introduction

In recent years, there has been ever increasing interest in statistical methods addressing the problem of structural stability in a time series environment. It is well known that ignoring the structural changes in a time series setting may lead to incorrect statistical procedures and consequently affect estimation and inference. In fact, this is a problem that arises in many areas of application, including economics, finance, climate data, engineering and health sciences, among others. The broad area of change-point detection is a subject of prime importance and an attractive field of research with many unresolved questions.

In very general terms, the change-point problem is as follows: we have a stochastic process $X_1, X_2, X_3, \cdots$ evolving in time. Initially the process is stationary; the random variables X_i are identically distributed, although not necessarily independent. There may be an unobserved time τ at which the behaviour of the process changes in some way - for example, there may be a change in mean or variance (a change in location or scale), or a change in the dependence structure of the X_i's. The goal is to detect the presence of an unobserved change-point.

Here, we consider one of the most widely applied time series models - *the causal linear process*, which is defined as follows: for $i \geq 0$,

$$X_i = \sum_{j \geq 0} a_j \xi_{i-j},$$

where (a_i) is a sequence of absolutely summable constants and (ξ_i) is a sequence of independent and identically distributed random variables (*the innovations*). This summability of the coefficients ensures that the linear model is *short range dependent*. Under very general conditions, we will consider the null hypothesis of "no-change" and the alternative that there is a time at which there is a change in the marginal distribution of the X_i's. Our model allows for a change in the coefficients (a_i) and/or a change in location (mean) or scale (variance) of the innovations ξ_j.

From a statistical perspective, the change-point problem is approached in one of two ways, depending on the available data: a sequential or a retrospective (ex-post) scheme. In the first case, the decision on whether or not there has been a change-point is made "on line" with the observations. More precisely, with the arrival of each new data point X_i, a decision of "change" or "no change" is made based on the information available up to that time. In the retrospective approach, the decision is made after observing a sample $(X_1, \ldots, X_n)$ of a fixed length. In this thesis, we will take the retrospective approach.

The literature on change-point detection is vast, including many different time series models, both short and long range dependent, sequential and retrospective detection, single or multiple change-points. It is impossible to present a comprehensive literature review, and so we restrict ourselves to a brief overview that illustrates the wide range of recent research in the area. Change-point analysis was originally developed under the assumption that the X_i's were independent random variables. One of the fundamental references for this theory is the book by Csörgő and Horváth [11]. More recently, many more general time series models have been explored. Tests for structural changes in univariate (multivariate) time series include Horváth et al. [28], Vogelsang [57, 58] and Shao [56] for a change in the mean; Inclán and Tiao [31], Lee and Park [38], Gombay, Horváth and Hušková [24] for a change in the marginal variance; Giraitis et al. [20], and Inoue [32] for a change in the marginal distribution function; Giraitis and Leipis [19] and Lavielle and Ludena [36] for a change in the spectrum; Berkes et al. [3], Galeano and Peña [18] for a change in the autocovariance structure. The tests for change-point in time series models or regression models with dependent errors are covered by Andrews [2], Lee et al. [37], Ling [39] and Gombay [22] and [23], among others. For further methods and developments, we refer to the books by Csörgő and Horváth [11], Hackl and Westlund (1991), Tong (1990) and the review articles by Bhattacharya [6], Deshayes and Picard [13] and the references therein.

Among the references above, the most relevant to our work are [20] and [32], in which a change in the marginal distribution of the X_i's is considered. We will discuss these papers in more detail shortly.

The theory of change-point detection in the marginal distribution F of the X_i's relies on a detailed analysis of the asymptotic behaviour of the sequential empirical distribution $F_{[ns]}(x) := \frac{1}{[ns]} \sum_{i=1}^{[ns]} I(X_i \leq x)$, for $-\infty < x < \infty$ and $0 \leq s \leq 1$. This in turn depends on whether the process is short or long range dependent (equivalently, the process is said to have short or long memory). The linear process we consider is short range dependent, and in much of the literature on short range dependent time series, an assumption of "mixing" is made. Heuristically, the process (X_i) is mixing if X_i and X_j are approximately independent for large values of $|i-j|$. Mixing assumptions lead to a very nice asymptotic theory (see Bradley [7, 8], Csörgő and Horváth [11], Dehling and Philipp [12], Doukhan [14], Peligrad [46], Philipp [48], Rio [54]). However, mixing conditions can be difficult to check and can impose unnecessary restrictions on the model. Furthermore, many authors have given counterexamples where mixing conditions are not satisfied - see, for example, Andrews [1], Chernick [10], Eberlein and Taqqu [16], Ibragimov and Linnik [29], Ibragimov and Rosanov [30] and Radulovic [53].

An alternative approach to deriving the asymptotic behaviour of the empirical distribution of a stationary process is based on an elegant martingale technique first introduced by Gordin [25], which allows one to approximate the empirical process $\sqrt{n}(F_n(x) - F(x))$ with a martingale and then apply one of the classic martingale central limit theorems (see Heyde [27], Maxwell and Woodroofe [41] and McLeish [43, 44] among others). This was the approach taken by Doukhan and Surgailis [15] who proved an invariance principle for the empirical process of the linear model described earlier under minimal conditions that include non-mixing sequences.

In this thesis, we use the martingale approach since our goal is to develop a simple and widely applicable testing method by striking a balance between the generality of the model, the nature of the change, and the simplicity of the conditions imposed. The linear model includes a wide range of ARMA processes and we are able to prove an invariance principle (functional central limit theorem) for the sequential empirical process (both with and without a change) under exactly the same conditions as Doukhan and Surgailis [15]. These conditions are expressed in terms of existence of moments of any order for the innovations ξ_j combined with a corresponding condition on summability of the coefficients a_j; there are no assumptions of mixing or association. Furthermore, we are able to test simultaneously for changes in the coefficients (a_j) or in the location or scale of the innovations.

We now return to some related references. We shall mention for instance that a linear model is considered in [3] and although mixing is not required, the summability and moment conditions imposed are stronger than ours and the change considered is only reflected in the covariance structure of a linear process. In [20] and [32], as in our case, the authors propose tests for change in the marginal distribution function of a time series. The key to developing the asymptotic behaviour of the test statistics is a functional central limit theorem (FCLT) for the sequential empirical process both with and without a change-point. On one hand, although no particular model is imposed in [20], a FCLT is assumed a priori and there is no discussion of how to find critical values for the test statistics. On the other hand, Inoue [32] proposes a nonparametric testing approach for a very specific change-point model that requires quite complex assumptions and mixing properties.

While of theoretical interest, the Gaussian limit in the FCLT for the sequential empirical process generated by a short memory stationary time series has a complex covariance structure that cannot be evaluated from a single realization of the process. This necessitates a bootstrap procedure for applications - i.e. we need a consistent bootstrapping technique that preserves the interdependence within the sample both under the hypothesis and the alternative in order to tabulate appropriate critical values for our test statistics.

An excellent general reference for various bootstrap techniques for stationary sequences is [35]. A comprehensive review of the moving block bootstrap (MBB) is given by Radulović in [51]; sufficient conditions for a bootstrap empirical CLT are given in terms of α or β-mixing coefficients. More recently, Radulović revisits the disjoint block bootstrap (DBB) in [52], proving that under mild β-mixing conditions, if an empirical central limit theorem exists, then there is also a DBB empirical central limit theorem. Other bootstrap techniques are discussed by Politis et al. in [50] and Bühlmann in [9]. In the references above, and indeed in most of the available literature, assumptions of mixing or association are made. In the case of the MBB empirical central limit theorem, a notable exception is Theorem 2.2 of Peligrad [47] where sufficient conditions for convergence of finite dimensional distributions and tightness of the bootstrapped empirical process are expressed in terms of moments of the empirical distribution of the stationary sequence.

We will see that conditions on the innovations and coefficients of the linear process similar to those of Doukhan and Surgailis [15] are sufficient to ensure that Peligrad's moment conditions hold. These simple assumptions allow us to handle non-mixing linear processes. Furthermore, even when the linear process can be shown to be mixing, our conditions improve on the sharpest available sufficient mixing rates for the empirical MBB FCLT. Additionally, we must also prove a sequential version of Peligrad's theorem and show that the bootstrap is consistent under the alternative. As in [32], this will be proven under the assumption of a converging alternative. While the "converging alternative" model of [32] can include multiple change-points, it is very complicated, involving specific rates at which change occurs in both the marginal distribution of X_1 and the joint distributions of (X_1, X_{1+d}), $d \geq 1$. In addition, a very strong mixing rate is required for consistency of the bootstrap. In contrast, we assume only that the change occurs relatively early or late during the observation period and no mixing is required.

We proceed as follows: we introduce the model, based on a stationary linear process, and the simple conditions of [15] in the next chapter. A functional central limit theorem for the sequential empirical process will then be stated and proven.

In Chapter 2, we present the change-point model. A functional central limit theorem for the sequential empirical process in the change-point framework is proven in Section 2.1. Finally, the last section is dedicated to developing a nonparametric approach for testing for a change in the marginal distribution function. We end the chapter with an analysis of the asymptotic behaviour under the null and the alternative hypotheses of both a Kolmogorov-Smirnov type statistic and a Cramér-Von Mises statistic.

Chapter 3 addresses the validity of the weak convergence of the bootstrap empirical process. A brief introduction to the moving blocks bootstrap technique and the necessary assumptions will be discussed in Section 3.1. The main result in this chapter, Theorem 3.2.2 is of independent interest and establishes the validity of the moving block bootstrap in the case of the linear model. It will be then followed by a direct application.

An extension of the results of the last chapter will be detailed in Chapter 4: we will prove functional limit theorems for the bootstrap sequential empirical process under both the null hypothesis and a local alternative. The last section will be devoted to examining the asymptotic behaviour of the bootstrapped test statistics.

Finally, the proposed framework will be illustrated by simulations in Chapter 5. We investigate the power of our tests and as well illustrate that our test can detect a gradual change in the marginal distribution, provided there is rapid convergence to stationarity after the change. We conclude with a summary of the contributions of the thesis and a discussion of some directions for further research. Additional simulations and R code are presented in the Appendix.

■

Chapter 1

Functional Central Limit Theorem for the Sequential Empirical Process

In this chapter, we investigate the asymptotic behaviour of the sequential empirical process for a stationary causal linear process. To that end, we introduce the model and the necessary background in the first section. The second section will be devoted to proving a sequential version of the functional central limit theorem of Doukhan and Surgailis [15] for the sequential empirical process.

1.1 The model

We are given a stationary causal linear process

$$X_i = \sum_{j \geq 0} a_j \xi_{i-j}, \quad i \in \mathbb{Z}, \tag{1.1.1}$$

where $\{a_j, j \in \mathbb{Z}\}$ is a sequence of non-random weights and the ξ_j are independent and identically distributed (i.i.d) innovations.

Let F and F_ξ denote the respective distribution functions of X_0 and ξ_0. In the sequel, we will proceed under the following assumptions as in Doukhan and Surgailis [15].

Assumptions 1.1.1

1. Let $\{a_j, j \in \mathbb{Z}\}$ be a sequence of non-random weights, infinitely many of which are non-zero, satisfying
$$\sum_{j \geq 0} |a_j|^\gamma < \infty \quad \text{for some} \quad \gamma \in (0, 1].$$
2. There exist constants $C < \infty$ and $\Delta \in \left(\frac{2}{3}, 1\right]$ such that for all $u \in \mathbb{R}$
$$|E \exp(iu\xi_0)| \leq \frac{C}{(1 + |u|)^\Delta}.$$
3. $E[|\xi_0|^{4\gamma}] < \infty$.

Comments 1.1.2

- Note that $E[X_i]$ might not exist and $Var(X_i) < \infty$ if and only if $E[\xi_0^2] < \infty$. In this case, X_i has short memory (is short range dependent) in the sense that $\sum_{i \in \mathbb{Z}} |Cov(X_0, X_i)| < \infty$.
- Remark that the more general the moment condition in Assumption 1.1.1.3 , the more restrictive the summability condition Assumption 1.1.1.1.
- Assumption 1.1.1.2 implies that the distribution function of ξ_0 satisfies the Hölder condition $|F_\xi(x) - F_\xi(y)| < C|x-y|^\Delta$. It also implies that the distribution function of a partial sum of the $a_j \xi_{i-j}$ terms is differentiable with a bounded density satisfying a uniform Lipschitz condition, provided that sufficient terms with non-zero a_j are included in the moving average (cf. [15]).
  We note that Doukhan and Surgailis assumed that $\Delta \in \left(\frac{1}{2}, 1\right]$, but in fact there

is a small error in the tightness argument [1]. To be specific, the right hand side of equation (17) of [15] should be $CN^2|x-y|^{3\Delta/2}$. If F_ξ has a bounded Lipschitz density, then Assumption 1.1.1.2 is not required.

Before we state the functional central limit theorem for the sequential empirical process, we need to introduce some basic notation:

$$\begin{aligned} R_i(x) &:= \mathbf{I}(X_i \le x) - F(x), \\ F_n(x) &:= \frac{1}{n}\sum_{i=1}^{n} \mathbf{I}(X_i \le x). \end{aligned}$$

Define now the two-parameter sequential process for $(x,s) \in \mathbb{R} \times [0,1]$

$$\begin{aligned} W_n(x,s) &:= \frac{[ns]}{\sqrt{n}}(F_{[ns]}(x) - F(x)) \\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]} R_i(x). \end{aligned}$$

We consider the weak convergence, denoted by $\stackrel{\mathbf{D}}{\to}$, of random elements taking values in the space $D(\mathbb{R} \times [0,1])$ equipped with Skorokhod's J_1-topology [2].

1.2 FCLT for the sequential empirical process

Here we provide a functional central limit theorem for the sequential empirical process. This is the main result in the first chapter and is a sequential version of the functional central limit theorem of [15].

Theorem 1.2.1 *Assume Assumptions 1.1.1.1-1.1.1.3 hold. Then, as* $n \to \infty$*,*

$$W_n(\cdot,\cdot) \stackrel{\mathbf{D}}{\to} W^{(1)}(\cdot,\cdot),$$

[1] We thank Professor Donatas Surgailis for this clarification, made in a private communication.

[2] See [4] for more details.

where $W^{(1)}(\cdot,\cdot)$ *is a centred Gaussian process with covariance*

$$\begin{aligned}\sigma((x,s),(y,t)) &:= Cov\left(W^{(1)}(x,s),W^{(1)}(y,t)\right)\\ &= (s\wedge t)\sum_{i\in\mathbf{Z}} Cov\left(R_0(x),R_i(y)\right)\\ &= (s\wedge t)\sum_{i\in\mathbf{Z}} Cov\left(\mathbf{I}(X_0\le x),\mathbf{I}(X_i\le y)\right).\end{aligned}$$

Proof

Let $\mathcal{F}_i$ be the σ-algebra defined by

$$\mathcal{F}_i=\{\xi_j\ :\ j\le i\}.$$

Define for $h\ge 0$ the martingale differences

$$U_{i,h}(x):=P(X_i\le x|\mathcal{F}_{i-h})-P(X_i\le x|\mathcal{F}_{i-h-1}).$$

The $U_{i,h}(x)$ are stationary in i for a fixed h. Furthermore,

$$E[U_{i,h}(x)U_{j,h'}(y)]=0\quad\text{for}\quad i-h\neq j-h',x,y\in\mathbb{R}.\tag{1.2.1}$$

Define for $N<\infty$,

$$\begin{aligned}R_i^N(x) &:= \sum_{h=0}^{N}U_{i,h}(x)\\ &= \mathbf{I}(X_i\le x)-P(X_i\le x|\mathcal{F}_{i-N-1}).\end{aligned}\tag{1.2.2}$$

This series converges in L^p for all $p>0$. In fact, we can observe that:

$$\begin{aligned}R_i^N(x) &\overset{a.s}{\to} \mathbf{I}(X_i\le x)-P(X_i\le x|\bigcap_{\ell}\mathcal{F}_\ell)\\ &= \mathbf{I}(X_i\le x)-F(x)\\ &= R_i(x),\end{aligned}\tag{1.2.3}$$

where the first and the second lines follow respectively using the reverse martingale convergence theorem and the 0-1 law.

Thus, we have almost surely

$$R_i(x)=\lim_{N\to\infty}R_i^N(x)=\sum_{h\ge 0}U_{i,h}(x).$$

Moreover, since $0 \leq P(X_i \leq x|\mathcal{F}_{i-N-1}) \leq 1$, we have convergence in L^p for all $p > 0$. Write now

$$R_i(x) = R_i^N(x) + \widetilde{R}_i^N(x), \text{ where } \widetilde{R}_i^N(x) = \sum_{h>N} U_{i,h}(x).$$

Similarly, write

$$W_n(x,s) = W_n^N(x,s) + \widetilde{W}_n^N(x,s),$$

where

$$W_n^N(x,s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} R_i^N(x) \text{ and } \widetilde{W}_n^N(x,s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} \widetilde{R}_i^N(x).$$

The proof of the theorem will be based on the following

(a) Convergence of finite dimensional distributions using the Cramér-Wold device.

(b) A tightness argument for the sequence $W_n(\cdot,\cdot)$.

Proof of (a) We will give an illustration of the proof by showing the result for only two points since the general proof follows similarly.

We know from [15] that

$$\sqrt{[ns]}\left(F_{[ns]}(x) - F(x)\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2(x)),$$

where $\xrightarrow{d}$ denotes convergence in distribution and

$$\sigma^2(x) = \sum_{j\in\mathbb{Z}} cov(R_0(x), R_j(x)).$$

On the other hand, we know that

$$\frac{\sqrt{[ns]}}{\sqrt{n}} \to \sqrt{s} \text{ as } n \to \infty.$$

Therefore,

$$W_n(x,s) \xrightarrow{d} \mathcal{N}(0, s\sigma^2(x)) \text{ as } n \to \infty.$$

For all $a, b \in \mathbb{R}$

$$aW_n(x,s) + bW_n(y,t) = aW_n^N(x,s) + bW_n^N(y,t) + a\widetilde{W}_n^N(x,s) + b\widetilde{W}_n^N(y,t).$$

In order to prove **(a)**, we will show

(a1) $aW_n^N(x,s)+bW_n^N(y,t)\xrightarrow{d}\mathcal{N}(0,\sigma_N^2)$, as $n\to\infty$, where

$$\begin{aligned}\sigma_N^2 &= a^2s\sum_{j\in\mathbb{Z}}Cov(R_0^N(x),R_j^N(x))+2ab(s\wedge t)\sum_{j\in\mathbb{Z}}Cov(R_0^N(x),R_j^N(y))\\ &\quad +b^2t\sum_{j\in\mathbb{Z}}Cov(R_0^N(y),R_j^N(y)). \qquad (1.2.4)\end{aligned}$$

(a2) $\sigma_N^2\to\sigma_*^2=a^2s\sigma^2(x)+2ab\sigma((x,s),(y,t))+b^2t\sigma^2(y)$, as $N\to\infty$.

(a3) $Var(a\widetilde{W}_n^N(x,s)+b\widetilde{W}_n^N(y,t))\leq\delta(N)$, where $\delta(N)\to0$ as $N\to\infty$.

Hence,

$$aW_n^N(x,s)+bW_n^N(y,t)\xrightarrow{d}\mathcal{N}(0,\sigma_N^2)\xrightarrow{d}\mathcal{N}(0,\sigma_*^2),$$

where the first limit corresponds to $n\to\infty$ and the second limit is taken as $N\to\infty$. The finite dimensional convergence follows from [5] Theorem 3.2, as (a3) implies that

$$\lim_{N\to\infty}\limsup_{n\to\infty}P(|a\widetilde{W}_n^N(x,s)+b\widetilde{W}_n^N(y,t)|\geq\varepsilon)=0,\quad\forall\varepsilon>0.$$

Thus,

$$aW_n(x,s)+bW_n(y,t)\xrightarrow{d}\mathcal{N}(0,\sigma_*^2)\quad\text{as } n\to\infty,$$

and **(a)** follows by the Cramér-Wold device.

Proof of (a1) Using the same techniques developed in [15], we write

$$\begin{aligned}\sqrt{n}W_n^N(x,s) &= \sum_{i=1}^{[ns]}R_i^N(x)=\sum_{i=1}^{[ns]}\sum_{h=0}^{N}U_{i,h}(x)\\ &= \sum_{h=0}^{N}\sum_{i=1}^{h}U_{i,h}(x)+\sum_{h=0}^{N}\sum_{i=h+1}^{[ns]}U_{i,h}(x)\\ &= \sum_{h=0}^{N}\sum_{i=1}^{h}U_{i,h}(x)+\sum_{h=0}^{N}\sum_{i=h+1}^{[ns]+h}U_{i,h}(x)-\sum_{h=0}^{N}\sum_{i=[ns]+1}^{[ns]+h}U_{i,h}(x)\end{aligned}$$

$$= \sum_{h=0}^{N}\sum_{i=1}^{h} U_{i,h}(x) + \sum_{h=0}^{N}\sum_{i=1}^{[ns]} U_{i+h,h}(x) - \sum_{h=0}^{N}\sum_{i=[ns]+1}^{[ns]+h} U_{i,h}(x)$$

$$= \sum_{i=1}^{[ns]} M_i^N(x) + Q_n^{N,s}(x),$$

where

$$M_i^N(x) = \sum_{h=0}^{N} U_{i+h,h}(x) \quad \text{and} \quad Q_n^{N,s}(x) = \sum_{h=0}^{N}\sum_{i=1}^{h} U_{i,h}(x) - \sum_{h=0}^{N}\sum_{i=[ns]+1}^{[ns]+h} U_{i,h}(x).$$

Similarly, we have

$$\sqrt{n}W_n^N(y,t) = \sum_{i=1}^{[nt]} M_i^N(y) + Q_n^{N,t}(y).$$

In what follows, C will denote a generic constant throughout the proof which may be different at each appearance. It was shown in [15] that

- $(M_i^N(x), \mathcal{F}_i)$ is a martingale difference sequence.
- for a fixed N
$$\max_{1\le i\le n}\left|\frac{1}{\sqrt{n}}M_i^N(x)\right| \le \frac{C}{\sqrt{n}}. \qquad (1.2.5)$$
- $$\frac{1}{n}\sum_{i=1}^{n}(M_i^N(x))^2 \overset{a.s}{\to} \sigma_N^2(x) \quad \text{as} \quad n\to\infty, \qquad (1.2.6)$$
  where
$$\sigma_N^2(x) = E\left[(M_0^N(x))^2\right] = \sum_{j\in\mathbb{Z}} cov(R_0^N(x), R_j^N(x)).$$
- $$\sigma_N^2(x) \to \sigma^2(x) = \sum_{j\in\mathbb{Z}} cov(R_0(x), R_j(x)), \text{ as } N\to\infty. \qquad (1.2.7)$$

Suppose now, for instance, that $s \le t$ and consider the sequence

$$K_i^N(x,y) = a\mathbf{I}(i\le[ns])M_i^N(x) + bM_i^N(y).$$

Hence

1. $\{K_i^N(x,y), \mathcal{F}_i\}$ is a martingale difference sequence.

2. For a fixed N, we have

$$\begin{aligned}\max_{1\le i\le [nt]}\left|\frac{1}{\sqrt{n}}K_i^N(x,y)\right| &\le \max_{1\le i\le n}\left|\frac{a}{\sqrt{n}}M_i^N(x)\right| + \max_{1\le i\le n}\left|\frac{b}{\sqrt{n}}M_i^N(y)\right| \\ &\le \frac{C}{\sqrt{n}}.\end{aligned}$$

3.

$$\begin{aligned}\frac{1}{n}\sum_{i=1}^{[nt]}(K_i^N(x,y))^2 &= \frac{1}{n}\left[\sum_{i=1}^{[ns]}\left(aM_i^N(x)+bM_i^N(y)\right)^2 + \sum_{i=[ns]+1}^{[nt]}(bM_i^N(y))^2\right] \\ &\overset{a.s}{\to} sE\left[\left(aM_0^N(x)+bM_0^N(y)\right)^2\right] + (t-s)E\left[\left(bM_0^N(y)\right)^2\right],\end{aligned}$$

where the last line follows from the ergodic theorem (see [5] for more details).

Applying Theorem 2.3 of McLeish [42] and combining that with equation (1.2.6) leads to

$$\frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]}M_i^N(x) + \sum_{i=1}^{[nt]}M_i^N(y)\right] \overset{d}{\to} N(0,\sigma_N^2), \quad \text{as } n\to\infty, \tag{1.2.8}$$

where σ_N^2 was defined previously in (1.2.4).

Here we used the fact that

$$\begin{aligned}E[M_0^N(x)M_0^N(y)] &= E\left[\sum_{h=0}^{N}\sum_{h'=0}^{N}U_{h,h}(x)U_{h',h'}(y)\right] \\ &= \sum_{h=0}^{N}\sum_{h'=0}^{N}E\left[U_{0,h}(x)U_{h'-h,h'}(y)\right] \quad \text{by stationarity} \\ &= \sum_{j\in\mathbb{Z}}\sum_{h=0}^{N}\sum_{h'=0}^{N}E[U_{0,h}(x)U_{j,h'}(y)] \quad \text{using (1.2.1)} \\ &= \sum_{j\in\mathbb{Z}}Cov(R_0^N(x),R_j^N(y)).\end{aligned} \tag{1.2.9}$$

Recall

$$Q_n^{N,s}(x) = \sum_{h=0}^{N}\sum_{i=1}^{h} U_{i,h}(x) - \sum_{h=0}^{N}\sum_{i=[ns]+1}^{[ns]+h} U_{i,h}(x).$$

We want to show that $\frac{1}{\sqrt{n}}Q_n^{N,s}(x)$ converges to 0 in probability. For h fixed, we can easily see that

$$\begin{aligned} E\left[\left(\frac{1}{\sqrt{n}}\sum_{i=[ns]+1}^{[ns]+h} U_{i,h}(x)\right)^2\right] &= \frac{1}{n}E\left[\sum_{i=[ns]+1}^{[ns]+h}\sum_{j=[ns]+1}^{[ns]+h} U_{i,h}(x)U_{j,h}(x)\right] \\ &= \frac{1}{n}\sum_{i=[ns]+1}^{[ns]+h} E\left[U_{i,h}^2(x)\right] \quad \text{using (1.2.1)} \\ &\leq \frac{h}{n} \quad \text{since } |U_{i,h}(x)| \leq 1 \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned}$$

This implies that the second term in $\frac{1}{\sqrt{n}}Q_n^{N,s}(x)$ converges to 0 in probability as $n \to \infty$ since it consists of a finite sum of terms, each of which converges to 0. The first term also converges to 0 in probability by the same argument. Thus, for N fixed

$$\frac{1}{\sqrt{n}}Q_n^{N,s}(x) \xrightarrow{p} 0 \quad \left(\text{similarly, } \frac{1}{\sqrt{n}}Q_n^{N,t}(y) \xrightarrow{p} 0\right), \quad \text{as} \quad n \to \infty. \tag{1.2.10}$$

Combining now (1.2.8) and (1.2.10) completes the proof of **(a1)**.

□

Proof of (a2) To prove (a2) we need the following lemmata:

Lemma 1.2.2 *Under the conditions of Theorem 1.2.1, there exists an $h_0 < \infty$ such that, provided $h > h_0$,*

$$|U_{i,h}(x)| \leq C|a_h|^{\gamma}(1 + |\xi_{i-h}|^{\gamma}),$$

where C does not depend on i or h.

The proof of the above lemma will be omitted here since a similar one can be found in [15].

Lemma 1.2.3 *Under the conditions of Theorem 1.2.1 there exists a sequence* $b_j \geq 0$, $j \in \mathbb{Z}$, *independent of* N *and such that* $\sum_{j\in\mathbb{Z}} b_j < \infty$ *and*

$$|Cov(R_0^N(x), R_j^N(y))| \leq b_j.$$

This is equation (10) of [15]. For completeness, we provide a detailed proof.

Proof For $j > 0$

$$\begin{aligned}
\left|Cov(R_0^N(x), R_j^N(y))\right| &= \left|E\left[\sum_{h=0}^{N}\sum_{h'=0}^{N} U_{0,h}(x)U_{j,h'}(y)\right]\right| \\
&= \left|E\left[\sum_{h=0}^{N} U_{0,h}(x)U_{j,j+h}(y)\right]\right| \\
&\leq \sum_{h=0}^{N} E\left[|U_{0,h}(x)U_{j,j+h}(y)|\right] \\
&\leq \sum_{h=0}^{N} E^{\frac{1}{2}}[U_{0,h}^2(x)]E^{\frac{1}{2}}[U_{j,j+h}^2(y)],
\end{aligned}$$

where the second line follows from (1.2.1).

From Lemma 1.2.2 and since $|U_{0,h}(x)| \leq 1$, for $j > h_0$

$$\begin{aligned}
\sum_{h=0}^{h_0} E^{\frac{1}{2}}[U_{0,h}^2(x)]E^{\frac{1}{2}}[U_{j,j+h}^2(y)] &\leq \sum_{h=0}^{h_0} E^{\frac{1}{2}}[U_{j,j+h}^2(y)] \\
&\leq \sum_{h=0}^{h_0} E^{\frac{1}{2}}\left[C^2|a_{j+h}|^{2\gamma}(1+|\xi_{-h}|^{\gamma})^2\right] \\
&\leq C\sum_{h=0}^{h_0} |a_{j+h}|^{\gamma}(1+E^{\frac{1}{2}}[|\xi_0|^{2\gamma}]) \\
&\leq C\sum_{h=0}^{h_0} |a_{j+h}|^{\gamma} \\
&= b'_j.
\end{aligned}$$

Using again Lemma 1.2.2 for $j > h_0$

$$\begin{aligned}
\sum_{h=h_0+1}^{N} E^{\frac{1}{2}}[U_{0,h}^2(x)]E^{\frac{1}{2}}[U_{j,j+h}^2(y)] &\leq C \sum_{h=h_0+1}^{N} |a_h|^\gamma |a_{j+h}|^\gamma (1 + E[|\xi_0|^{2\gamma}]) \\
&\leq C \sum_{h=h_0+1}^{\infty} |a_h|^\gamma |a_{j+h}|^\gamma \\
&= b_j''.
\end{aligned}$$

By Assumption 1.1.1.1

$$\begin{aligned}
\sum_{j>h_0} (b_j' + b_j'') &\leq C \left[\sum_{j\geq 0} \sum_{h=0}^{h_0} |a_{j+h}|^\gamma + \sum_{j\geq 0} \sum_{h=h_0+1}^{\infty} |a_h|^\gamma |a_{j+h}|^\gamma \right] \\
&= C \left[\sum_{h=0}^{h_0} \sum_{j\geq 0} |a_{j+h}|^\gamma + \sum_{h=h_0+1}^{\infty} |a_h|^\gamma \sum_{j\geq 0} |a_{j+h}|^\gamma \right] \\
&\leq C \left[\sum_{h=0}^{h_0} \sum_{j\geq 0} |a_{j+h}|^\gamma + \sum_{h=0}^{\infty} |a_h|^\gamma \right] \\
&< \infty,
\end{aligned}$$

where the last line follows from Assumption 1.1.1.1 and remarking that the number of terms in the first sum is finite.

Thus for $j > 0$

$$|Cov(R_0^N(x), R_j^N(y))| \leq b_j,$$

where $b_j = b_j' + b_j''$ for $j > h_0$ and $b_j = 1$ for $0 \leq j \leq h_0$, since we know from equation (1.2.2) that $|R_j^N(x)| \leq 1$ for all N.

Remark now, using stationarity, that for $j < 0$

$$\begin{aligned}
Cov(R_0^N(x), R_j^N(y)) &= E\left[\sum_{h=0}^{N} \sum_{h'=0}^{N} U_{0,h}(x) U_{j,h'}(y) \right] \\
&= E\left[\sum_{h=0}^{N} \sum_{h'=0}^{N} U_{-j,h}(x) U_{0,h'}(y) \right] \\
&= Cov(R_0^N(x), R_{-j}^N(y)).
\end{aligned}$$

This completes the proof of Lemma 1.2.3.

□

Recall equation (1.2.3)

$$R_0^N(x) \to R_0(x) \quad \text{a.s.}$$

$$R_j^N(y) \to R_j(y) \quad \text{a.s.}$$

By the bounded convergence theorem

$$Cov(R_0^N(x),\ R_j^N(y)) \to Cov(R_0(x),\ R_j(y)),$$

and since the covariances are absolutely summable, by Lemma 1.2.3, we can exchange limits and summations to obtain

$$\sum_{j\in\mathbb{Z}} Cov(R_0^N(x),\ R_j^N(y)) \to \sum_{j\in\mathbb{Z}} Cov(R_0(x),\ R_j(y)). \tag{1.2.11}$$

Now **(a2)** follows from (1.2.7) and (1.2.11).

□

Proof of (a3) We know from [15] that

$$Var(\widetilde{W}_n^N(x,s)) \leq \delta_1(N), \qquad \text{where } \ \delta_1(N) \to 0 \ \text{ as } \ N \to 0,$$
$$Var(\widetilde{W}_n^N(y,t)) \leq \delta_2(N), \qquad \text{where } \ \delta_2(N) \to 0 \ \text{ as } \ N \to 0.$$

Consequently,

$$\begin{aligned} Var(a\widetilde{W}_n^N(x,s) + b\widetilde{W}_n^N(y,t)) &= a^2 Var(\widetilde{W}_n^N(x,s)) + 2abCov(\widetilde{W}_n^N(x,s), \widetilde{W}_n^N(y,t)) \\ &\quad + b^2 Var(\widetilde{W}_n^N(y,t)) \\ &\leq a^2\delta_1(N) + 2ab(\delta_1(N)\delta_2(N))^{\frac{1}{2}} + b^2\delta_2(N) \\ &= \delta(N) \to 0 \quad \text{as} \quad N \to \infty. \end{aligned}$$

This completes the proof of **(a3)** as well as **(a)**.

□

Proof of (b) Let us now address the problem of the tightness of the sequence $\{W_n(\cdot,\cdot), n \geq 1\}$. To that end, let $T = \mathbb{R} \times [0,1]$, $\mathbf{x} = (x,s)$, $\mathbf{y} = (y,t)$ and define the modulus of continuity for $\delta > 0$

$$w(W_n, \delta) = \sup \{|W_n(\mathbf{x}) - W_n(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in T, \|\mathbf{x} - \mathbf{y}\| \leq \delta\},$$

where $\|\mathbf{x}\| = \max(|x|, |s|)$.
We shall show that for all $\varepsilon > 0$

$$\lim_{\delta \to 0} \limsup_n P\{w(W_n, \delta) \geq \varepsilon\} = 0.$$

In fact, we will prove that for each positive ε and η there exists a δ, $0 < \delta < 1$, such that for all n sufficiently large

$$P\{w(W_n, \delta) \geq \varepsilon\} \leq \eta. \tag{1.2.12}$$

Let ε and η be two positive reals and consider the block $A = (x,y] \times (s,t]$ such that $|x-y| \leq 1$. The increment of W_n around A is defined to be

$$W_n(A) = W_n(x,s) - W_n(x,t) - W_n(y,s) + W_n(y,t).$$

The proof of (1.2.12) is a consequence of **(b1)** and **(b2)** following:

(b1) We first establish the following inequality

$$\begin{aligned} &E\left[W_n^4(A)\right] \\ &\leq \; C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}(|t-s||x-y|)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta}\right). \end{aligned}$$

(b2) Secondly, we show that there exists $0 < \delta < 1$ such that

$$P\left\{ \sup_{\substack{x \leq y \leq x+\delta \\ s \leq t \leq s+\delta}} |W_n(y,t) - W_n(x,s)| \geq 5\varepsilon \right\} < \eta\delta^2,$$

for all sufficiently large n.

Then, arguing as in the proof of Theorem 8.3 in [5], we define

$$A_{x,s} = \left\{ \sup_{\substack{x \le y \le x+\delta \\ s \le t \le s+\delta}} |W_n(y,t) - W_n(x,s)| \ge 5\varepsilon \right\},$$

and

$$A^* = \left\{ \sup_{\substack{|x-y|<\delta \\ |t-s|<\delta}} |W_n(y,t) - W_n(x,s)| \ge 20\varepsilon \right\}.$$

If $|x-y| < \delta$ and $|t-s| < \delta$ then one of the following cases occurs for some i and j:

- $i\delta \le x, y \le (i+1)\delta$ and $j\delta \le s, t \le (j+1)\delta$. In this case, we have

$$|W_n(y,t) - W_n(x,s)| \le |W_n(y,t) - W_n(i\delta, j\delta)| + |W_n(x,s) - W_n(i\delta, j\delta)| .$$

- $i\delta \le x, y \le (i+1)\delta$ and $j\delta \le s \le (j+1)\delta \le t \le (j+2)\delta$. In this case, we have

$$\begin{aligned} |W_n(y,t) - W_n(x,s)| &\le |W_n(y,t) - W_n(i\delta, (j+1)\delta)| \\ &\quad + |W_n(i\delta, (j+1)\delta) - W_n(i\delta, j\delta)| + |W_n(x,s) - W_n(i\delta, j\delta)| . \end{aligned}$$

- $i\delta \le x \le (i+1)\delta \le y \le (i+2)\delta$ and $j\delta \le s, t \le (j+1)\delta$. In this case, we have

$$\begin{aligned} |W_n(y,t) - W_n(x,s)| &\le |W_n(y,t) - W_n((i+1)\delta, j\delta)| \\ &\quad + |W_n((i+1)\delta, j\delta) - W_n(i\delta, j\delta)| + |W_n(x,s) - W_n(i\delta, j\delta)| . \end{aligned}$$

- $i\delta \le x \le (i+1)\delta \le y \le (i+2)\delta$ and $j\delta \le s \le (j+1)\delta \le t \le (j+2)\delta$. In this case, we have

$$\begin{aligned} |W_n(y,t) - W_n(x,s)| &\le |W_n(y,t) - W_n((i+1)\delta, (j+1)\delta)| \\ &\quad + |W_n((i+1)\delta, (j+1)\delta) - W_n(i\delta, j\delta)| + |W_n(x,s) - W_n(i\delta, j\delta)| . \end{aligned}$$

- $i\delta \le x \le (i+1)\delta \le y \le (i+2)\delta$ and $j\delta \le t \le (j+1)\delta \le s \le (j+2)\delta$. In this case, we have

$$|W_n(y,t) - W_n(x,s)| \le |W_n(y,t) - W_n((i+1)\delta, j\delta)|$$

$$+\left|W_n((i+1)\delta, j\delta) - W_n(i\delta, j\delta)\right| + \left|W_n(i\delta, j\delta) - W_n(i\delta, (j+1)\delta)\right|$$
$$+\left|W_n(i\delta, (j+1)\delta) - W_n(x,s)\right|.$$

We conclude that $A^* \subset \bigcup_{i,j<\delta^{-1}} A_{i\delta, j\delta}$. Therefore, for n sufficiently large, we get

$$\begin{aligned} P\left\{ \sup_{\substack{|x-y|<\delta \\ |t-s|<\delta}} |W_n(y,t) - W_n(x,s)| \geq 20\varepsilon \right\} &= P(A^*) \\ &\leq P\left(\bigcup_{i,j<\delta^{-1}} A_{i\delta,j\delta} \right) \\ &\leq \sum_{i,j<\delta^{-1}} P(A_{i\delta,j\delta}) \\ &\leq (1+[\delta^{-1}])^2 \eta \delta^2 \\ &\leq 4\eta, \end{aligned}$$

which is (1.2.12) except for the factors preceding ε and η. It remains to prove **(b1)** and **(b2)**.

Proof of (b1) Given a function $g(x)$, $x \in \mathbb{R}$, define $g(x,y) = g(x) - g(y)$ and suppose for instance that $s \leq t$. Then

$$\begin{aligned} W_n(A) &= W_n(x,s) - W_n(x,t) - W_n(y,s) + W_n(y,t) \\ &= \frac{1}{\sqrt{n}} \left[\sum_{i=1}^{[ns]} R_i(x) - \sum_{i=1}^{[nt]} R_i(x) - \sum_{i=1}^{[ns]} R_i(y) + \sum_{i=1}^{[nt]} R_i(y) \right] \\ &= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} (R_i(y) - R_i(x)) \\ &= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} R_i(y,x) \\ &= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} \sum_{h\geq 0} U_{i,h}(y,x). \end{aligned}$$

Next define for $k \le [nt]$

$$V_k(x) = \sum_{i=([ns]+1)\vee k}^{[nt]} U_{i,i-k}(x).$$

We will make use of the methodology and the results obtained in [15] and [33]. First, we establish a formula for the fourth moment via the sums $V_k(x)$.

Write

$$\begin{aligned} W_n(A) &= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} \sum_{h\ge 0} U_{i,h}(y,x) \\ &= \frac{1}{\sqrt{n}} \sum_{k\le [nt]} \sum_{i=([ns]+1)\vee k}^{[nt]} U_{i,i-k}(y,x) \\ &= \frac{1}{\sqrt{n}} \sum_{k\le [nt]} V_k(y,x). \end{aligned}$$

Hence

$$\begin{aligned} W_n^4(A) &= \frac{1}{n^2} \left(\sum_{k\le [nt]} V_k(y,x) \right)^4 \\ &= \frac{1}{n^2} \left(\sum_{k\le [nt]} V_k(x,y) \right)^4 \\ &:= \frac{1}{n^2}(4I_1 + 6I_2 + 4I_3 + I_4), \end{aligned}$$

where

$$I_j = \sum_{k\le [nt]} \left(\sum_{\ell \le k-1} V_\ell(x,y) \right)^{4-j} (V_k(x,y))^j, \quad \text{for} \quad j = 1,2,3,4.$$

Remark that $V_k(x,y)$ is $\mathcal{F}_k$-measurable and $E[V_k(x,y)|\mathcal{F}_{k-1}] = 0$, hence

$$E[I_1] = 0. \tag{1.2.13}$$

For the other terms, we are going to develop the desired bound using some results from [15]. For the sake of easy reference, we restate equations (14) and (15) from [15]:

$$\text{For} \quad h > h_0 \qquad |U_{i,h}(x,y)| \le C|x-y||a_h|^\gamma(1 + |\xi_{i-h}|^\gamma). \tag{1.2.14}$$

For $0 \le h \le h_0$ and $p \ge 1$ $\qquad E[|U_{i,h}(x,y)|^p|\mathcal{F}_{i-h-1}] \le C|x-y|^{\Delta}. \qquad (1.2.15)$

Consider now the term I_2 and remark that for $k \le -h_0$

$$\begin{aligned} E[V_k^2(x,y)|\mathcal{F}_{k-1}] &= \sum_{i,j=([ns]+1)\vee k}^{[nt]} E[U_{i,i-k}(x,y)U_{j,j-k}(x,y)|\mathcal{F}_{k-1}] \\ &\le C|x-y|^2 \sum_{i,j=[ns]+1}^{[nt]} |a_{i-k}|^{\gamma}|a_{j-k}|^{\gamma}, \end{aligned} \qquad (1.2.16)$$

where the second line follows from (1.2.14), the fact that ξ_k is independent of $\mathcal{F}_{k-1}$ and $E[|\xi_0|^{2\gamma}] < \infty$.

On the other hand, for $-h_0 < k \le [nt]$, we need to take into consideration the position of $i-k$ and $j-k$ with respect to h_0. In each case, we will make use of the equation(s) (1.2.14) and (or) (1.2.15), the fact that ξ_k is independent of $\mathcal{F}_{k-1}$ and $E[|\xi_0|^{2\gamma}] < \infty$.

- $i-k > h_0$ and $j-k > h_0$

$$\begin{aligned} E[U_{i,i-k}(x,y)U_{j,j-k}(x,y)|\mathcal{F}_{k-1}] &\le C|x-y|^2|a_{i-k}|^{\gamma}|a_{j-k}|^{\gamma}E\left[(1+|\xi_k|^{\gamma})^2\right] \\ &\le C|x-y|^2|a_{i-k}|^{\gamma}|a_{j-k}|^{\gamma}. \end{aligned}$$

- $0 \le i-k \le h_0$ and $0 \le j-k \le h_0$

$$\begin{aligned} &E\left[U_{i,i-k}(x,y)U_{j,j-k}(x,y)|\mathcal{F}_{k-1}\right] \\ &\le E^{\frac{1}{2}}\left[U_{i,i-k}^2(x,y)|\mathcal{F}_{k-1}\right]E^{\frac{1}{2}}\left[U_{j,j-k}^2(x,y)|\mathcal{F}_{k-1}\right] \\ &\le C|x-y|^{\Delta}. \end{aligned}$$

- $0 \le i-k \le h_0$ and $j-k > h_0$

$$\begin{aligned} &E\left[U_{i,i-k}(x,y)U_{j,j-k}(x,y)|\mathcal{F}_{k-1}\right] \\ &\le E^{\frac{1}{2}}\left[U_{i,i-k}^2(x,y)|\mathcal{F}_{k-1}\right]E^{\frac{1}{2}}\left[U_{j,j-k}^2(x,y)|\mathcal{F}_{k-1}\right] \\ &\le C|x-y|^{\frac{\Delta}{2}}E^{\frac{1}{2}}\left[|x-y|^2|a_{j-k}|^{2\gamma}(1+|\xi_k|^{\gamma})^2\right] \\ &\le C|x-y|^{1+\frac{\Delta}{2}}|a_{j-k}|^{\gamma} \\ &\le C|x-y|^{\Delta}|a_{j-k}|^{\gamma}. \end{aligned}$$

Similarly, we get for

- $i - k > h_0$ and $0 \le j - k \le h_0$

$$E\left[U_{i,i-k}(x,y)U_{j,j-k}(x,y)|\mathcal{F}_{k-1}\right] \le C|x-y|^{\Delta}|a_{i-k}|^{\gamma}.$$

Thus, for $-h_0 < k \le [nt]$

$$\begin{aligned} E[V_k^2(x,y)|\mathcal{F}_{k-1}] &= \sum_{i,j=([ns]+1)\vee k}^{[nt]} E\left[U_{i,i-k}(x,y)U_{j,j-k}(y,x)|\mathcal{F}_{k-1}\right] \\ &\le C \sum_{i,j=([ns]+1)\vee(h_0+k+1)}^{[nt]} |x-y|^2|a_{i-k}|^{\gamma}|a_{j-k}|^{\gamma} \\ &\quad + C \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |x-y|^{\Delta} \\ &\quad + C \sum_{i=([ns]+1)\vee(h_0+k+1)}^{[nt]} \sum_{j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |x-y|^{\Delta}|a_{i-k}|^{\gamma} \\ &\le C \sum_{i,j=[ns]+1}^{[nt]} |x-y|^2|a_{i-k}|^{\gamma}|a_{j-k}|^{\gamma} \\ &\quad + C \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |x-y|^{\Delta} \\ &\quad + C \sum_{i=[ns]+1}^{[nt]} \sum_{j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |x-y|^{\Delta}|a_{i-k}|^{\gamma}. \end{aligned} \tag{1.2.17}$$

For the sake of convenience, we denote the upper bounds found in (1.2.16) and (1.2.17) by $T_k^{(1)}$ and $T_k^{(2)}$ respectively and we define

$$T_k = \begin{cases} T_k^{(1)} & \text{if } k \le -h_0 \\ T_k^{(2)} & \text{otherwise.} \end{cases}$$

By orthogonality we obtain for $k \le [nt]$

$$E\left[\left(\sum_{\ell\le k-1} V_\ell(x,y)\right)^2\right] = \sum_{\ell\le k-1} E\left[V_\ell^2(x,y)\right] \le \sum_{\ell\le[nt]} E\left[V_\ell^2(x,y)\right].$$

Therefore,

$$
\begin{aligned}
E[I_2] &= E\left[\sum_{k\le[nt]}\left(\sum_{\ell\le k-1}V_\ell(x,y)\right)^2 V_k^2(x,y)\right]\\
&= \sum_{k\le[nt]}E\left[\left(\sum_{\ell\le k-1}V_\ell(x,y)\right)^2 E[V_k^2(x,y)|\mathcal{F}_{k-1}]\right]\\
&\le \sum_{k\le[nt]}T_kE\left[\left(\sum_{\ell\le k-1}V_\ell(x,y)\right)^2\right]\\
&\le \sum_{k\le[nt]}T_k\sum_{\ell\le[nt]}E\left[V_\ell^2(x,y)\right]\\
&\le \sum_{k\le[nt]}T_k\sum_{\ell\le[nt]}T_\ell\\
&= \left(\sum_{k\le[nt]}T_k\right)^2 \qquad (1.2.18)\\
&= \left(\sum_{k\le -h_0}T_k+\sum_{-h_0<k\le[nt]}T_k\right)^2\\
&\le C\left\{\sum_{k\le[nt]}|x-y|^2\sum_{i,j=[ns]+1}^{[nt]}|a_{i-k}|^\gamma|a_{j-k}|^\gamma\right.\\
&+\sum_{-h_0<k\le[nt]}\sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|x-y|^\Delta\\
&\left.+\sum_{-h_0<k\le[nt]}\sum_{i=[ns]+1}^{[nt]}\sum_{j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|x-y|^\Delta|a_{i-k}|^\gamma\right\}^2. \qquad (1.2.19)
\end{aligned}
$$

Let's now analyze the last two sums in (1.2.19) and notice that

$$
\sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}1=\sum_{j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}1=0 \quad \text{if} \quad [ns]+1>h_0+k,
$$

hence

$$\begin{aligned}
\sum_{-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 &= \sum_{[ns]-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \\
&= \sum_{[ns]-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&+ \sum_{[ns]-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\ge[nt]-[ns]). \qquad (1.2.20)
\end{aligned}$$

Remark now that

$$\begin{aligned}
&\sum_{[ns]-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&= \sum_{[ns]-h_0<k\le[ns]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\le[nt]-h_0} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\le \sum_{[ns]-h_0<k\le[ns]} \sum_{i=k}^{h_0+k} \sum_{j=[ns]+1}^{[nt]} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\le[nt]-h_0} \sum_{i,j=k}^{h_0+k} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\le[nt]} \sum_{i=k}^{h_0+k} \sum_{j=[ns]+1}^{[nt]} \mathbf{I}(h_0<[nt]-[ns]) \\
&= 2h_0(h_0+1)([nt]-[ns])\mathbf{I}(h_0<[nt]-[ns]) \\
&\quad +(h_0+1)^2([nt]-[ns]-h_0)\mathbf{I}(h_0<[nt]-[ns]) \\
&\le C([nt]-[ns])\mathbf{I}(h_0<[nt]-[ns]). \qquad (1.2.21)
\end{aligned}$$

Similarly, we get a bound for the second term in (1.2.20)

$$
\begin{aligned}
&\sum_{[ns]-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
= &\sum_{[ns]-h_0<k\le[nt]-h_0} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
&+ \sum_{[nt]-h_0<k\le[ns]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
&+ \sum_{[ns]<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
\le &\sum_{[ns]-h_0<k\le[nt]-h_0} \sum_{i,j=k}^{h_0+k} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
&+ \sum_{[nt]-h_0<k\le[ns]} \sum_{i=k}^{h_0+k} \sum_{j=[ns]+1}^{[nt]} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
&+ \sum_{[ns]<k\le[nt]} \sum_{i,j=k}^{h_0+k} \mathbf{I}(h_0 \ge [nt]-[ns]) \\
= &\left[2(h_0+1)^2 + (h_0+[ns]-[nt])(h_0+1)\right]([nt]-[ns])\mathbf{I}(h_0 \ge [nt]-[ns]) \\
\le &\ C([nt]-[ns])\mathbf{I}(h_0 \ge [nt]-[ns]). \qquad (1.2.22)
\end{aligned}
$$

Therefore, combining (1.2.20), (1.2.21) and (1.2.22) yields

$$
\sum_{-h_0<k\le[nt]} \sum_{i,j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \le C([nt]-[ns]). \qquad (1.2.23)
$$

To establish a bound for the last sum in (1.2.19), we note that

$$
\begin{aligned}
\sum_{-h_0<k\le[nt]} \sum_{i=[ns]+1}^{[nt]} \sum_{j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |a_{i-k}|^\gamma &\le \sum_{-h_0<k\le[nt]} \sum_{i=[ns]+1}^{[nt]} \sum_{j=k}^{h_0+k} |a_{i-k}|^\gamma \\
&\le (h_0+1) \sum_{i=[ns]+1}^{[nt]} \sum_{k\le[nt]} |a_{i-k}|^\gamma \\
&\le C([nt]-[ns]). \qquad (1.2.24)
\end{aligned}
$$

Substituting (1.2.23) and (1.2.24) in equation (1.2.19) and since $|x-y| \leq 1$, we further get

$$
\begin{aligned}
E[I_2] &\leq C \left\{ |x-y|^2 \sum_{k \leq [nt]} \left(\sum_{i=[ns]+1}^{[nt]} |a_{i-k}|^{\gamma} \right)^2 + ([nt]-[ns])|x-y|^{\Delta} \right\}^2 \\
&\leq C \left[|x-y|^2 \sum_{i=[ns]+1}^{[nt]} \sum_{k \leq [nt]} |a_{i-k}|^{\gamma} + ([nt]-[ns])|x-y|^{\Delta} \right]^2 \\
&\leq C \left[([nt]-[ns]) \left(|x-y|^2 + |x-y|^{\Delta} \right) \right]^2 \\
&\leq C([nt]-[ns])^2 |x-y|^{2\Delta}, \qquad (1.2.25)
\end{aligned}
$$

From (1.2.18) and (1.2.25), we deduce the following inequality which will be used later in the proof:

$$
\sum_{k \leq [nt]} T_k \leq C([nt]-[ns])|x-y|^{\Delta}. \qquad (1.2.26)
$$

Next, consider

$$
E[I_3] = E\left[\sum_{k \leq [nt]} \left(\sum_{\ell \leq k-1} V_{\ell}(x,y) \right) V_k^3(x,y) \right].
$$

In the following, we will use repeatedly equations (1.2.14) and (1.2.15) in addition to having ξ_k independent of $\mathcal{F}_{k-1}$ and $E[|\xi_0|^{4\gamma}] < \infty$.

If $k \leq -h_0$, then

$$
\begin{aligned}
E[\left|V_k^3(x,y)\right| |\mathcal{F}_{k-1}] &\leq \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]} E\left[\prod_{j=1}^{3} |U_{i_j,i_j-k}(x,y)| |\mathcal{F}_{k-1} \right] \\
&\leq C|x-y|^3 \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]} \left(\prod_{j=1}^{3} |a_{i_j-k}|^{\gamma} \right) \\
&\leq C|x-y|^3 \left(\sum_{i=[ns]+1}^{[nt]} |a_{i-k}|^{\gamma} \right)^3 \\
&\leq C|x-y|^3 \sum_{i=[ns]+1}^{[nt]} |a_{i-k}|^{\gamma}. \qquad (1.2.27)
\end{aligned}
$$

If $-h_0 < k \le [nt]$, then we need to consider the position of $i_j - k$ with respect to h_0 for $j = 1, 2, 3$.

• $0 \le i_j - k \le h_0$ for $j = 1, 2, 3$

$$\begin{aligned} & E\left[\prod_{\ell=1}^{3} |U_{i_\ell, i_\ell - k}(x,y)||\mathcal{F}_{k-1}\right] \\ \le\ & E^{\frac{1}{2}}\left[U^2_{i_1, i_1-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\ell=2}^{3} U^2_{i_\ell, i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \le\ & E^{\frac{1}{2}}\left[U^2_{i_1, i_1-k}(x,y)|\mathcal{F}_{k-1}\right] \prod_{\ell=2}^{3} E^{\frac{1}{4}}\left[U^4_{i_\ell, i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \le\ & C|x-y|^{\frac{\Delta}{2}} \prod_{\ell=2}^{3} |x-y|^{\frac{\Delta}{4}} \\ =\ & C|x-y|^{\Delta}. \end{aligned}$$

• $\exists! j \in \{1, 2, 3\}$ such that $0 \le i_j - k \le h_0$

$$\begin{aligned} & E\left[\prod_{\ell=1}^{3} |U_{i_\ell, i_\ell - k}(x,y)||\mathcal{F}_{k-1}\right] \\ \le\ & E^{\frac{1}{2}}\left[U^2_{i_j, i_j-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\substack{\ell=1 \\ \ell \ne j}}^{3} U^2_{i_\ell, i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \le\ & C|x-y|^{\frac{\Delta}{2}} |x-y|^2 \prod_{\substack{\ell=1 \\ \ell \ne j}}^{3} |a_{i_\ell - k}|^{\gamma} \\ \le\ & C|x-y|^{\Delta} \prod_{\substack{\ell=1 \\ \ell \ne j}}^{3} |a_{i_\ell - k}|^{\gamma}. \end{aligned}$$

• $\exists! j \in \{1, 2, 3\}$ such that $i_j - k > h_0$

$$\begin{aligned} & E\left[\prod_{\ell=1}^{3} |U_{i_\ell, i_\ell - k}(x,y)||\mathcal{F}_{k-1}\right] \\ \le\ & E^{\frac{1}{2}}\left[U^2_{i_j, i_j-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\substack{\ell=1 \\ \ell \ne j}}^{3} U^2_{i_\ell, i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \end{aligned}$$

$$\begin{aligned}
&\leq E^{\frac{1}{2}}\left[U^2_{i_j,i_j-k}(x,y)|\mathcal{F}_{k-1}\right]\prod_{\substack{\ell=1\\ \ell\neq j}}^{3}E^{\frac{1}{4}}\left[U^4_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right]\\
&\leq C|x-y||a_{i_j-k}|^{\gamma}\prod_{\substack{\ell=1\\ \ell\neq j}}^{3}|x-y|^{\frac{\Delta}{4}}\\
&\leq C|x-y|^{1+\frac{\Delta}{2}}|a_{i_j-k}|^{\gamma}\\
&\leq C|x-y|^{\Delta}|a_{i_j-k}|^{\gamma}.
\end{aligned}$$

- $i_j-k>h_0$ for $j=1,2,3$

$$\begin{aligned}
E\left[\prod_{j=1}^{3}|U_{i_j,i_j-k}(x,y)||\mathcal{F}_{k-1}\right] &\leq C|x-y|^3\prod_{j=1}^{3}|a_{i_j-k}|^{\gamma}E\left[(1+|\xi_k|^{\gamma})^3\right]\\
&\leq C|x-y|^3\prod_{j=1}^{3}|a_{i_j-k}|^{\gamma}.
\end{aligned}$$

Thus, for $-h_0<k\leq[nt]$

$$\begin{aligned}
&E[\left|V^3_k(x,y)\right||\mathcal{F}_{k-1}] \leq \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]}E\left[\prod_{j=1}^{3}|U_{i_j,i_j-k}(x,y)||\mathcal{F}_{k-1}\right]\\
&\leq C\sum_{i_1,i_2,i_3=([ns]+1)\vee(h_0+k+1)}^{[nt]}|x-y|^3\prod_{\ell=1}^{3}|a_{i_\ell-k}|^{\gamma}+C\sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|x-y|^{\Delta}\\
&\quad+C\sum_{j=1}^{3}\sum_{i_j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}\sum_{\substack{i_\ell,i_p=([ns]+1)\vee(h_0+k+1)\\ \ell,p\in\{1,2,3\}\setminus\{j\},\ell<p}}^{[nt]}|x-y|^{\Delta}|a_{i_\ell-k}|^{\gamma}|a_{i_p-k}|^{\gamma}\\
&\quad+C\sum_{j=1}^{3}\sum_{i_j=([ns]+1)\vee(h_0+k+1)}^{[nt]}\sum_{\substack{i_\ell,i_p=([ns]+1)\vee k\\ \ell,p\in\{1,2,3\}\setminus\{j\},\ell<p}}^{[nt]\wedge(h_0+k)}|x-y|^{\Delta}|a_{i_j-k}|^{\gamma}\\
&\leq C\sum_{i=[ns]+1}^{[nt]}|x-y|^3|a_{i-k}|^{\gamma}+C\sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|x-y|^{\Delta}\\
&\quad+C\sum_{j=1}^{3}\sum_{i_j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}\sum_{\substack{i_\ell,i_p=[ns]+1\\ \ell,p\in\{1,2,3\}\setminus\{j\},\ell<p}}^{[nt]}|x-y|^{\Delta}|a_{i_\ell-k}|^{\gamma}|a_{i_p-k}|^{\gamma}
\end{aligned}$$

$$+ C\sum_{j=1}^{3}\sum_{i_j=[ns]+1}^{[nt]}\sum_{\substack{i_\ell,i_p=([ns]+1)\vee k\\ \ell,p\in\{1,2,3\}\setminus\{j\},\ell<p}}^{[nt]\wedge(h_0+k)}|x-y|^{\Delta}|a_{i_j-k}|^{\gamma}. \tag{1.2.28}$$

If we denote the upper bounds defined earlier in (1.2.27) and (1.2.28) by $B_k^{(1)}$ and $B_k^{(2)}$ respectively and we define

$$B_k = \begin{cases} B_k^{(1)} & \text{if } k \le -h_0 \\ B_k^{(2)} & \text{otherwise,} \end{cases}$$

then, using (1.2.27) and (1.2.28), we have

$$\begin{aligned}
\sum_{k\le[nt]} B_k &= \sum_{k\le -h_0} B_k^{(1)} + \sum_{-h_0<k\le[nt]} B_k^{(2)} \\
&= C\sum_{k\le[nt]}\sum_{i=[ns]+1}^{[nt]}|x-y|^3|a_{i-k}|^{\gamma} + C\sum_{-h_0<k\le[nt]}\sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|x-y|^{\Delta} \\
&\quad + C\sum_{-h_0<k\le[nt]}\sum_{j=1}^{3}\sum_{i_j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}\sum_{\substack{i_\ell,i_p=[ns]+1\\ \ell,p\in\{1,2,3\}\setminus\{j\},\ell<p}}^{[nt]}|x-y|^{\Delta}|a_{i_\ell-k}|^{\gamma}|a_{i_p-k}|^{\gamma} \\
&\quad + C\sum_{-h_0<k\le[nt]}\sum_{j=1}^{3}\sum_{i_j=[ns]+1}^{[nt]}\sum_{\substack{i_\ell,i_p=([ns]+1)\vee k\\ \ell,p\in\{1,2,3\}\setminus\{j\},\ell<p}}^{[nt]\wedge(h_0+k)}|x-y|^{\Delta}|a_{i_j-k}|^{\gamma}.
\end{aligned} \tag{1.2.29}$$

Remark now that

$$\begin{aligned}
\sum_{-h_0<k\le[nt]}\sum_{i_1=[ns]+1}^{[nt]}\sum_{i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|a_{i_1-k}|^{\gamma} &\le C\sum_{-h_0<k\le[nt]}\sum_{i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \\
&\le C([nt]-[ns]),
\end{aligned} \tag{1.2.30}$$

where the last line follows from (1.2.23).

By using (1.2.24), we get

$$\sum_{-h_0<k\le[nt]}\sum_{i_1=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}\sum_{i_2,i_3=[ns]+1}^{[nt]}|a_{i_2-k}|^{\gamma}|a_{i3-k}|^{\gamma}$$

$$
\begin{aligned}
&= \sum_{-h_0<k\le[nt]} \sum_{i=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \left(\sum_{j=[ns]+1}^{[nt]} |a_{j-k}|^\gamma \right)^2 \\
&\le C \sum_{-h_0<k\le[nt]} \sum_{i=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \sum_{j=[ns]+1}^{[nt]} |a_{j-k}|^\gamma \\
&\le C([nt]-[ns]). \qquad (1.2.31)
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
&\sum_{-h_0<k\le[nt]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 = \sum_{[ns]-h_0<k\le[nt]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \\
&= \sum_{[ns]-h_0<k\le[nt]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&+ \sum_{[ns]-h_0<k\le[nt]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\ge[nt]-[ns]). \qquad (1.2.32)
\end{aligned}
$$

As before, remark that

$$
\begin{aligned}
&\sum_{-h_0<k\le[nt]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&= \sum_{[ns]-h_0<k\le[ns]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\le[nt]-h_0} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\le[nt]} \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\le \sum_{[ns]-h_0<k\le[ns]} \sum_{i_1,i_2=k}^{h_0+k} \sum_{i_3=[ns]+1}^{[nt]} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\le[nt]-h_0} \sum_{i_1,i_2,i_3=k}^{h_0+k} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\le[nt]} \sum_{i_1,i_2=k}^{h_0+k} \sum_{i_3=[ns]+1}^{[nt]} \mathbf{I}(h_0<[nt]-[ns])
\end{aligned}
$$

$$
\begin{aligned}
&= 2h_0(h_0+1)^2([nt]-[ns])\mathbf{I}(h_0<[nt]-[ns]) \\
&\quad +(h_0+1)^3([nt]-[ns]-h_0)\mathbf{I}(h_0<[nt]-[ns]) \\
&\le C([nt]-[ns])\mathbf{I}(h_0<[nt]-[ns]). \qquad (1.2.33)
\end{aligned}
$$

Similarly, we get a bound for the second term:

$$
\begin{aligned}
&\sum_{[ns]-h_0<k\le[nt]} \; \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&= \sum_{[ns]-h_0<k\le[nt]-h_0} \; \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\le[ns]} \; \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\le[nt]} \; \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&\le \sum_{[ns]-h_0<k\le[nt]-h_0} \; \sum_{i_1,i_2,i_3=k}^{h_0+k} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\le[ns]} \; \sum_{i_1,i_2=k}^{h_0+k} \; \sum_{i_3=[ns]+1}^{[nt]} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\le[nt]} \; \sum_{i_1,i_2,i_3=k}^{h_0+k} \mathbf{I}(h_0\ge[nt]-[ns]) \\
&= 2(h_0+1)^3([nt]-[ns])\mathbf{I}(h_0\ge[nt]-[ns]) \\
&\quad +(h_0+[ns]-[nt])(h_0+1)^2([nt]-[ns])\mathbf{I}(h_0\ge[nt]-[ns]) \\
&\le C([nt]-[ns])\mathbf{I}(h_0\ge[nt]-[ns]). \qquad (1.2.34)
\end{aligned}
$$

Using now (1.2.32), (1.2.33) and (1.2.34) to get

$$
\sum_{-h_0<k\le[nt]} \; \sum_{i_1,i_2,i_3=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \le C([nt]-[ns]), \qquad (1.2.35)
$$

and combining (1.2.29), (1.2.30), (1.2.31) and (1.2.35) yields

$$\begin{aligned}\sum_{k\le[nt]} B_k &\le C\sum_{k\le[nt]}\sum_{i=[ns]+1}^{[nt]} |x-y|^3|a_{i-k}|^\gamma + C([nt]-[ns])|x-y|^\Delta \\ &\le C([nt]-[ns])(|x-y|^3+|x-y|^\Delta) \\ &\le C([nt]-[ns])|x-y|^\Delta. \qquad (1.2.36)\end{aligned}$$

Therefore, (1.2.26) and (1.2.36) imply that

$$\begin{aligned}E[I_3] &= E\left[\sum_{k\le[nt]}\left(\sum_{\ell\le k-1} V_\ell(x,y)\right)V_k^3(x,y)\right] \\ &\le \sum_{k\le[nt]} E\left[\left|\sum_{\ell\le k-1} V_\ell(x,y)\right| E[|V_k^3(x,y)||\mathcal{F}_{k-1}]\right] \\ &\le \sum_{k\le[nt]} B_k E\left[\left|\sum_{\ell\le k-1} V_\ell(x,y)\right|\right] \\ &\le \sum_{k\le[nt]} B_k E^{\frac{1}{2}}\left[\left(\sum_{\ell\le k-1} V_\ell(x,y)\right)^2\right] \\ &\le \sum_{k\le[nt]} B_k\left(\sum_{\ell\le[nt]} E\left[V_\ell^2(x,y)\right]\right)^{\frac{1}{2}} \\ &\le \sum_{k\le[nt]} B_k\left(\sum_{\ell\le[nt]} T_\ell\right)^{\frac{1}{2}} \\ &\le C([nt]-[ns])^{\frac{3}{2}}|x-y|^{\frac{3\Delta}{2}}. \qquad (1.2.37)\end{aligned}$$

Finally, consider

$$E[I_4] = \sum_{k\le[nt]} E\left[V_k^4(x,y)\right].$$

We use (1.2.14) and (or) (1.2.15) in the following. Since ξ_k is independent of $\mathcal{F}_{k-1}$ and $E[|\xi_0|^{4\gamma}] < \infty$, observe that for $k \leq -h_0$

$$\begin{aligned} E\left[V_k^4(x,y)|\mathcal{F}_{k-1}\right] &= \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]} E\left[\prod_{j=1}^{4} U_{i_j,i_j-k}(x,y)|\mathcal{F}_{k-1}\right] \\ &\leq C|x-y|^4 \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]} \left(\prod_{j=1}^{4}|a_{i_j-k}|^\gamma\right) \\ &\leq C|x-y|^4 \sum_{i=[ns]+1}^{[nt]} |a_{i-k}|^\gamma. \end{aligned} \tag{1.2.38}$$

For $-h_0 < k \leq [nt]$, the upper bound will depend on the position of $i_j - k$ with respect to h_0 for $j = 1, 2, 3, 4$.

- $i_j - k > h_0$ for $j = 1, 2, 3, 4$

$$\begin{aligned} E\left[\prod_{j=1}^{4} U_{i_j,i_j-k}(x,y)|\mathcal{F}_{k-1}\right] &\leq C|x-y|^4 \prod_{j=1}^{4}|a_{i_j-k}|^\gamma E\left[(1+|\xi_k|^\gamma)^4\right] \\ &\leq C|x-y|^4 \prod_{j=1}^{4}|a_{i_j-k}|^\gamma. \end{aligned}$$

- $\exists! j \in \{1,2,3,4\}$ such that $0 \leq i_j - k \leq h_0$

$$\begin{aligned} &E\left[\prod_{\ell=1}^{4} U_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ &\leq E^{\frac{1}{2}}\left[U^2_{i_j,i_j-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\substack{\ell=1\\ \ell\neq j}}^{4} U^2_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ &\leq C|x-y|^{\frac{\Delta}{2}}|x-y|^3 \prod_{\substack{\ell=1\\ \ell\neq j}}^{4}|a_{i_\ell-k}|^\gamma \\ &\leq C|x-y|^{\Delta} \prod_{\substack{\ell=1\\ \ell\neq j}}^{4}|a_{i_\ell-k}|^\gamma. \end{aligned}$$

• $\exists! j \in \{1,2,3\}$ such that $i_j - k > h_0$

$$\begin{aligned} & E\left[\prod_{\ell=1}^{4} U_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \leq\ & E^{\frac{1}{2}}\left[U^2_{i_j,i_j-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\substack{\ell=1\\ \ell\neq j}}^{4} U^2_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \leq\ & C|x-y||a_{i_j-k}|^\gamma |x-y|^{\frac{\Delta}{2}} \\ \leq\ & C|x-y|^\Delta |a_{i_j-k}|^\gamma. \end{aligned}$$

• $\exists j, j' \in \{1,2,3,4\}$, $j \neq j'$, such that $i_j - k > h_0$, $i_{j'} - k > h_0$ and $i_\ell - k \leq h_0$ for $\ell \neq j, j'$

$$\begin{aligned} & E\left[\prod_{\ell=1}^{4} U_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \leq\ & E^{\frac{1}{2}}\left[U^2_{i_j,i_j-k}(x,y)U^2_{i_{j'},i_{j'}-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\substack{\ell=1\\ \ell\neq j,j'}}^{4} U^2_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \leq\ & C|x-y|^2|a_{i_j-k}|^\gamma |a_{i_{j'}-k}|^\gamma |x-y|^{\frac{\Delta}{2}} \\ \leq\ & C|x-y|^\Delta |a_{i_j-k}|^\gamma |a_{i_{j'}-k}|^\gamma. \end{aligned}$$

• $0 \leq i_j - k \leq h_0$ for $j = 1,2,3,4$

$$\begin{aligned} & E\left[\prod_{\ell=1}^{4} |U_{i_\ell,i_\ell-k}(x,y)||\mathcal{F}_{k-1}\right] \\ \leq\ & E^{\frac{1}{2}}\left[\prod_{\ell=1}^{2} U^2_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] E^{\frac{1}{2}}\left[\prod_{\ell=3}^{4} U^2_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \leq\ & \prod_{\ell=1}^{4} E^{\frac{1}{4}}\left[U^4_{i_\ell,i_\ell-k}(x,y)|\mathcal{F}_{k-1}\right] \\ \leq\ & C|x-y|^\Delta. \end{aligned}$$

Hence for $-h_0 < k \leq [nt]$

$$E[V_k^4(x,y)|\mathcal{F}_{k-1}] = \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]} E\left[\prod_{j=1}^{4} U_{i_j,i_j-k}(x,y)|\mathcal{F}_{k-1}\right]$$

$$
\begin{aligned}
&\leq C \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee(h_0+k+1)}^{[nt]} |x-y|^4 \prod_{j=1}^{4} |a_{i_j-k}|^\gamma \\
&+ C \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |x-y|^\Delta \\
&+ C \sum_{j=1}^{4} \sum_{i_j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \sum_{\substack{i_\ell,i_p,i_m=([ns]+1)\vee(h_0+k+1) \\ \ell,p,m\in\{1,2,3,4\}\setminus\{j\},\ell<p<m}}^{[nt]} |x-y|^\Delta \prod_{\substack{\ell=1 \\ \ell\neq j}}^{4} |a_{i_\ell-k}|^\gamma \\
&+ C \sum_{j=1}^{4} \sum_{i_j=([ns]+1)\vee(h_0+k+1)}^{[nt]} \sum_{\substack{i_\ell,i_p,i_m=([ns]+1)\vee k \\ \ell,p,m\in\{1,2,3,4\}\setminus\{j\},\ell<p<m}}^{[nt]\wedge(h_0+k)} |x-y|^\Delta |a_{i_j-k}|^\gamma \\
&+ C \sum_{\substack{j,\ell=1 \\ j<\ell}}^{4} \sum_{i_j,i_\ell=([ns]+1)\vee(h_0+k+1)}^{[nt]} \sum_{\substack{i_p,i_m=([ns]+1)\vee k \\ p,m\in\{1,2,3,4\}\setminus\{j,\ell\},p<m}}^{[nt]\wedge(h_0+k)} |x-y|^\Delta |a_{i_j-k}|^\gamma |a_{i_\ell-k}|^\gamma \\
&\leq C \sum_{i=[ns]+1}^{[nt]} |x-y|^4 |a_{i-k}|^\gamma + C \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} |x-y|^\Delta \\
&+ C \sum_{j=1}^{4} \sum_{i_j=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \sum_{\substack{i_\ell,i_p,i_m=[ns]+1 \\ \ell,p,m\in\{1,2,3,4\}\setminus\{j\},\ell<p<m}}^{[nt]} |x-y|^\Delta \prod_{\substack{\ell=1 \\ \ell\neq j}}^{4} |a_{i_\ell-k}|^\gamma \\
&+ C \sum_{j=1}^{4} \sum_{i_j=[ns]+1}^{[nt]} \sum_{\substack{i_\ell,i_p,i_m=([ns]+1)\vee k \\ \ell,p,m\in\{1,2,3,4\}\setminus\{j\},\ell<p<m}}^{[nt]\wedge(h_0+k)} |x-y|^\Delta |a_{i_j-k}|^\gamma \\
&+ C \sum_{\substack{j,\ell=1 \\ j<\ell}}^{4} \sum_{i_j,i_\ell=[ns]+1}^{[nt]} \sum_{\substack{i_p,i_m=([ns]+1)\vee k \\ p,m\in\{1,2,3,4\}\setminus\{j,\ell\},p<m}}^{[nt]\wedge(h_0+k)} |x-y|^\Delta |a_{i_j-k}|^\gamma |a_{i_\ell-k}|^\gamma. \qquad (1.2.39)
\end{aligned}
$$

We denote the upper bounds found above in (1.2.38) and (1.2.39) respectively by $J_k^{(1)}$ and $J_k^{(2)}$ and we define

$$
J_k = \begin{cases} J_k^{(1)} & \text{if } k \leq -h_0 \\ J_k^{(2)} & \text{otherwise.} \end{cases}
$$

Then, we have

$$\begin{aligned}
&\sum_{-h_0<k\le[nt]}\ \sum_{i_1=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}\ \sum_{i_2,i_3,i_4=[ns]+1}^{[nt]}\prod_{\ell=2}^{4}|a_{i_\ell-k}|^\gamma \\
\le\ & \sum_{-h_0<k\le[nt]}\ \sum_{i_1=k}^{h_0+k}\ \sum_{i_2,i_3,i_4=[ns]+1}^{[nt]}\prod_{\ell=2}^{4}|a_{i_\ell-k}|^\gamma \\
=\ & \sum_{-h_0<k\le[nt]}\ \sum_{i_1=k}^{h_0+k}\left(\sum_{j=[ns]+1}^{[nt]}|a_{j-k}|^\gamma\right)^3 \\
=\ & (h_0+1)\sum_{-h_0<k\le[nt]}\left(\sum_{j=[ns]+1}^{[nt]}|a_{j-k}|^\gamma\right)^3 \\
\le\ & C\sum_{j=[ns]+1}^{[nt]}\ \sum_{k\le[nt]}|a_{j-k}|^\gamma \\
\le\ & C([nt]-[ns]), \qquad (1.2.40)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{-h_0<k\le[nt]}\ \sum_{i_1=[ns]+1}^{[nt]}\ \sum_{i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}|a_{i_1-k}|^\gamma \\
\le\ & \sum_{-h_0<k\le[nt]}\ \sum_{i_1=[ns]+1}^{[nt]}\ \sum_{i_2,i_3,i_4=k}^{h_0+k}|a_{i_1-k}|^\gamma \\
\le\ & (h_0+1)^3\sum_{i_1=[ns]+1}^{[nt]}\ \sum_{k\le[nt]}|a_{i_1-k}|^\gamma \\
\le\ & C([nt]-[ns]), \qquad (1.2.41)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{-h_0<k\le[nt]}\ \sum_{i_1,i_2=[ns]+1}^{[nt]}\ \sum_{i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)}\prod_{\ell=1}^{2}|a_{i_\ell-k}|^\gamma \\
\le\ & \sum_{-h_0<k\le[nt]}\ \sum_{i_3,i_4=k}^{h_0+k}\left(\sum_{\ell=[ns]+1}^{[nt]}|a_{\ell-k}|^\gamma\right)^2
\end{aligned}$$

$$
\begin{aligned}
&\leq (h_0+1)^2 \sum_{k\leq [nt]} \left(\sum_{\ell=[ns]+1}^{[nt]} |a_{\ell-k}|^\gamma \right)^2 \\
&\leq C \sum_{\ell=[ns]+1}^{[nt]} \sum_{k\leq [nt]} |a_{\ell-k}|^\gamma \\
&\leq C([nt]-[ns]). \qquad (1.2.42)
\end{aligned}
$$

On the other hand, arguing as before, we have

$$
\begin{aligned}
\sum_{-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 &= \sum_{[ns]-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \\
&= \sum_{[ns]-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&+ \sum_{[ns]-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\geq [nt]-[ns]). \qquad (1.2.43)
\end{aligned}
$$

Remark that

$$
\begin{aligned}
&\sum_{[ns]-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&= \sum_{[ns]-h_0<k\leq [ns]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\leq [nt]-h_0} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0<[nt]-[ns]) \\
&\leq \sum_{[ns]-h_0<k\leq [ns]} \sum_{i_1,i_2,i_3=k}^{h_0+k} \sum_{i_4=[ns]+1}^{[nt]} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\leq [nt]-h_0} \sum_{i_1,i_2,i_3,i_4=k}^{h_0+k} \mathbf{I}(h_0<[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\leq [nt]} \sum_{i_1,i_2,i_3=k}^{h_0+k} \sum_{i_4=[ns]+1}^{[nt]} \mathbf{I}(h_0<[nt]-[ns])
\end{aligned}
$$

$$
\begin{aligned}
&= 2h_0(h_0+1)^3([nt]-[ns])\mathbf{I}(h_0<[nt]-[ns]) \\
&\quad +(h_0+1)^4([nt]-[ns]-h_0)\mathbf{I}(h_0<[nt]-[ns]) \\
&\leq C([nt]-[ns])\mathbf{I}(h_0<[nt]-[ns]). \qquad (1.2.44)
\end{aligned}
$$

Similarly, we get a bound for the second term of (1.2.43):

$$
\begin{aligned}
&\sum_{[ns]-h_0<k\leq[nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&= \sum_{[ns]-h_0<k\leq[nt]-h_0} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\leq[ns]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\leq[nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&\leq \sum_{[ns]-h_0<k\leq[nt]-h_0} \sum_{i_1,i_2,i_3,i_4=k}^{h_0+k} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&\quad + \sum_{[nt]-h_0<k\leq[ns]} \sum_{i_1,i_2,i_3=k}^{h_0+k} \sum_{i_4=[ns]+1}^{[nt]} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&\quad + \sum_{[ns]<k\leq[nt]} \sum_{i_1,i_2,i_3,i_4=k}^{h_0+k} \mathbf{I}(h_0\geq[nt]-[ns]) \\
&= 2(h_0+1)^4([nt]-[ns])\mathbf{I}(h_0\geq[nt]-[ns]) \\
&\quad +(h_0+[ns]-[nt])(h_0+1)^3([nt]-[ns])\mathbf{I}(h_0\geq[nt]-[ns]) \\
&\leq C([nt]-[ns])\mathbf{I}(h_0\geq[nt]-[ns]). \qquad (1.2.45)
\end{aligned}
$$

Therefore, we get by substituting (1.2.44) and (1.2.45) into (1.2.43)

$$
\sum_{-h_0<k\leq[nt]} \sum_{i_1,i_2,i_3,i_4=([ns]+1)\vee k}^{[nt]\wedge(h_0+k)} 1 \leq C([nt]-[ns]). \qquad (1.2.46)
$$

Combining now (1.2.39), (1.2.40), (1.2.41), (1.2.42) and (1.2.46) to get

$$\sum_{-h_0<k\leq[nt]} J_k^{(2)} \leq C \sum_{-h_0<k\leq[nt]} \sum_{i=[ns]+1}^{[nt]} |x-y|^4 |a_{i-k}|^\gamma + C([nt]-[ns])|x-y|^\Delta.$$

Moreover, we can get using (1.2.38)

$$\begin{aligned} E[I_4] &= \sum_{k\leq[nt]} E\left[E\left[V_k^4(x,y)|\mathcal{F}_{k-1}\right]\right] \\ &\leq \sum_{k\leq -h_0} J_k^{(1)} + \sum_{-h_0\leq k\leq[nt]} J_k^{(2)} \\ &\leq C\sum_{k\leq[nt]} \sum_{i=[ns]+1}^{[nt]} |x-y|^4|a_{i-k}|^\gamma + C([nt]-[ns])|x-y|^\Delta \\ &\leq C([nt]-[ns])\left[|x-y|^4+|x-y|^\Delta\right] \\ &\leq C([nt]-[ns])|x-y|^\Delta. \end{aligned} \tag{1.2.47}$$

Recall

$$W_n^4(A) = \frac{1}{n^2}(4I_1+6I_2+4I_3+I_4),$$

hence, from (1.2.13), (1.2.25), (1.2.37) and (1.2.47) we can see that

$$\begin{aligned} &E\left[W_n^4(A)\right] \\ &\leq \frac{C}{n^2}\left(([nt]-[ns])^2|x-y|^{2\Delta} + ([nt]-[ns])^{\frac{3}{2}}|x-y|^{\frac{3\Delta}{2}}\right) \\ &+ \frac{C}{n^2}\left(([nt]-[ns])|x-y|^\Delta\right) \\ &\leq C\left(\left(t-s+\frac{1}{n}\right)^2|x-y|^{2\Delta} + \frac{1}{\sqrt{n}}\left(t-s+\frac{1}{n}\right)^{\frac{3}{2}}|x-y|^{\frac{3\Delta}{2}}\right) \\ &+ \frac{C}{n}\left(t-s+\frac{1}{n}\right)|x-y|^\Delta \\ &\leq C\left(\left((t-s)^2+\frac{1}{n^2}\right)|x-y|^{2\Delta} + \frac{1}{\sqrt{n}}\left((t-s)^{\frac{3}{2}}+\frac{1}{n^{\frac{3}{2}}}\right)|x-y|^{\frac{3\Delta}{2}}\right) \\ &+ \frac{C}{n}\left(t-s+\frac{1}{n}\right)|x-y|^\Delta \\ &\leq C\left((t-s)^{\frac{3}{2}}|x-y|^{\frac{3\Delta}{2}} + \frac{1}{n}(t-s)|x-y|^\Delta + \frac{1}{n^2}|x-y|^\Delta\right) \end{aligned}$$

$$\leq \; C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}(|t-s||x-y|)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta} \right). \qquad (1.2.48)$$

This completes the proof of **(b1)**.

□

Proof of (b2) We extend the argument on [5], pg 198-199, to two dimensions. Recall that $|x-y| \leq 1$ and assume without loss of generality that $0 < \varepsilon < 1$.
If $\frac{\varepsilon}{n} \leq (|x-y||t-s|)^{\frac{3\Delta}{4}}$, then

$$\begin{aligned} E\left[W_n^4(A)\right] &\leq C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{(|t-s||x-y|)^{\frac{3\Delta}{2}}}{n(|t-s||x-y|)^{\frac{\Delta}{2}}} \right) \\ &+ \frac{C(|t-s||x-y|)^{\frac{3\Delta}{2}}}{n^2|t-s|^{\frac{3\Delta}{2}}|x-y|^{\frac{\Delta}{2}}} \\ &\leq \frac{C}{\varepsilon^2}(|t-s|\,|x-y|)^{\frac{3\Delta}{2}}, \end{aligned} \qquad (1.2.49)$$

and the so called condition (β, γ) in [4] is in force with $\beta = \frac{3\Delta}{2}$ and $\gamma = 4$. Let p be a number satisfying $\left(\frac{\varepsilon}{n}\right)^{\frac{2}{3\Delta}} \leq p \leq 1$, and remark that for $i, j = 1, \ldots, m$

$$|x+ip-x|^{\frac{3\Delta}{4}}|s+jp-s|^{\frac{3\Delta}{4}} \geq p^{\frac{3\Delta}{2}} \geq \frac{\varepsilon}{n}.$$

Now apply the method used in the proof of Theorem 1 of [4] to get

$$\begin{aligned} P\left\{ \max_{i,j\leq m} |W_n(x+ip, s+jp) - W_n(x,s)| \geq \lambda \right\} &\leq \frac{C}{\varepsilon^2\lambda^4}\left(m^2p^2\right)^{\frac{3\Delta}{2}} \\ &\leq \frac{C}{\varepsilon^2\lambda^4}(mp)^{3\Delta}. \end{aligned} \qquad (1.2.50)$$

We want now to prove that for $x \leq y \leq x+p$ and $s \leq t \leq s+p$

$$|W_n(y,t) - W_n(x,s)| \leq |W_n(x+p, s+p) - W_n(x,s)| + C\sqrt{n}p + \frac{1}{\sqrt{n}}. \qquad (1.2.51)$$

Let

$$U_n(x,s) = \sum_{i=1}^{[ns]} \mathbf{I}(X_i \leq x)$$

be the number among $X_1, \ldots, X_{[ns]}$ that satisfy $X_i \leq x$.

Then

$$\begin{aligned}
&U_n(y,t) - [nt]F(y) - U_n(x,s) + [ns]F(x) \\
&\quad\leq\; U_n(x+p,s+p) - [n(s+p)]F(x+p) - U_n(x,s) + [ns]F(x) \\
&\qquad +[n(s+p)]F(x+p) - [nt]F(y) \\
&\quad\leq\; |U_n(x+p,s+p) - [n(s+p)]F(x+p) - U_n(x,s) + [ns]F(x)| \\
&\qquad +[n(s+p)]F(x+p) - [ns]F(x),
\end{aligned}$$

and

$$\begin{aligned}
&U_n(x,s) - [ns]F(x) - U_n(y,t) + [nt]F(y) \\
&\quad\leq\; [nt]F(y) - [ns]F(x) \\
&\quad\leq\; |U_n(x+p,s+p) - [n(s+p)]F(x+p) - U_n(x,s) + [ns]F(x)| \\
&\qquad +[n(s+p)]F(x+p) - [ns]F(x).
\end{aligned}$$

Thus,

$$\begin{aligned}
&|U_n(y,t) - [nt]F(y) - U_n(x,s) + [ns]F(x)| \\
&\quad\leq\; |U_n(x+p,s+p) - [n(s+p)]F(x+p) - U_n(x,s) + [ns]F(x)| \\
&\qquad + \left([n(s+p)] - [ns]\right) F(x+p) + [ns]\left(F(x+p) - F(x)\right) \\
&\quad\leq\; |U_n(x+p,s+p) - [n(s+p)]F(x+p) - U_n(x,s) + [ns]F(x)| \\
&\qquad +np + 1 + n\left(F(x+p) - F(x)\right).
\end{aligned}$$

We complete the proof of (1.2.51) using the fact that F is Lipschitz. This will be proven later.

Since F is Lipschitz, then

$$\begin{aligned}
&|U_n(y,t) - [nt]F(y) - U_n(x,s) + [ns]F(x)| \\
&\quad\leq\; |U_n(x+p,s+p) - [n(s+p)]F(x+p) - U_n(x,s) + [ns]F(x)| \\
&\qquad +Cnp + 1,
\end{aligned}$$

and (1.2.51) follows immediately. Therefore,

$$\begin{aligned}
&\sup_{\substack{x\le y\le x+mp\\ s\le t\le s+mp}} |W_n(y,t)-W_n(x,s)| \\
&\le \max_{1\le i,j\le m} \sup_{\substack{x-(i-1)p\le y\le x+ip\\ s-(j-1)p\le t\le s+jp}} |W_n(y,t)-W_n(x,s)| \\
&\le \max_{1\le i,j\le m} \sup_{\substack{x-(i-1)p\le y\le x+ip\\ s-(j-1)p\le t\le s+jp}} [|W_n(y,t)-W_n\left(x+(i-1)p,s+(j-1)p\right)| \\
&\quad + |W_n(x+(i-1)p,s+(j-1)p)-W_n(x,s)|] \\
&\le \max_{1\le i,j\le m} [|W_n(x+ip,s+jp)-W_n(x+(i-1)p,s+(j-1)p)| \\
&\quad + |W_n(x+(i-1)p,s+(j-1)p)-W_n(x,s)|] + C\sqrt{n}p + \frac{1}{\sqrt{n}} \\
&\le \max_{1\le i,j\le m} [|W_n(x+ip,s+jp)-W_n(x,s)| \\
&\quad +2\,|W_n(x+(i-1)p,s+(j-1)p)-W_n(x,s)|] + C\sqrt{n}p + \frac{1}{\sqrt{n}} \\
&\le 3\max_{1\le i,j\le m} |W_n(x+ip,s+jp)-W_n(x,s)| + C\sqrt{n}p + \frac{1}{\sqrt{n}}.
\end{aligned}$$

We know that $\frac{1}{\sqrt{n}} < \varepsilon$ for all sufficiently large n. In addition, if

$$\left(\frac{\varepsilon}{n}\right)^{\frac{2}{3\Delta}} \le p < \frac{\varepsilon}{C\sqrt{n}}, \tag{1.2.52}$$

then (1.2.50) applies and we get

$$\begin{aligned}
&P\left\{\sup_{\substack{x\le y\le x+mp\\ s\le t\le s+mp}} |W_n(y,t)-W_n(x,s)| \ge 5\varepsilon\right\} \\
&\le P\left\{3\max_{1\le i,j\le m} |W_n(x+ip,s+jp)-W_n(x,t)| + Cp\sqrt{n} + \frac{1}{\sqrt{n}} \ge 5\varepsilon\right\} \\
&\le P\left\{\max_{1\le i,j\le m} |W_n(x+ip,s+jp)-W_n(x,s)| \ge \varepsilon\right\} \\
&\le \frac{C}{\varepsilon^6}(mp)^{3\Delta}.
\end{aligned} \tag{1.2.53}$$

Choose δ such that $\dfrac{C\delta^{3\Delta-2}}{\varepsilon^6} < \eta$, then it follows from (1.2.53) that

$$P\left\{\sup_{\substack{x\le y\le x+\delta\\ s\le t\le s+\delta}} |W_n(y,t)-W_n(x,s)| \ge 5\varepsilon\right\} < \eta\delta^2, \tag{1.2.54}$$

provided there exists a real p satisfying (1.2.52) and an integer m such that $\delta = mp$. This is equivalent to the existence of an integer m such that

$$C\delta\left(\frac{\sqrt{n}}{\varepsilon}\right) < m \le \delta\left(\frac{n}{\varepsilon}\right)^{\frac{2}{3\Delta}},$$

which is true for all sufficiently large n. This completes the proof of tightness and that of Theorem 1.2.1.

□

Finally, we prove that F is Lipschitz.

Lemma 1.2.4 *F, the distribution function of X_0, is Lipschitz continuous.*

Proof For $h > h_0$, put

$$X_0 = X_0^h + \widetilde{X}_0^h \quad \text{where} \quad X_0^h = \sum_{i=0}^{h} a_i\xi_{-i} \quad \text{and} \quad \widetilde{X}_0^h = \sum_{i>h} a_i\xi_{-i}.$$

Also, denote by F_h and $\widetilde{F}_h$ the distribution functions of X_0^h and $\widetilde{X}_0^h$ respectively. It was pointed out in Comments 1.1.2 that the density function f_h of X_0^h is bounded. Hence

$$\begin{aligned}
|F(x)-F(y)| &= |P(X_0\le x) - P(X_0 \le y)| \\
&= \left|P(X_0^h \le x-\widetilde{X}_0^h) - P(X_0^h \le y - \widetilde{X}_0^h)\right| \\
&= \left|\int_{\mathbb{R}} (F_h(x-u)-F_h(y-u))\, d(\widetilde{F}_h(u))\right| \\
&= \left|\int_{\mathbb{R}}\int_y^x f_h(v-u)dvd(\widetilde{F}_h(u))\right| \\
&\le \int_{\mathbb{R}}\int_y^x |f_h(v-u)|dvd(\widetilde{F}_h(u))
\end{aligned}$$

$$\leq \quad C|x-y|.$$

This completes the proof of Lemma 1.2.4.

■

Chapter 2

Asymptotics and Testing for the Sequential Empirical Process with a Change-Point

In this chapter, we establish a functional central limit theorem for the sequential empirical process, including a change point, of a causal linear process. The change-point model and the limit theorem are given in Section 2.1. In Section 2.2, the result will be applied to detecting a change point in the marginal distribution of a linear process and we will demonstrate that the proposed test statistics are consistent.

2.1 FCLT for the sequential empirical process with a change-point

Let $\{\theta_n, n \in \mathbb{N}\}$ be a sequence of real numbers satisfying:

$$0 \leq \theta_n \leq 1 \quad \text{and} \quad \lim_{n\to\infty} \theta_n = \theta.$$

In this section, we will consider a causal linear process with a change-point at $[n\theta_n]$. More precisely, consider the following stationary processes for $i \in \mathbb{Z}$ and $b \in \mathbb{R}$

$$Y_i = \sum_{j\geq 0} a_j^{(1)} \xi_{i-j} \quad \text{and} \quad Z_i = \sum_{j\geq 0} a_j^{(2)} \left(\xi_{i-j} + b\right),$$

and denote by F and G the respective distribution functions of Y_0 and Z_0.
Define

$$X_i = \begin{cases} Y_i & \text{for} \quad i \leq [n\theta_n] \\ Z_i & \text{otherwise.} \end{cases}$$

The model considered here is quite general and readily includes the case of a fixed change-point ($\theta_n = \theta$) and that of converging alternatives $\theta_n \to 0$ or 1 as $n \to \infty$. It also covers different types of change in the marginal distribution such as a change in the parameters, a change in the scale and (or) a change in the location of the innovations.

Similarly to the methodology followed in the previous chapter, we define for $h \geq 0$ the martingale differences

$$\begin{aligned} U_{i,h}^F(x) &:= P(Y_i \leq x|\mathcal{F}_{i-h}) - P(Y_i \leq x|\mathcal{F}_{i-h-1}), \\ U_{i,h}^G(x) &:= P(Z_i \leq x|\mathcal{F}_{i-h}) - P(Z_i \leq x|\mathcal{F}_{i-h-1}), \end{aligned}$$

and

$$\begin{aligned} R_i^F(x) &:= \mathbf{I}(Y_i \leq x) - F(x), \\ R_i^G(x) &:= \mathbf{I}(Z_i \leq x) - G(x). \end{aligned}$$

Define, for $0 \leq s \leq 1$

$$H^{(n)}(x,s) := (s \wedge \theta_n)F(x) + (s - \theta_n)^+ G(x),$$

where $s^+ = \max(0, s)$ and let for $(x,s) \in \mathbb{R} \times [0,1]$

$$W_n(x,s) \;\; := \;\; \frac{1}{\sqrt{n}} \left([ns]F_{[ns]}(x) - nH^{(n)}(x,s)\right)$$

$$= \frac{1}{\sqrt{n}} \left[\sum_{i=1}^{[ns]} \mathbf{I}(X_i \leq x) - nH^{(n)}(x,s) \right],$$

where

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{I}(X_i \leq x).$$

Assumptions 2.1.1

1. Let $\{a_j^{(\ell)}, j \in \mathbb{Z}\}$ be sequences of non-random weights, infinitely many of which are non-zero, satisfying

$$\sum_{j \geq 0} |a_j^{(\ell)}|^\gamma < \infty, \ \ell = 1, 2 \ \text{ for some } \ \gamma \in (0, 1].$$

2. There exist constants $C < \infty$ and $\Delta \in \left(\frac{2}{3}, 1 \right]$ such that for all $u \in \mathbb{R}$

$$|E \exp(iu\xi_0)| \leq \frac{C}{(1 + |u|)^\Delta}.$$

3. $E[|\xi_0|^{4\gamma}] < \infty$.

Theorem 2.1.2 *Assume Assumptions 2.1.1.1-2.1.1.3 hold. Then, as* $n \to \infty$*,*

$$W_n(\cdot,\cdot) \xrightarrow{\mathbf{D}} W^{(2)}(\cdot,\cdot),$$

where $W^{(2)}(\cdot,\cdot)$ *is a centred Gaussian process with covariance*

$$\begin{aligned} \sigma((x,s),(y,t)) &:= Cov\left(W^{(2)}(x,s), W^{(2)}(y,t)\right) \\ &= (s \wedge t \wedge \theta) \sum_{i \in \mathbf{Z}} Cov\left(R_0^F(x), R_i^F(y)\right) \\ &+ ((s \wedge t) - \theta)^+ \sum_{i \in \mathbf{Z}} Cov\left(R_0^G(x), R_i^G(y)\right). \end{aligned}$$

Proof

Define for $N < \infty$

$$\begin{aligned} R_i^{F,N}(x) &:= \sum_{h=0}^{N} U_{i,h}^F(x) = \mathbf{I}(Y_i \le x) - P(Y_i \le x|\mathcal{F}_{i-N-1}) \\ &\overset{a.s}{\to} R_i^F(x) = \sum_{h\ge 0} U_{i,h}^F(x), \end{aligned}$$

and

$$\begin{aligned} R_i^{G,N}(x) &:= \sum_{h=0}^{N} U_{i,h}^G(x) = \mathbf{I}(Z_i \le x) - P(Z_i \le x|\mathcal{F}_{i-N-1}) \\ &\overset{a.s}{\to} R_i^G(x) = \sum_{h\ge 0} U_{i,h}^G(x). \end{aligned}$$

To simplify the notation, we also define for $T = F, G$ and $N < \infty$

$$\sigma_{T,N}(x,y) := \sum_{j\in\mathbb{Z}} Cov(R_0^{T,N}(x), R_j^{T,N}(y)) \text{ and } \sigma_{T,N}^2(x) := \sigma_{T,N}(x,x),$$

$$\sigma_T(x,y) := \sum_{j\in\mathbb{Z}} Cov(R_0^T(x), R_j^T(y)) \text{ and } \sigma_T^2(x) := \sigma_T(x,x).$$

We first remark that for $0 \le s \le 1$

$$\begin{aligned} W_n(x,s) &= W_n(x,s)\mathbf{I}(s \le \theta_n) + W_n(x,s)\mathbf{I}(s > \theta_n) \\ &= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]} \mathbf{I}(X_i \le x) - nsF(x)\right]\mathbf{I}(s \le \theta_n) \\ &+ \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]} \mathbf{I}(X_i \le x) - n\left(\theta_n F(x) + (s-\theta_n)G(x)\right)\right]\mathbf{I}(s > \theta_n) \\ &= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]} R_i^F(x) + ([ns]-ns)F(x)\right]\mathbf{I}(s \le \theta_n) \\ &+ \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[n\theta_n]} R_i^F(x) + \sum_{i=[n\theta_n]+1}^{[ns]} R_i^G(x)\right]\mathbf{I}(s > \theta_n) \\ &+ \frac{1}{\sqrt{n}}\left[\left([n\theta_n] - n\theta_n\right)F(x) + \left([ns] - ns - [n\theta_n] + n\theta_n\right)G(x)\right]\mathbf{I}(s > \theta_n). \end{aligned}$$

Define

$$\begin{aligned}
W_{n1}(x,s) &:= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]} R_i^F(x)\right]\mathbf{I}(s\le\theta_n)\\
W_{n2}(x,s) &:= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[n\theta_n]} R_i^F(x)+\sum_{i=[n\theta_n]+1}^{[ns]} R_i^G(x)\right]\mathbf{I}(s>\theta_n)\\
W_{n3}(x,s) &:= W_{n1}(x,s)+W_{n2}(x,s)\\
\varphi_{n1}(x,s) &:= \frac{[ns]-ns}{\sqrt{n}}F(x)\mathbf{I}(s\le\theta_n)\\
\varphi_{n2}(x,s) &:= \left[\frac{[n\theta_n]-n\theta_n}{\sqrt{n}}(F(x)-G(x))+\frac{[ns]-ns}{\sqrt{n}}G(x)\right]\mathbf{I}(s>\theta_n).
\end{aligned}$$

Therefore,

$$W_n(x,s)=W_{n3}(x,s)+\varphi_{n1}(x,s)+\varphi_{n2}(x,s).$$

Since

$$\sup_{\substack{x\in\mathbb{R}\\ s\in[0,1]}}(\varphi_{n1}(x,s)+\varphi_{n2}(x,s))\to 0,$$

it is sufficient to prove that

$$W_{n3}(\cdot,\cdot)\xrightarrow{\mathbf{D}}W^{(2)}(\cdot,\cdot).$$

We will proceed as follows:

(a) Convergence of finite dimensional distributions.

(b) Tightness of the sequence $W_{n3}(\cdot,\cdot)$.

Proof of (a) As before, we illustrate the proof of convergence of finite dimensional distributions with two points, (x,s) and (y,t). According to the expression of $W_n(\cdot,\cdot)$ and its covariance structure, the proof of the theorem will be demonstrated depending on the position of t and s with respect to θ. Three cases are to be discussed here: $0\le s,t<\theta$, $s<\theta<t$ and $\theta<s,t\le 1$. We will denote the limiting distribution of $W_{n3}(x,s)$, in the three cases, by the centred normal variable $\mathcal{N}_{x,s}$. The case $s=\theta$ or

$t = \theta$ will be discussed later.

i) $\underline{0 \leq s, t < \theta}$

Remark first the existence of ℓ in $\mathbb{N}$ such that $0 \leq s, t \leq \theta_n$ for $n \geq \ell$. In this case, we have

$$W_{n3}(x,s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[ns]} R_i^F(x) \quad \text{and} \quad W_{n3}(y,t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} R_i^F(y).$$

From Theorem 1.2.1, the finite dimensional convergence holds for this case and

$$\begin{aligned} Cov\left(\mathcal{N}_{x,s}, \mathcal{N}_{y,t}\right) &= (s \wedge t) \sum_{i \in \mathbf{Z}} Cov\left(R_0^F(x), R_i^F(y)\right) \\ &= (s \wedge t \wedge \theta)\sigma_F(x,y) + (s \wedge t - \theta)^+ \sigma_G(x,y). \end{aligned}$$

□

ii) $\underline{\theta < s, t \leq 1}$

Note again that there exists ℓ in $\mathbb{N}$ such that $\theta_n < s, t \leq 1$ for $n \geq \ell$. In this case, we have

$$W_{n3}(x,s) = \frac{1}{\sqrt{n}} \left[\sum_{i=1}^{[n\theta_n]} R_i^F(x) + \sum_{i=[n\theta_n]+1}^{[ns]} R_i^G(x) \right].$$

and

$$W_{n3}(y,t) = \frac{1}{\sqrt{n}} \left[\sum_{i=1}^{[n\theta_n]} R_i^F(y) + \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y) \right].$$

Following a similar procedure as in the proof of Theorem 1.2.1, we write for $T = F, G$

$$R_i^T(x) = R_i^{T,N}(x) + \widetilde{R}_i^{T,N}(x),$$

where $R_i^{T,N}(x)$ has been defined previously and $\widetilde{R}_i^{T,N}(x) = \sum_{h>N} U_{i,h}^T(x)$.

Write now

$$W_{n3}(x,s) = W_{n3}^N(x,s) + \widetilde{W}_{n3}^N(x,s),$$

where

$$W_{n3}^N(x,s) = \frac{1}{\sqrt{n}} \sum_{i=1}^{[n\theta_n]} R_i^{F,N}(x) + \frac{1}{\sqrt{n}} \sum_{i=[n\theta_n]+1}^{[ns]} R_i^{G,N}(x),$$

$$\widetilde{W}_{n3}^{N}(x,s) = \frac{1}{\sqrt{n}}\sum_{i=1}^{[n\theta_n]}\widetilde{R}_i^{F,N}(x) + \frac{1}{\sqrt{n}}\sum_{i=[n\theta_n]+1}^{[ns]}\widetilde{R}_i^{G,N}(x).$$

Arguing as in the proof of Theorem 1.2.1, write

$$\begin{aligned}\sum_{i=1}^{[n\theta_n]} R_i^F(x) &= \sum_{i=1}^{[n\theta_n]} R_i^{F,N}(x) + \sum_{i=1}^{[n\theta_n]}\widetilde{R}_i^{F,N}(x)\\ &= \sum_{i=1}^{[n\theta_n]} M_i^{F,N}(x) + Q_n^{F,N}(x) + \sum_{i=1}^{[n\theta_n]}\widetilde{R}_i^{F,N}(x),\end{aligned}$$

where

$$\begin{aligned}M_i^{F,N}(x) &= \sum_{h=0}^{N} U_{i+h,h}^F(x),\\ Q_n^{F,N}(x) &= \sum_{h=0}^{N}\sum_{i=1}^{h} U_{i,h}^F(x) - \sum_{h=0}^{N}\sum_{i=[n\theta_n]+1}^{[n\theta_n]+h} U_{i,h}^F(x).\end{aligned}$$

Similarly, we have

$$\begin{aligned}\sum_{i=[n\theta_n]+1}^{[ns]} R_i^G(x) &= \sum_{i=[n\theta_n]+1}^{[ns]} R_i^{G,N}(x) + \sum_{i=[n\theta_n]+1}^{[ns]}\widetilde{R}_i^{G,N}(x)\\ &= \sum_{i=[n\theta_n]+1}^{[ns]} M_i^{G,N}(x) + Q_n^{G,N}(x) + \sum_{i=[n\theta_n]+1}^{[ns]}\widetilde{R}_i^{G,N}(x),\end{aligned}$$

where

$$\begin{aligned}M_i^{G,N}(x) &= \sum_{h=0}^{N} U_{i+h,h}^G(x),\\ Q_n^{G,N}(x) &= \sum_{h=0}^{N}\sum_{i=[n\theta_n]+1}^{[n\theta_n]+h} U_{i,h}^G(x) - \sum_{h=0}^{N}\sum_{i=[ns]+1}^{[ns]+h} U_{i,h}^G(x).\end{aligned}$$

Analogously to the method used in the proof of Theorem 1.2.1, we will show

(aii1) $W_{n3}^N(x,s) \xrightarrow{d} \mathcal{N}(0, \theta\sigma_{F,N}^2(x) + (s-\theta)\sigma_{G,N}^2(x))$, as $n \to \infty$.

(aii2) $\theta\sigma_{F,N}^2(x) + (s-\theta)\sigma_{G,N}^2(x)) \to \theta\sigma_F^2(x) + (s-\theta)\sigma_G^2(x)$, as $N \to \infty$.

(aii3) $Var\left(\widetilde{W}_{n3}^{N}(x,s)\right) \le \delta(N)$, where $\delta(N) \to 0$ as $N \to \infty$.

Proof of (aii1) Consider the sequence $K_i^N(x)$, defined by

$$K_i^N(x) = \mathbf{I}(i \le [n\theta_n])M_i^{F,N}(x) + \mathbf{I}(i > [n\theta_n])M_i^{G,N}(x).$$

We will now proceed with the proof of the central limit theorem for the sequence $\frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]} K_i^N(x)$ where N is fixed. For this purpose, we are going to employ again the results stated in equations (1.2.5), (1.2.6) and (1.2.7).

1. $(K_i^N(x), \mathcal{F}_i)$ is a martingale difference sequence.

2. For fixed N

$$\begin{aligned} \max_{1\le i\le [ns]} \left| \frac{1}{\sqrt{n}} K_i^N(x) \right| &\le \max_{1\le i\le n} \left| \frac{1}{\sqrt{n}} M_i^{F,N}(x) \right| + \max_{1\le i\le n} \left| \frac{1}{\sqrt{n}} M_i^{G,N}(x) \right| \\ &\le \frac{C}{\sqrt{n}}. \end{aligned}$$

3.

$$\begin{aligned} \frac{1}{n}\sum_{i=1}^{[ns]} (K_i^N(x))^2 &= \frac{1}{n}\sum_{i=1}^{[n\theta_n]} \left(M_i^{F,N}(x)\right)^2 + \frac{1}{n}\sum_{i=[n\theta_n]+1}^{[ns]} \left(M_i^{G,N}(x)^2\right) \\ &\stackrel{a.s}{\to} \theta\sigma_{F,N}^2(x) + (s-\theta)\sigma_{G,N}^2(x), \quad \text{as} \quad n \to \infty. \end{aligned}$$

Theorem 2.3 of Mcleish [42] leads to the desired result:

$$\begin{aligned} \frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]} K_i^N(x) &= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[n\theta_n]} M_i^{F,N}(x) + \sum_{i=[n\theta_n]+1}^{[ns]} M_i^{G,N}(x)\right] \\ &\stackrel{d}{\to} \mathcal{N}(0, \theta\sigma_{F,N}^2(x) + (s-\theta)\sigma_{G,N}^2(x)), \quad \text{as} \quad n \to \infty. \end{aligned}$$

Now **(aii1)** follows since

$$\frac{1}{\sqrt{n}}(Q_n^{F,N}(x) + Q_n^{G,N}(x)) \stackrel{p}{\to} 0$$

by an argument similar to that used to prove equation (1.2.10).

□

Proof of (aii2) From equation (1.2.7), we get as $N \to \infty$ for $T = F, G$

$$\sigma^2_{T,N}(x) = \sum_{j\in\mathbb{Z}} Cov(R_0^{T,N}(x), R_j^{T,N}(x)) \to \sigma^2_T(x) = \sum_{j\in\mathbb{Z}} Cov(R_0^T(x), R_j^T(x)).$$

Hence,

$$\theta\sigma^2_{F,N}(x) + (s-\theta)\sigma^2_{G,N}(x)) \to \theta\sigma^2_F(x) + (s-\theta)\sigma^2_G(x).$$

This completes the proof of **(aii2)**.

□

Proof of (aii3) Recall

$$\widetilde{W}^N_{n3}(x) = \frac{1}{\sqrt{n}}\sum_{i=1}^{[n\theta_n]} \widetilde{R}_i^{F,N}(x) + \frac{1}{\sqrt{n}}\sum_{i=[n\theta_n]+1}^{[ns]} \widetilde{R}_i^{G,N}(x).$$

We know from [15] that

$$Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[n\theta_n]} \widetilde{R}_i^{F,N}(x)\right) \le \delta_1(N), \quad \text{where} \quad \delta_1(N) \to 0 \quad \text{as} \quad N \to \infty, \tag{2.1.1}$$

$$Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]-[n\theta_n]} \widetilde{R}_i^{G,N}(x)\right) \le \delta_2(N), \text{ where } \quad \delta_2(N) \to 0 \quad \text{as} \quad N \to \infty.$$

Using stationarity, we get

$$\begin{aligned} Var\left(\frac{1}{\sqrt{n}}\sum_{i=[n\theta_n]+1}^{[ns]} \widetilde{R}_i^{G,N}(x)\right) &= Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]-[n\theta_n]} \widetilde{R}_i^{G,N}(x)\right) \\ &\le \delta_2(N). \end{aligned} \tag{2.1.2}$$

Combining equations (2.1.1) and (2.1.2) completes the proof of **(aii3)**. Furthermore, we can conclude using Theorem 3.2 from [5] that

$$W_{n3}(x,s) \xrightarrow{d} N(0, \theta\sigma^2_F(x) + (s-\theta)\sigma^2_G(x)), \quad \text{as } n \to \infty.$$

The convergence of finite dimensional distributions follows in a similar approach using the Cramér-Wold technique as in the proof of Theorem 1.2.1 and provides the required result:

$$\begin{aligned} & Cov\left(\mathcal{N}_{x,s},\mathcal{N}_{y,t}\right) \\ &= \theta \sum_{i\in\mathbf{Z}} Cov\left(R_0^F(x),R_i^F(y)\right) + (s-\theta)\sum_{i\in\mathbf{Z}} Cov\left(R_0^G(x),R_i^G(y)\right) \\ &= (s\wedge t\wedge\theta)\sigma_F(x,y) + (s\wedge t-\theta)^+\sigma_G(x,y). \end{aligned}$$

□

iii) $s < \theta < t$

There exists ℓ in $\mathbb{N}$ such that $s \leq \theta_n \leq t$ for $n \geq \ell$. In this case, we present an illustration of the finite dimensional convergence using the Cramér Wold device for two points (x,s) and (y,t).

Let $\alpha,\beta \in \mathbb{R}$ and define

$$\begin{aligned} & \alpha W_{n3}(x,s) + \beta W_{n3}(y,t) \\ &= \frac{\alpha}{\sqrt{n}}\sum_{i=1}^{[ns]} R_i^F(x) + \frac{\beta}{\sqrt{n}}\left[\sum_{i=1}^{[n\theta_n]} R_i^F(y) + \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y)\right]. \end{aligned}$$

We want to prove that

$$\alpha W_{n3}(x,s) + \beta W_{n3}(y,t) \xrightarrow{d} \mathcal{N}\left(0,\sigma_*^2\right),$$

where

$$\sigma_*^2 = \alpha^2 s\sigma_F^2(x) + 2\alpha\beta s\sigma_F(x,y) + \beta^2(\theta\sigma_F^2(y) + (t-\theta)\sigma_G^2(y)).$$

Write now for fixed N

$$\alpha W_{n3}(x,s) + \beta W_{n3}(y,t) = \alpha W_{n3}^N(x,s) + \beta W_{n3}^N(y,t) + \alpha\widetilde{W}_{n3}(x,s) + \beta\widetilde{W}_{n3}(y,t),$$

where

$$\alpha W_{n3}^N(x,s) + \beta W_{n3}^N(y,t) \;=\; \frac{\alpha}{\sqrt{n}}\sum_{i=1}^{[ns]} R_i^{F,N}(x) + \frac{\beta}{\sqrt{n}}\sum_{i=1}^{[n\theta_n]} R_i^{F,N}(y)$$

$$+ \quad \frac{\beta}{\sqrt{n}} \sum_{i=[n\theta_n]+1}^{[nt]} R_i^{G,N}(y),$$

and

$$\begin{aligned} \alpha\widetilde{W}_{n3}(x,s) + \beta\widetilde{W}_{n3}(y,t) &= \frac{\alpha}{\sqrt{n}} \sum_{i=1}^{[ns]} \widetilde{R}_i^{F,N}(x) + \frac{\beta}{\sqrt{n}} \sum_{i=1}^{[n\theta_n]} \widetilde{R}_i^{F,N}(y) \\ &+ \frac{\beta}{\sqrt{n}} \sum_{i=[n\theta_n]+1}^{[nt]} \widetilde{R}_i^{G,N}(y). \end{aligned}$$

We will now make use of the martingale techniques utilized in the previous proofs; we will show that

(aiii1) $\alpha W_{n3}^N(x,s) + \beta W_{n3}^N(y,t) \xrightarrow{d} \mathcal{N}(0,\sigma_N^2)$, as $n \to \infty$, where

$$\sigma_N^2 = \alpha^2 s\sigma_{F,N}^2(x) + 2\alpha\beta s\sigma_{F,N}(x,y) + \beta^2(\theta\sigma_{F,N}^2(y) + (t-\theta)\sigma_{G,N}^2(y)).$$

(aiii2) $\sigma_N^2 \to \sigma_*^2$, as $N \to \infty$, where

$$\sigma_*^2 = \alpha^2 s\sigma_F^2(x) + 2\alpha\beta s\sigma_F(x,y) + \beta^2(\theta\sigma_F^2(y) + (t-\theta)\sigma_G^2(y)).$$

(aiii3) $Var\left(\alpha\widetilde{W}_{n3}(x,s) + \beta\widetilde{W}_{n3}(y,t)\right) \leq \delta(N)$, where $\delta(N) \to 0$ as $N \to \infty$.

Hence, Theorem 3.2 from [5] implies that

$$\alpha W_{n3}(x,s) + \beta W_{n3}(y,t) \xrightarrow{d} \mathcal{N}(0,\sigma_*^2),$$

as $n \to \infty$. In this case, we get

$$\begin{aligned} Cov\left(\mathcal{N}_{x,s}, \mathcal{N}_{y,t}\right) &= s \sum_{i\in\mathbf{Z}} Cov\left(R_0^F(x), R_i^F(y)\right) \\ &= (s \wedge t \wedge \theta)\sigma_F(x,y) + (s \wedge t - \theta)^+ \sigma_G(x,y). \end{aligned}$$

Proof of (aiii1) As in the proof of the previous case, write

$$\sum_{i=1}^{[ns]} R_i^{F,N}(x) = \sum_{i=1}^{[ns]} M_i^{F,N}(x) + Q_{n,s}^{F,N}(x),$$

where

$$M_i^{F,N}(x) = \sum_{h=0}^{N} U_{i+h,h}^F(x),$$

and

$$Q_{n,s}^{F,N}(x) = \sum_{h=0}^{N} \sum_{i=1}^{h} U_{i,h}^F(x) - \sum_{h=0}^{N} \sum_{i=[ns]+1}^{[ns]+h} U_{i,h}^F(x).$$

We also know that

$$\sum_{i=1}^{[n\theta_n]} R_i^{F,N}(y) = \sum_{i=1}^{[n\theta_n]} M_i^{F,N}(y) + Q_{n,\theta_n}^{F,N}(y).$$

Similarly, we have

$$\sum_{i=[n\theta_n]+1}^{[nt]} R_i^{G,N}(y) = \sum_{i=[n\theta_n]+1}^{[nt]} M_i^{G,N}(y) + Q_{n,t}^{G,N}(y),$$

where

$$M_i^{G,N}(y) = \sum_{h=0}^{N} U_{i+h,h}^G(y),$$

and

$$Q_{n,t}^{G,N}(y) = \sum_{h=0}^{N} \sum_{i=[n\theta_n]+1}^{[n\theta_n]+h} U_{i,h}^G(y) - \sum_{h=0}^{N} \sum_{i=[nt]+1}^{[nt]+h} U_{i,h}^G(y).$$

Therefore,

$$\begin{aligned}
&\alpha W_{n3}^N(x,s) + \beta W_{n3}^N(y,t) \\
&= \frac{1}{\sqrt{n}} \left[\alpha \sum_{i=1}^{[ns]} M_i^{F,N}(x) + \beta \sum_{i=1}^{[n\theta_n]} M_i^{F,N}(y) + \beta \sum_{i=[n\theta_n]+1}^{[nt]} M_i^{G,N}(y) \right] \\
&+ \frac{1}{\sqrt{n}} \left[\alpha Q_{n,s}^{F,N}(x) + \beta Q_{n,\theta_n}^{F,N}(y) + \beta Q_{n,t}^{G,N}(y) \right].
\end{aligned}$$

Define the sequence

$$\begin{aligned}
K_i^N(x,y) &= \mathbf{I}(i \le [ns])(\alpha M_i^{F,N}(x) + \beta M_i^{F,N}(y)) \\
&+ \mathbf{I}([ns] < i \le [n\theta_n])\beta M_i^{F,N}(y) + \mathbf{I}(i > [n\theta_n])\beta M_i^{G,N}(y).
\end{aligned}$$

Next, we apply a martingale central limit theorem to $\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]} K_i^N(x,y)$ for N fixed. To that end, equations (1.2.5), (1.2.6) and (1.2.7) will be of use again.

1. $(K_i^N(x,y), \mathcal{F}_i)$ is a martingale difference sequence.

2. For fixed N

$$\begin{aligned}
\max_{1\le i\le [nt]}\left|\frac{1}{\sqrt{n}}K_i^N(x)\right| &\le \max_{1\le i\le n}\left|\frac{\alpha}{\sqrt{n}}M_i^{F,N}(x)\right| + \max_{1\le i\le n}\left|\frac{\beta}{\sqrt{n}}M_i^{F,N}(y)\right| \\
&+ \max_{1\le i\le n}\left|\frac{\beta}{\sqrt{n}}M_i^{G,N}(y)\right| \\
&\le \frac{C}{\sqrt{n}}.
\end{aligned}$$

3.

$$\begin{aligned}
\frac{1}{n}\sum_{i=1}^{[nt]}(K_i^N(x,y))^2 &= \frac{1}{n}\sum_{i=1}^{[ns]}\left(\alpha M_i^{F,N}(x) + \beta M_i^{F,N}(y)\right)^2 \\
&+ \frac{1}{n}\sum_{i=[ns]+1}^{[n\theta_n]}(\beta M_i^{F,N}(y))^2 + \frac{1}{n}\sum_{i=[n\theta_n]+1}^{[nt]}(\beta M_i^{G,N}(y))^2 \\
&\stackrel{a.s}{\to} s\left(\alpha^2\sigma_{F,N}^2(x) + 2\alpha\beta\sigma_{F,N}(x,y) + \beta^2\sigma_{F,N}^2(y)\right) \\
&+ (\theta - s)\beta^2\sigma_{F,N}^2(y) + (t-\theta)\beta^2\sigma_{G,N}^2(y) \\
&= \sigma_N^2,
\end{aligned}$$

where the third and the fourth lines follow from the ergodic theorem and equation (1.2.9).

Apply Theorem 2.3 of Mcleish [42] to get

$$\frac{1}{\sqrt{n}}\left[\alpha\sum_{i=1}^{[ns]} M_i^{F,N}(x) + \beta\left(\sum_{i=1}^{[n\theta_n]} M_i^{F,N}(y) + \sum_{[n\theta_n]+1}^{[nt]} M_i^{G,N}(y)\right)\right] \stackrel{d}{\to} \mathcal{N}(0,\sigma_N^2).$$

Using an argument similar to that used to prove equation (1.2.10), we can see that

$$\frac{1}{\sqrt{n}}\left[\alpha Q_{n,s}^{F,N}(x) + \beta\left(Q_{n,\theta_n}^{F,N}(y) + Q_{n,t}^{G,N}(y)\right)\right] \stackrel{p}{\to} 0.$$

This completes the proof of **(aiii1)**.

□

Proof of (aiii2) From equations (1.2.7) and (1.2.11) respectively, we get as $N \to \infty$ for $T = F, G$ that $\sigma^2_{T,N}(x) \to \sigma^2_T(x)$ and $\sigma_{T,N}(x,y) \to \sigma_T(x,y)$.

Hence,

$$\sigma_N^2 \to \sigma_*^2 \quad \text{as} \quad N \to \infty,$$

which completes the proof of **(aiii2)**.

□

Proof of (aiii3) We know from [15] that

$$Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]}\widetilde{R}_i^{F,N}(x)\right) \leq \delta_1(N), \quad \text{where} \;\; \delta_1(N) \to 0 \;\; \text{as} \;\; N \to \infty, \tag{2.1.3}$$

and

$$Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[n\theta_n]}\widetilde{R}_i^{F,N}(y)\right) \leq \delta_2(N), \quad \text{where} \;\; \delta_2(N) \to 0 \;\; \text{as} \;\; N \to \infty, \tag{2.1.4}$$

and

$$Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]-[n\theta_n]}\widetilde{R}_i^{G,N}(x)\right) \leq \delta_3(N), \;\text{where} \quad \delta_3(N) \to 0 \quad \text{as} \;\; N \to \infty.$$

Using stationarity, we get

$$\begin{aligned} Var\left(\frac{1}{\sqrt{n}}\sum_{i=[n\theta_n]+1}^{[nt]}\widetilde{R}_i^{G,N}(y)\right) &= Var\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{[nt]-[n\theta_n]}\widetilde{R}_i^{G,N}(y)\right) \\ &\leq \delta_3(N). \end{aligned} \tag{2.1.5}$$

Combining equations (2.1.3), (2.1.4) and (2.1.5) completes the proof of **(aiii3)** and that of the finite dimensional convergence for this case.

□

Comment 2.1.3 As mentioned in the beginning of the proof of **(a)**, the case $s = \theta$ or $t = \theta$ needs to be given more consideration. In fact, say, $s = \theta$, then limits can be taken over the sets $\{n,\, \theta_n \leq \theta\}$ and $\{n,\, \theta_n > \theta\}$. In the first case, we proceed as for $s > \theta$ while we use the analysis for $s < \theta$ in the second case. In both cases, we get the same limiting distribution; this completes the proof of **(a)**.

Proof of (b) In order to get the functional limit, we address the problem of tightness of the sequence $\{W_{n3}(\cdot,\cdot), n \geq 1\}$. The demonstration will be based on the techniques in the proof of Theorem 1.2.1. For this purpose, consider a block of the form $A = (x, y] \times (s, t]$ in $T = \mathbb{R} \times [0, 1]$, where $|x - y| \leq 1$ and $|t - s| \leq 1$. We will proceed, when it is needed, by bounding the fourth moment of the increment of W_{n3} around A for the three cases according to s, t and θ. In fact, we will prove that:

(b1)

$$\begin{aligned} & E\left[W_{n3}^4(A)\right] \\ \leq\ & C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}\left(|t-s||x-y|\right)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta} \right). \end{aligned}$$

(b2) There exists $0 < \delta < 1$ such that

$$P\left\{ \sup_{\substack{x \leq y \leq x+\delta \\ s \leq t \leq s+\delta}} |W_n(y,t) - W_n(x,s)| \geq 5\varepsilon \right\} < \eta\delta^2,$$

for all sufficiently large n.

The remainder of the proof will be omitted since it is analogous to what has been presented previously to show tightness in Theorem 1.2.1.

Proof of (b1) We need to discuss three cases according to the position of s and t with respect to θ_n.

i) $\underline{0 \leq s \leq t \leq \theta_n}$

We have in this case

$$W_{n3}(A) \;=\; W_{n3}(x,s) - W_{n3}(x,t) - W_{n3}(y,s) + W_{n3}(y,t)$$

$$\begin{aligned}
&= W_{n1}(x,s) - W_{n1}(x,t) - W_{n1}(y,s) + W_{n1}(y,t) \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]} R_i^F(x) - \sum_{i=1}^{[nt]} R_i^F(x) - \sum_{i=1}^{[ns]} R_i^F(y) + \sum_{i=1}^{[nt]} R_i^F(y)\right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} \left(R_i^F(y) - R_i^F(x)\right) \\
&= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} R_i^F(y,x).
\end{aligned}$$

Therefore, we get using (1.2.48)

$$E\left[(W_{n3}(A))^4\right] \leq C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}\left(|t-s||x-y|\right)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta}\right). \tag{2.1.6}$$

ii) $\underline{\theta_n < s \leq t \leq 1}$

Remark that, in this case, the increment of W_{n3} around A is

$$\begin{aligned}
W_{n3}(A) &= W_{n3}(x,s) - W_{n3}(x,t) - W_{n3}(y,s) + W_{n3}(y,t) \\
&= W_{n2}(x,s) - W_{n2}(x,t) - W_{n2}(y,s) + W_{n2}(y,t) \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=[n\theta_n]+1}^{[ns]} R_i^G(x) - \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(x)\right] \\
&+ \frac{1}{\sqrt{n}}\left[-\sum_{i=[n\theta_n]+1}^{[ns]} R_i^G(y) + \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y)\right] \\
&= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} \left(R_i^G(y) - R_i^G(x)\right) \\
&= \frac{1}{\sqrt{n}} \sum_{i=[ns]+1}^{[nt]} R_i^G(y,x).
\end{aligned}$$

Using (1.2.48) again, we get

$$E\left[(W_{n3}(A))^4\right] \leq C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}\left(|t-s||x-y|\right)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta}\right). \tag{2.1.7}$$

iii) $\underline{s \leq \theta_n < t}$

The increment of W_{n3} around A is given by

$$\begin{aligned}
W_{n3}(A) &= W_{n1}(x,s) - W_{n1}(y,s) - W_{n2}(x,t) + W_{n2}(y,t) \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]} R_i^F(x) - \sum_{i=1}^{[ns]} R_i^F(y) - \sum_{i=1}^{[n\theta_n]} R_i^F(x) - \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(x)\right] \\
&+ \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[n\theta_n]} R_i^F(y) + \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y)\right] \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=[ns]+1}^{[n\theta_n]} R_i^F(y,x) + \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y,x)\right].
\end{aligned}$$

Hence, using (1.2.48) repeatedly leads to

$$\begin{aligned}
&E\left[(W_{n3}(A))^4\right] \\
&= \frac{1}{n^2} E\left[\left(\sum_{i=[ns]+1}^{[n\theta_n]} R_i^F(y,x) + \sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y,x)\right)^4\right] \\
&\leq \frac{C}{n^2} E\left[\left(\sum_{i=[ns]+1}^{[n\theta_n]} R_i^F(y,x)\right)^4\right] + \frac{C}{n^2} E\left[\left(\sum_{i=[n\theta_n]+1}^{[nt]} R_i^G(y,x)\right)^4\right] \\
&\leq C\left((|\theta_n - s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}(|\theta_n - s||x-y|)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta}\right) \\
&+ C\left((|t-\theta_n||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}(|t-\theta_n||x-y|)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta}\right) \\
&\leq C\left((|t-s||x-y|)^{\frac{3\Delta}{2}} + \frac{1}{n}(|t-s||x-y|)^{\Delta} + \frac{1}{n^2}|x-y|^{\Delta}\right). \qquad (2.1.8)
\end{aligned}$$

Combining (2.1.6), (2.1.7) and (2.1.8) completes the proof of **(b1)**.

□

Proof of (b2) Let $0 < \varepsilon < 1$ and $\left(\frac{\varepsilon}{n}\right)^{\frac{2}{3\Delta}} \leq p \leq 1$, then (1.2.50) holds.

We want to show that for $x \leq y \leq x+p$ and $s \leq t \leq s+p$

$$\begin{aligned}
|W_{n3}(y,t) - W_{n3}(x,s)| &\leq |W_{n3}(x+p, s+p) - W_{n3}(x,s)| \\
&\quad + C\sqrt{n}p + \frac{2}{\sqrt{n}}. \qquad (2.1.9)
\end{aligned}$$

Define

$$V^* := \sqrt{n}\left(W_{n3}(x+p,s+p) - W_{n3}(x,s)\right),$$

and recall

$$U_n(x,s) = \sum_{i=1}^{[ns]} \mathbf{I}(X_i \le x),$$

the number among $X_1, \ldots, X_{[ns]}$ that satisfy $X_i \le x$.

i) $\underline{0 \le s \le t \le s+p \le \theta_n}$

In this case, we have

$$W_{n3}(y,t) - W_{n3}(x,s) = \frac{1}{\sqrt{n}}\left[U_n(y,t) - [nt]F(y) - U_n(x,s) + [ns]F(x)\right].$$

Therefore, using the results presented previously to prove equation (1.2.51), we know that

$$\begin{aligned} |W_{n3}(y,t) - W_{n3}(x,s)| \;\; &\le \;\; |W_{n3}(x+p,s+p) - W_{n3}(x,s)| \\ &\quad + C\sqrt{n}p + \frac{1}{\sqrt{n}}. \end{aligned} \tag{2.1.10}$$

ii) $\underline{0 \le s \le t \le \theta_n \le s+p}$

In this case, we have

$$W_{n3}(y,t) - W_{n3}(x,s) = \frac{1}{\sqrt{n}}\left[U_n(y,t) - [nt]F(y) - U_n(x,s) + [ns]F(x)\right],$$

and

$$\begin{aligned} W_{n3}(x+p,s+p) - W_{n3}(x,s) &= \frac{1}{\sqrt{n}}\left[U_n(x+p,s+p) - U_n(x,s)\right] \\ - \;\; \frac{1}{\sqrt{n}}&\left[[n\theta_n]F(x+p) + \left([n(s+p)] - [n\theta_n]\right)G(x+p) - [ns]F(x)\right]. \end{aligned}$$

Using the fact that F is Lipschitz, we remark that

$$\begin{aligned} &U_n(y,t) - [nt]F(y) - U_n(x,s) + [ns]F(x) \\ &\le \;\; V^* + [n\theta_n]F(x+p) + \left([n(s+p)] - [n\theta_n]\right)G(x+p) - [nt]F(y) \\ &\le \;\; |V^*| + [n\theta_n](F(x+p) - F(y)) + \left([n\theta_n] - [nt]\right)F(y) \end{aligned}$$

$$\begin{aligned} &+ \quad ([n(s+p)] - [n\theta_n])\, G(x+p) \\ &\leq \quad |V^*| + Cnp + 2, \end{aligned}$$

and

$$\begin{aligned} & U_n(x,s) - [ns]F(x) - U_n(y,t) + [nt]F(y) \\ &\quad \leq \quad [nt]F(y) - [ns]F(x) \\ &\quad \leq \quad |V^*| + ([nt]-[ns])F(y) + [ns](F(y)-F(x)) \\ &\quad \leq \quad |V^*| + Cnp + 1. \end{aligned}$$

Thus,

$$\begin{aligned} |W_{n3}(y,t) - W_{n3}(x,s)| \quad &\leq \quad |W_{n3}(x+p,s+p) - W_{n3}(x,s)| \\ &\quad + C\sqrt{n}p + \frac{2}{\sqrt{n}}. \end{aligned} \tag{2.1.11}$$

iii) $\underline{0 \leq s \leq \theta_n \leq t \leq s+p}$

In this case, we have

$$\begin{aligned} & W_{n3}(y,t) - W_{n3}(x,s) \\ &\quad = \quad \frac{1}{\sqrt{n}} \left[U_n(y,t) - [n\theta_n]F(y) - ([nt]-[n\theta_n])G(y) - U_n(x,s) + [ns]F(x) \right], \end{aligned}$$

and

$$\begin{aligned} W_{n3}(x+p,s+p) - W_{n3}(x,s) &= \frac{1}{\sqrt{n}} \left[U_n(x+p,s+p) - U_n(x,s) \right] \\ - \quad \frac{1}{\sqrt{n}} & \left[[n\theta_n]F(x+p) + ([n(s+p)] - [n\theta_n])\, G(x+p) - [ns]F(x) \right]. \end{aligned}$$

Since F is Lipschitz, then

$$\begin{aligned} & U_n(y,t) - [n\theta_n]F(y) - ([nt]-[n\theta_n])G(y) - U_n(x,s) + [ns]F(x) \\ &\quad \leq \quad V^* + [n\theta_n]F(x+p) + ([n(s+p)] - [n\theta_n])\, G(x+p) \\ &\quad\quad\; - [n\theta_n]F(y) - ([nt]-[n\theta_n])G(y) \end{aligned}$$

$$\begin{aligned}
&\leq |V^*| + [n\theta_n]F(x+p) + ([n(s+p)] - [n\theta_n])\,G(x+p) \\
&\quad -[n\theta_n]F(x) \\
&\leq |V^*| + [n\theta_n](F(x+p) - F(x)) + n(s+p-\theta_n) + 1 \\
&\leq |V^*| + Cnp + 1,
\end{aligned}$$

and

$$\begin{aligned}
&U_n(x,s) - [ns]F(x) - U_n(y,t) + [n\theta_n]F(y) + ([nt] - [n\theta_n])G(y) \\
&\leq [n\theta_n]F(y) + ([nt] - [n\theta_n])G(y) - [ns]F(x) \\
&\leq |V^*| + n(s+p)F(x+p) + n(t-\theta_n) + 1 + (-ns+1)F(x) \\
&\leq |V^*| + ns(F(x+p) - F(x)) + 2np + 2 \\
&\leq |V^*| + Cnp + 2.
\end{aligned}$$

Thus,

$$\begin{aligned}
|W_{n3}(y,t) - W_{n3}(x,s)| &\leq |W_{n3}(x+p, s+p) - W_{n3}(x,s)| \\
&\quad + C\sqrt{n}p + \frac{2}{\sqrt{n}}.
\end{aligned} \tag{2.1.12}$$

iv) $\underline{0 \leq \theta_n \leq s \leq t \leq s+p}$

In this case, we have

$$\begin{aligned}
W_{n3}(y,t) - W_{n3}(x,s) &= \frac{1}{\sqrt{n}}\left[U_n(y,t) - [n\theta_n]F(y) - ([nt] - [n\theta_n])G(y)\right] \\
&- \frac{1}{\sqrt{n}}\left[U_n(x,s) - [n\theta_n]F(x) - ([ns] - [n\theta_n])G(x)\right].
\end{aligned}$$

and

$$\begin{aligned}
&W_{n3}(x+p, s+p) - W_{n3}(x,s) = \frac{1}{\sqrt{n}}\left[U_n(x+p, s+p) - U_n(x,s)\right] \\
&- \frac{1}{\sqrt{n}}\left[[n\theta_n]F(x+p) + ([n(s+p)] - [n\theta_n])\,G(x+p)\right] \\
&+ \frac{1}{\sqrt{n}}\left[[n\theta_n]F(x) + ([ns] - [n\theta_n])G(x)\right].
\end{aligned}$$

Since both F and G are Lipschitz,

$$\begin{aligned} & U_n(y,t) - [n\theta_n]F(y) - ([nt]-[n\theta_n])G(y) - U_n(x,s) + [n\theta_n]F(x) \\ + & \quad ([ns]-[n\theta_n])G(x) \\ \leq & \quad V^* + [n\theta_n]F(x+p) + ([n(s+p)]-[n\theta_n])\,G(x+p) \\ & \quad -[n\theta_n]F(y) - ([nt]-[n\theta_n])G(y) \\ \leq & \quad |V^*| + [n\theta_n](F(x+p)-F(y)) + [n\theta_n](G(y)-G(x+p)) \\ + & \quad [n(s+p)](G(x+p)-G(y)) + ([n(s+p)]-[nt])\,G(y) \\ \leq & \quad |V^*| + Cnp + 1, \end{aligned}$$

and

$$\begin{aligned} & U_n(x,s) - [n\theta_n]F(x) - ([ns]-[n\theta_n])G(x) - U_n(y,t) + [n\theta_n]F(y) \\ + & \quad ([nt]-[n\theta_n])G(y) \\ \leq & \quad [n\theta_n]F(y) + ([nt]-[n\theta_n])G(y) - [n\theta_n]F(x) - ([ns]-[n\theta_n])G(x) \\ \leq & \quad |V^*| + [n\theta_n](F(y)-F(x)) + [n\theta_n](G(x)-G(y)) \\ + & \quad [nt](G(y)-G(x)) + ([nt]-[ns])G(x) \\ \leq & \quad |V^*| + Cnp + 1. \end{aligned}$$

Thus,

$$\begin{aligned} |W_{n3}(y,t) - W_{n3}(x,s)| \quad \leq & \quad |W_{n3}(x+p,s+p) - W_{n3}(x,s)| \\ & \quad + C\sqrt{n}p + \frac{2}{\sqrt{n}}. \end{aligned} \qquad (2.1.13)$$

Therefore, (2.1.9) holds and as mentioned before, the arguments presented in part **(b)** of the proof of tightness in Theorem 1.2.1 can be utilized in a similar way to complete the proof of tightness here.

□

2.2 Testing for the sequential empirical process with a change-point

Here, we apply Theorems 1.2.1 and 2.1.2 to the change-point problem. Used by several authors, as in [20] and [32], the Kolmogorov-Smirnov type statistic and the Cramér-Von Mises statistic or their weighted versions will be the key to detect any change in the marginal distribution of the linear process.

The approach to testing the change-point problem in the marginal distribution function F will be based on the limiting distributions, $W^{(1)}$ and $W^{(2)}$, which were determined in Theorem 1.2.1 and Theorem 2.1.2 respectively. To avoid confusion, we denote in the following the two-parameter sequential empirical processes defined in this chapter and the previous one by $W_n^{(2)}(\cdot,\cdot)$ and $W_n^{(1)}(\cdot,\cdot)$ respectively.
Recall the stationary processes Y_i and Z_i satisfying Assumptions 2.1.1, for $i \in \mathbb{Z}$ and $b \in \mathbb{R}$

$$Y_i = \sum_{j \geq 0} a_j^{(1)} \xi_{i-j} \quad \text{and} \quad Z_i = \sum_{j \geq 0} a_j^{(2)} \left(\xi_{i-j} + b\right),$$

and denote by F and G the respective distribution functions of Y_0 and Z_0.

In the following, we will make use of the methodology proposed in [20]. We introduce the class $\mathbf{\Psi}_n(\theta_n, F, G)$ of all random vectors $\mathbf{X}_n = (X_1, \ldots, X_n)$ such that for $0 \leq \theta_n \leq 1$

$$X_i = \begin{cases} Y_i & \text{if} \quad 1 \leq i \leq [n\theta_n] \\ Z_i & \text{if} \quad [n\theta_n] < i \leq n, \end{cases}$$

where $\theta_n \to \theta$ as $n \to \infty$.
We note here that the change-point of the marginal distribution of the sample $\mathbf{X}_n$ occurs at $i = [n\theta_n] + 1$. We also denote the class of all vectors $(X_1, \ldots, X_n)$ having the same marginal distribution F by $\mathbf{\Psi}_n(F) := \mathbf{\Psi}_n(1, F)$.
We aim to test the null hypothesis of no change in the marginal distribution function of the X_i's, namely, there exists a distribution function F such that $P(X_i \leq x) = F(x)$.

Two cases arise in our discussion depending on the prior knowledge of the marginal distribution function.

2.2.1 Testing the null hypothesis for known marginal distribution

Here, we want to test

$$\begin{aligned} H_0 &: \{\mathbf{X}_n \in \mathbf{\Psi}_n(F)\} \\ H_1 &: \{\exists \theta_n \in (0,1)\ \exists G \neq F \text{ such that } \mathbf{X}_n \in \mathbf{\Psi}_n(\theta_n, F, G)\}, \end{aligned}$$

for some known distribution F.

The test is based on the process:

$$V_n^{(1)}(x,s) = \frac{n-[ns]}{\sqrt{n}} \left(F^*_{n-[ns]}(x) - F(x) \right),$$

where

$$F^*_{n-m}(x) = \frac{1}{n-m} \sum_{i=m+1}^{n} \mathbf{I}(X_i \leq x)$$

is the empirical distribution function based on $X_{m+1}, \ldots, X_n$.

Consider now the following test statistics based on the preceding process:

- Kolmogorov-Smirnov statistic:

$$T_1 = \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left| V_n^{(1)}(x,s) \right|.$$

- Cramér-Von Mises statistic:

$$T_2 = \int_0^1 \int_{\mathbb{R}} \left| V_n^{(1)}(x,s) \right|^2 dF_n(x) ds.$$

Hence, we reject the null hypothesis H_0 when $T_i > c$, for $i = 1, 2$ and c a constant.

Proposition 2.2.1 *Under the null hypothesis H_0, we have for every $c > 0$*

$$\lim_{n\to\infty} P\{T_1 > c\} = P\left\{\sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W^{(1)}(x,s)\right| > c\right\}, \tag{2.2.1}$$

$$\lim_{n\to\infty} P\{T_2 > c\} = P\left\{\int_0^1\int_{\mathbb{R}} |W^{(1)}(x,s)|^2 dF(x)ds > c\right\}, \tag{2.2.2}$$

where $W^{(1)}(\cdot,\cdot)$ is the limiting process in Theorem 1.2.1.

Proof

We know that

$$\begin{aligned}
V_n^{(1)}(x,s) &= \frac{n-[ns]}{\sqrt{n}}\left(F^*_{n-[ns]}(x) - F(x)\right) \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=[ns]+1}^{n} \mathbf{I}(X_i \le x) - (n-[ns])F(x)\right] \\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{n} \mathbf{I}(X_i \le x) - nF(x) - \sum_{i=1}^{[ns]} \mathbf{I}(X_i \le x) + [ns]F(x)\right] \\
&= W_n^{(1)}(x,1) - W_n^{(1)}(x,s).
\end{aligned}$$

By stationarity, we have

$$\begin{aligned}
V_n^{(1)}(x,s) &\overset{d}{=} W_n^{(1)}\left(x, 1-\frac{[ns]}{n}\right) - W_n^{(1)}(x,0) \\
&= W_n^{(1)}\left(x, 1-\frac{[ns]}{n}\right).
\end{aligned}$$

Thus, by Theorem 1.2.1,

$$V_n^{(1)}(\cdot,\cdot) \overset{\mathbf{D}}{\to} V^{(1)}(\cdot,\cdot),$$

where $V^{(1)}(x,s) = W^{(1)}(x,1-s)$ and $\overset{\mathbf{D}}{\to}$ denotes weak convergence in the space $D(\mathbb{R}\times[0,1])$ under the Skorohod topology.

Applying the continuous mapping theorem, since $\sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W_n^{(1)}(x,s)\right|$ is a continuous functional on $D(\mathbb{R}\times[0,1])$, leads to

$$T_1 \overset{d}{\to} \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W^{(1)}(x,1-s)\right| = \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W^{(1)}(x,s)\right|,$$

and (2.2.1) follows immediately.

Similarly,

$$T_2 \xrightarrow{d} \int_0^1 \int_{\mathbb{R}} |W^{(1)}(x, 1-s)|^2 dF(x) ds = \int_0^1 \int_{\mathbb{R}} |W^{(1)}(x, s)|^2 dF(x) ds.$$

Here, the proof is based partly on the continuous mapping theorem and partly on Theorem 4.4 in [5] to show that $(V_n^{(1)}(\cdot,\cdot), F_n(\cdot)) \xrightarrow{\mathbf{D}} (V^{(1)}(\cdot,\cdot), F(\cdot))$. This completes the proof of (2.2.2).

□

The following proposition proves the consistency of the statistics T_1 and T_2.

Proposition 2.2.2 *Suppose the sequence $\{\theta_n : n \in \mathbb{N}\}$ satisfies one of the following assumptions*

1. $\theta_n \to \theta$ *and* $\theta \in [0, 1)$
2. $\theta_n \to 1$ *and* $\sqrt{n}(1 - \theta_n) \to \infty$.

Then, under the alternative H_1, $T_i \xrightarrow{p} \infty$ for $i = 1, 2$.

Proof

Recall first

$$H^{(n)}(x, s) = (s \wedge \theta_n) F(x) + (s - \theta_n)^+ G(x).$$

Write now

$$\begin{aligned} V_n^{(1)}(x, s) &= \frac{n - [ns]}{\sqrt{n}} (F^*_{n-[ns]}(x) - F(x)) \\ &= \frac{1}{\sqrt{n}} \left[\sum_{i=[ns]+1}^{n} \mathbf{I}(X_i \le x) - (n - [ns]) F(x) \right] \\ &= \frac{1}{\sqrt{n}} \left[\sum_{i=1}^{n} \mathbf{I}(X_i \le x) - n H^{(n)}(x, 1) - \sum_{i=1}^{[ns]} \mathbf{I}(X_i \le x) + n H^{(n)}(x, s) \right] \\ &+ \frac{1}{\sqrt{n}} \left[n H^{(n)}(x, 1) - n H^{(n)}(x, s) - (n - [ns]) F(x) \right] \end{aligned}$$

$$\begin{aligned} &= W_n^{(2)}(x,1) - W_n^{(2)}(x,s) + \sqrt{n}\left(H^{(n)}(x,1) - H^{(n)}(x,s)\right) - \frac{n-[ns]}{\sqrt{n}}F(x) \\ &= K_n^{(1)}(x,s) + A_n(x,s), \end{aligned}$$

where

$$\begin{aligned} K_n^{(1)}(x,s) &:= W_n^{(2)}(x,1) - W_n^{(2)}(x,s) \\ \text{and} \qquad A_n(x,s) &:= \frac{1}{\sqrt{n}}\left[nH^{(n)}(x,1) - nH^{(n)}(x,s) - (n-[ns])F(x)\right]. \end{aligned}$$

Suppose first that $\theta_n \to \theta$ such that $\theta \in (0,1)$. Two cases are to be discussed here depending on the position of s with respect to θ:

• $0 < s < \theta$

There exists ℓ in $\mathbb{N}$ such that $s \leq \theta_n$ for $n \geq \ell$. In this case,

$$\begin{aligned} A_n(x,s) &= \frac{1}{\sqrt{n}}\left[n\theta_n F(x) + n(1-\theta_n)G(x) - nsF(x) - (n-[ns])F(x)\right] \\ &= \sqrt{n}(1-\theta_n)(G(x)-F(x)) + \frac{[ns]-ns}{\sqrt{n}}F(x). \end{aligned} \tag{2.2.3}$$

• $\theta < s < 1$

There exists ℓ in $\mathbb{N}$ such that $\theta_n \leq s$ for $n \geq \ell$. Here, we have

$$\begin{aligned} A_n(x,s) &= \frac{1}{\sqrt{n}}\left[n\theta_n F(x) + n(1-\theta_n)G(x) - n\theta_n F(x) - n(s-\theta_n)G(x)\right] \\ &- \frac{1}{\sqrt{n}}(n-[ns])F(x) \\ &= \sqrt{n}(1-s)(G(x)-F(x)) - \frac{ns-[ns]}{\sqrt{n}}F(x). \end{aligned} \tag{2.2.4}$$

Remark now that if $\theta_n \to 0$, then (2.2.4) holds for all $s > 0$. On the other hand, equation (2.2.3) follows immediately for all $s < 1$ provided that $\theta_n \to 1$.

We can conclude that if the assumptions of Proposition 2.2.2 are satisfied by the sequence $\{\theta_n : n \in \mathbb{N}\}$, then we can easily see that $\sup_{(x,s)\in\mathbb{R}\times[0,1]} |A_n(x,s)| \to \infty$, as $n \to \infty$.

On the other hand, by Theorem 2.1.2, we get

$$K_n^{(1)}(\cdot,\cdot) \xrightarrow{\mathbf{D}} K^{(1)}(\cdot,\cdot),$$

where $K^{(1)}(x,s) = W^{(2)}(x,1) - W^{(2)}(x,s)$.

Hence,

$$\sup_{(x,s)\in\mathbb{R}\times[0,1]} |K_n^{(1)}(x,s)| \xrightarrow{d} \sup_{(x,s)\in\mathbb{R}\times[0,1]} |K^{(1)}(x,s)|.$$

Remark now that

$$\sup_{(x,s)\in\mathbb{R}\times[0,1]} |A_n(x,s)| \leq T_1 + \sup_{(x,s)\in\mathbb{R}\times[0,1]} |K_n^{(1)}(x,s)|.$$

Therefore, $T_1 \xrightarrow{p} \infty$ as $n \to \infty$.

Similarly,

$$\int_0^1 \int_{\mathbb{R}} |A_n(x,s)|^2 dF_n(x) ds \leq 2T_2 + 2\int_0^1 \int_{\mathbb{R}} |K_n^{(1)}(x,s)|^2 dF_n(x) ds.$$

Using again Theorem 4.4 in [5] and the continuous mapping theorem to get

$$\int_0^1 \int_{\mathbb{R}} |K_n^{(1)}(x,s)|^2 dF_n(x) ds \xrightarrow{d} \int_0^1 \int_{\mathbb{R}} |K^{(1)}(x,s)|^2 dH(x) ds,$$

where $H = \theta F + (1-\theta)G$. Hence, $T_2 \xrightarrow{p} \infty$, as $n \to \infty$.

□

2.2.2 Testing the null hypothesis for unknown marginal distribution

Consider now the pair of hypotheses defined by

$$\begin{aligned} H_0 &: \{\exists F \quad \text{such that} \quad \mathbf{X}_n \in \mathbf{\Psi}_n(F)\} \\ H_1 &: \{\exists \theta_n \in (0,1)\ \exists F^{(1)} \neq F^{(2)} \text{ such that } \mathbf{X}_n \in \mathbf{\Psi}_n(\theta_n, F^{(1)}, F^{(2)})\}. \end{aligned}$$

Because of the lack of any knowledge of F, the testing method for this case will be based on the following process:

$$V_n^{(2)}(x,s) := \frac{[ns](n-[ns])}{n^{\frac{3}{2}}} \left(F_{[ns]}(x) - F^*_{n-[ns]}(x) \right),$$

which involves the distance based on the difference between the prechange and the postchange empirical distributions

$$\left| \frac{1}{[ns]} \sum_{i=1}^{[ns]} \mathbf{I}(X_i \leq x) - \frac{1}{n-[ns]} \sum_{i=[ns]+1}^{n} \mathbf{I}(X_i \leq x) \right|.$$

To test the pair (H_0, H_1), we use the following statistics:

- Weighted Kolmogorov-Smirnov statistic:

$$T_3 = \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left| V_n^{(2)}(x,s) \right|.$$

- Weighted Cramér-Von Mises statistic:

$$T_4 = \int_0^1 \int_{\mathbb{R}} \left| V_n^{(2)}(x,s) \right|^2 dF_n(x) ds.$$

Again, we reject the null hypothesis H_0 when $T_i > c$, for $i = 1, 2$ and c a constant.

Proposition 2.2.3 *Assume the null hypothesis H_0 is true. Then we have for every $c > 0$*

$$\lim_{n\to\infty} P\{T_3 > c\} = P\left\{ \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left| W^{(1)}(x,s) - sW^{(1)}(x,1) \right| > c \right\}, \tag{2.2.5}$$

$$\lim_{n\to\infty} P\{T_4 > c\} = P\left\{ \int_0^1 \int_{\mathbb{R}} |W^{(1)}(x,s) - sW^{(1)}(x,1)|^2 dF(x) ds > c \right\}, \tag{2.2.6}$$

where $W^{(1)}(\cdot,\cdot)$ is the limiting process in Theorem 1.2.1.

Proof

Remark that

$$\begin{aligned} V_n^{(2)}(x,s) &= \frac{[ns](n-[ns])}{n^{\frac{3}{2}}} \left(F_{[ns]}(x) - F^*_{n-[ns]}(x) \right) \\ &= \frac{1}{n^{\frac{3}{2}}} \left[(n-[ns]) \sum_{i=1}^{[ns]} \mathbf{I}(X_i \leq x) - [ns] \sum_{i=[ns]+1}^{n} \mathbf{I}(X_i \leq x) \right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{n^{\frac{3}{2}}}\left[n\sum_{i=1}^{[ns]}\mathbf{I}(X_i \le x) - [ns]\sum_{i=1}^{n}\mathbf{I}(X_i \le x)\right] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{[ns]}(\mathbf{I}(X_i \le x) - F(x)) - \frac{[ns]}{n^{\frac{3}{2}}}\sum_{i=1}^{n}(\mathbf{I}(X_i \le x) - F(x)) \\
&= W_n^{(1)}(x,s) - \frac{[ns]}{n}W_n^{(1)}(x,1).
\end{aligned}
$$

Hence,

$$V_n^{(2)}(\cdot,\cdot) \xrightarrow{\mathbf{D}} V^{(2)}(\cdot,\cdot),$$

where $V^{(2)}(x,s) = W^{(1)}(x,s) - sW^{(1)}(x,1)$. Therefore, using arguments similar to those in Proposition 2.2.1, we get

$$T_3 \xrightarrow{d} \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W^{(1)}(x,s) - sW^{(1)}(x,1)\right|,$$

and

$$T_4 \xrightarrow{d} \int_0^1\int_{\mathbb{R}} |W^{(1)}(x,s) - sW^{(1)}(x,1)|^2 dF(x)ds.$$

This completes the proof of Proposition 2.2.3.

□

We shall now be concerned with the consistency of the test statistics T_3 and T_4.

Proposition 2.2.4 *Suppose the sequence $\{\theta_n : n \in \mathbb{N}\}$ satisfies one of the following assumptions*

1. $\theta_n \to \theta$ *and* $\theta \in (0,1)$
2. $\theta_n \to 0$ *and* $\sqrt{n}\theta_n \to \infty$
3. $\theta_n \to 1$ *and* $\sqrt{n}(1-\theta_n) \to \infty$.

Then, under the alternative H_1, $T_i \xrightarrow{p} \infty$ for $i = 3, 4$.

Proof

Write

$$
\begin{aligned}
V_n^{(2)}(x,s) &= \frac{[ns](n-[ns])}{n^{\frac{3}{2}}}\left(F_{[ns]}(x) - F^*_{n-[ns]}(x)\right)\\
&= \frac{1}{n^{\frac{3}{2}}}\left[(n-[ns])\sum_{i=1}^{[ns]}\mathbf{I}(X_i \le x) - [ns]\sum_{i=[ns]+1}^{n}\mathbf{I}(X_i \le x)\right]\\
&= \frac{1}{n^{\frac{3}{2}}}\left[n\sum_{i=1}^{[ns]}\mathbf{I}(X_i \le x) - [ns]\sum_{i=1}^{n}\mathbf{I}(X_i \le x)\right]\\
&= \frac{1}{\sqrt{n}}\left[\sum_{i=1}^{[ns]}\mathbf{I}(X_i \le x) - nH^{(n)}(x,s)\right] - \frac{[ns]}{n^{\frac{3}{2}}}\left[\sum_{i=1}^{n}\mathbf{I}(X_i \le x) - nH^{(n)}(x,1)\right]\\
&+ \sqrt{n}H^{(n)}(x,s) - \frac{[ns]}{\sqrt{n}}H^{(n)}(x,1)\\
&= W_n^{(2)}(x,s) - \frac{[ns]}{n}W_n^{(2)}(x,1) + \sqrt{n}H^{(n)}(x,s) - \frac{[ns]}{\sqrt{n}}H^{(n)}(x,1)\\
&= K_n^{(2)}(x,s) + B_n(x,s),
\end{aligned}
$$

where

$$
\begin{aligned}
K_n^{(2)}(x,s) &:= W_n^{(2)}(x,s) - \frac{[ns]}{n}W_n^{(2)}(x,1)\\
\text{and} \qquad B_n(x,s) &:= \sqrt{n}H^{(n)}(x,s) - \frac{[ns]}{\sqrt{n}}H^{(n)}(x,1).
\end{aligned}
$$

As in the proof of Proposition 2.2.2, we first consider the case where $\theta_n \to \theta$ for some $\theta \in (0,1)$. Two subcases are to be discussed here.

• $0 < s < \theta$

There exists ℓ in $\mathbb{N}$ such that $s \le \theta_n$ for $n \ge \ell$. In this case, we have

$$
\begin{aligned}
B_n(x,s) &= \sqrt{n}sF^{(1)}(x) - \frac{[ns]}{\sqrt{n}}\left[\theta_n F^{(1)}(x) + (1-\theta_n)F^{(2)}(x)\right]\\
&= \frac{1}{\sqrt{n}}(ns - [ns]\theta_n)F^{(1)}(x) - \frac{[ns]}{\sqrt{n}}(1-\theta_n)F^{(2)}(x)\\
&= \frac{ns-[ns]}{\sqrt{n}}F^{(1)}(x) + \frac{[ns]}{\sqrt{n}}(1-\theta_n)(F^{(1)}(x) - F^{(2)}(x)). \qquad (2.2.7)
\end{aligned}
$$

• $\theta < s < 1$

There exists ℓ in $\mathbb{N}$ such that $\theta_n \leq s$ for $n \geq \ell$. Here, we have

$$\begin{aligned} B_n(x,s) &= \sqrt{n}\left[\theta_n F^{(1)}(x) + (s-\theta_n)F^{(2)}(x)\right] \\ &- \frac{[ns]}{\sqrt{n}}\left[\theta_n F^{(1)}(x) + (1-\theta_n)F^{(2)}(x)\right] \\ &= \frac{ns-[ns]}{\sqrt{n}}[\theta_n F^{(1)}(x) + (1-\theta_n)F^{(2)}(x)] \\ &+ \sqrt{n}\left[\theta_n F^{(1)}(x) + (s-\theta_n)F^{(2)}(x) - s\theta_n F^{(1)}(x) - s(1-\theta_n)F^{(2)}(x)\right] \\ &= \frac{ns-[ns]}{\sqrt{n}}[\theta_n F^{(1)}(x) + (1-\theta_n)F^{(2)}(x)] \\ &+ \sqrt{n}\theta_n(1-s)(F^{(1)}(x) - F^{(2)}(x)). \end{aligned} \tag{2.2.8}$$

If $\theta_n \to 1$, then equation (2.2.7) holds. On the other hand, equation (2.2.8) holds if $\theta_n \to 0$. Thus, $\sup_{(x,s)\in\mathbb{R}\times[0,1]} |B_n(x,s)| \to \infty$, as $n \to \infty$, provided that the conditions in Proposition 2.2.4 are satisfied.

Moreover, Theorem 2.1.2 yields

$$K_n^{(2)}(\cdot,\cdot) \xrightarrow{\mathbf{D}} K^{(2)}(\cdot,\cdot),$$

where $K^{(2)}(x,s) = W^{(2)}(x,s) - sW^{(2)}(x,1)$.

Using now similar arguments as in the proof of Proposition 2.2.2, we can conclude that

$$T_i \xrightarrow{p} \infty \quad \text{for} \quad i = 3,4.$$

□

Remark 2.2.5 In real world situations, our knowledge of past information regarding the stochastic process sometimes strengthens our belief that the change occurred at certain value $[n\theta]$, where $\theta \in (0,1)$ is known. In this case, it would be appropriate to reformulate the test as follows:

$$H_0^{(\theta)} \quad : \quad \{\exists F \quad \text{such that} \quad \mathbf{X}_n \in \mathbf{\Psi}_n(F)\}$$

$$H_1^{(\theta)} \quad : \quad \{\exists F^{(1)} \neq F^{(2)} \text{ such that } \mathbf{X}_n \in \mathbf{\Psi}_n(\theta, F^{(1)}, F^{(2)})\}.$$

Analogous to the case that has been studied before, the testing method will be based on the following process

$$V_n^{(2,\theta)}(x) := \frac{[n\theta](n-[n\theta])}{n^{\frac{3}{2}}} \left(F_{[n\theta]}(x) - F^*_{n-[n\theta]}(x) \right).$$

To test the pair $\left(H_0^{(\theta)}, H_1^{(\theta)}\right)$, we use the following statistics:

$$T_{3,\theta} = \sup_{x\in\mathbb{R}} \left| V_n^{(2,\theta)}(x) \right|$$

or

$$T_{4,\theta} = \int_{\mathbb{R}} \left| V_n^{(2,\theta)}(x) \right|^2 dF_n(x).$$

Again, we reject the null hypothesis $H_0^{(\theta)}$ when $T_{i,\theta} > c$, for $i = 1, 2$ and c a constant.

Proposition 2.2.6 *Assume the null hypothesis $H_0^{(\theta)}$ is true. Then we have for every $c > 0$*

$$\lim_{n\to\infty} P\{T_{3,\theta} > c\} = P\left\{ \sup_{x\in\mathbb{R}} \left| W^{(1)}(x,\theta) - \theta W^{(1)}(x,1) \right| > c \right\}, \tag{2.2.9}$$

$$\lim_{n\to\infty} P\{T_{4,\theta} > c\} = P\left\{ \int_{\mathbb{R}} |W^{(1)}(x,\theta) - \theta W^{(1)}(x,1)|^2 dF(x) > c \right\}, \tag{2.2.10}$$

where $W^{(1)}(\cdot,\cdot)$ is the limiting process in Theorem 1.2.1.

Proposition 2.2.7 *Under the alternative $H_1^{(\theta)}$, $T_{i,\theta} \xrightarrow{p} \infty$ for $i = 3, 4$.*

Arguing as in the proof of Proposition 2.2.3 and Proposition 2.2.4, it is easy to see that the results in the last two propositions hold. In fact, one has only to replace s by θ and the results will follow.

For future reference, we denote the Gaussian limiting distribution of $V_n^{(2,\theta)}(\cdot)$ under $H_0^{(\theta)}$ by

$$L_{W^{(1)},\theta}(\cdot) := W^{(1)}(\cdot,\theta) - \theta W^{(1)}(\cdot,1), \tag{2.2.11}$$

with covariance function

$$Cov\left(L_{W^{(1)},\theta}(x), L_{W^{(1)},\theta}(y)\right) = \theta(1-\theta)\sum_{i\in\mathbf{Z}} Cov\left(\mathbf{I}(X_0 \leq x), \mathbf{I}(X_i \leq y)\right).$$

■

Chapter 3

Functional Central Limit Theorem for the Bootstrapped Empirical Process

As shown in the previous chapters, the covariance structure of the limiting process in Theorem 1.2.1 makes appropriate critical values difficult to determine. To deal with situation, we will use a block version of the bootstrap technique. This method will allow us to capture the dependency without imposing any particular linear model. The remainder of this chapter will be devoted to proving a functional central limit for the bootstrapped empirical process. Our main result, Theorem 3.2.2, gives the validity of the moving block bootstrap for the linear process under conditions similar to Assumptions 1.1.1. In fact, this is a result of independent interest since no mixing assumption is required. This result appeared in [17]. Bootstrapping the sequential empirical process will be investigated in Chapter 4.

3.1 Blockwise bootstrap approach

We first revisit a method called the Moving Blocks Bootstrap (MBB), that was introduced independently by Künsch in [34] and Liu and Singh in [40]. The technique is based on the selection of k blocks of l consecutive observations with replacement from the blocks of observations $(X_{i+1}, X_{i+2}, \ldots, X_{i+l})$, $i = 0, 1, \ldots, n-l$. In fact, we are going to use a slightly modified version of the natural MBB, as in [47], which is defined as follows.

Consider first a stationary sequence X_i, $i = 1, \ldots, n$ such that $n = lk$ for some integers l and k. Secondly, we extend our sample of size n by the first $l-1$ observations, namely, $X_1, \ldots, X_{l-1}$ to introduce the sequence X_{ni}, $i = 1, \ldots, n+l-1$, defined as follows:

$$X_{ni} := \begin{cases} X_i & \text{if} \quad 1 \le i \le n \\ X_{i-n} & \text{if} \quad n+1 \le i \le n+l-1. \end{cases}$$

Let $I_{n1}, I_{n2}, \ldots, I_{nk}$ be independent and identically distributed random variables each having uniform distribution on $\{1, 2, \ldots, n\}$. The intuitive idea behind the MBB is to collect k randomly chosen blocks of size l of the form $\{X_{nI_j}, X_{n,I_j+1}, \ldots, X_{n,I_j+l-1}\}$, $1 \le j \le k$, and construct the bootstrap sample of size n,

$$\left(X_1^{(b)}, \ldots, X_n^{(b)}\right) = \left(X_{nI_{n1}}, \ldots, X_{n,I_{n1}+l-1}, \ldots, X_{nI_{nk}}, \ldots, X_{n,I_{nk}+l-1}\right).$$

Let us reconsider the stationary causal linear process (1.1.1) and recall some notation seen previously in Chapter 1:

$$\begin{aligned} U_{i,h}(x) &= P(X_i \le x|\mathcal{F}_{i-h}) - P(X_i \le x|\mathcal{F}_{i-h-1}) \\ R_i(x) &= \sum_{h \ge 0} U_{i,h}(x) = \mathbf{I}(X_i \le x) - F(x) \\ R_i(x,y) &= R_i(x) - R_i(y). \end{aligned}$$

Define now, for $x \in \mathbb{R}$, the bootstrapped empirical process as

$$W_n^{(b)}(x) := \sqrt{n}\left[F_n^{(b)}(x) - F_n(x)\right], \tag{3.1.1}$$

where

$$F_n^{(b)}(x) := \frac{1}{k}\sum_{j=1}^{k}\frac{1}{l}\sum_{i=I_{nj}}^{I_{nj}+l-1}\mathbf{I}(X_{ni} \le x) \tag{3.1.2}$$

is the bootstrapped empirical distribution.

For the sake of consistency with the conditions used in the previous chapters, we will develop our method under the following assumptions:

Assumptions 3.1.1

1. Let $\{a_j, j \in \mathbb{Z}\}$ be a sequence of non-random weights, infinitely many of which are non-zero, such that for some $\gamma \in (0, 1]$

$$\sum_{j\ge 0} j|a_j|^\gamma = \sum_{j\ge 1} A_j(\gamma) < \infty,$$

   where $A_j(\gamma) = \sum_{i\ge j} |a_i|^\gamma$.

2. There exist constants $C < \infty$ and $\Delta > 0$ such that for all $u \in \mathbb{R}$

$$|E\exp(iu\xi_0)| \le \frac{C}{(1+|u|)^\Delta}.$$

3. $E[|\xi_0|^{2\gamma}] < \infty$.

We remark briefly that only a $2\gamma-$moment will be needed for the analysis of the bootstrapped empirical process while we used a $4\gamma-$moment in Chapter 1 for the empirical process based on the initial observations. Also, here we only require $\Delta > 0$ rather than $\Delta > \frac{2}{3}$ as in Chapter 1 and Doukhan and Surgailis [15]. On the other hand, a stronger summability condition is imposed on the sequence $\{a_i\}_{i\in\mathbb{Z}}$ in order to get a functional central limit theorem for the bootstrapped empirical process. However, both Assumptions 1.1.1 and Assumptions 3.1.1 are satisfied if the sequence $\{|a_j|\}_{j\in\mathbb{N}}$ is non-increasing such that $\sum_{j\ge 0} |a_j|^{\frac{\gamma}{2}} < \infty$ and $E[|\xi_0|^{2\gamma}] < \infty$ for some $\gamma \in (0, 1]$, and Assumption 1.1.1.2 holds.

The fact that Assumptions 1.1.1 and Assumptions 2.1.1 are not the same is not particularly surprising, since it has been observed by Peligrad [47] and Radulović [51, 52] that the bootstrap CLT may be valid in situations when the original process does not satisfy a CLT, and vice versa. When proving bootstrap empirical CLTs under mixing assumptions, convergence of the finite dimensional distributions follows immediately from the MBB CLT for the mean, since the mixing coefficients apply equally to the original sequence (X_i) and to functionals $(f(X_i))$; proving tightness is the principal challenge. However, here the a_i's are not mixing coefficients: the stronger assumption 3.1.1.1 is used to control the conditional variance of the bootstrapped empirical process and is also sufficient for tightness.

In the following, we denote by P^* the conditional probability given the sample $(X_1, X_2, \dots, X_n)$: we will examine the weak convergence to a Brownian bridge of $W_n^{(b)}(x)$ in the Skorohod topology on $\mathbb{R}$, P^*-almost surely.

3.2 FCLT for the bootstrapped empirical process

Before stating the functional central limit theorem for the bootstrapped empirical process, we recall Theorem 2.2 from [47] on which our proof will be based. We follow [47] in using the notation $\ll$ to replace the Vinogradov symbol 0.

Theorem 3.2.1 *[47]*

Let $\{X_n\}_{n\in\mathbb{Z}}$, be a stationary sequence of random variables. Let l_n, k_n be sequences of natural numbers satisfying

$$n^h \ll l \ll n^{\frac{1}{3}-a} \quad \text{for some} \quad 0 < h < \frac{1}{3} - a, \quad 0 < a < \frac{1}{3}, \tag{3.2.1}$$

$l_n = l_{2^k}$ for $2^k \le n < 2^{k+1}$, $l_n \to \infty$ as $n \to \infty$ and $n = k_n l_n$.

Assume there are two constants C_1 and C_2 such that, for some $\gamma > 0$ and every

$0 \le s, t \le 1$,

$$\sup_{n>m} \left| \sum_{i=m}^{n} Cov(\mathbf{I}(s < X_0 \le t), \mathbf{I}(s < X_i \le t)) \right| \le C_1 m^{-\gamma}, \tag{3.2.2}$$

and for every $1 \le m \le n$,

$$Var\left(\sum_{i=1}^{m} Y_{l_n i}^2(s,t) \right) \le C_2 m l_n^4, \tag{3.2.3}$$

where

$$Y_{l_n i}(s,t) := \sum_{j=i}^{i+l_n-1} \mathbf{I}(s < X_j \le t) - (F(t) - F(s)).$$

Then

$$W_n^{(b)}(\cdot) \xrightarrow{\mathbf{D}} W(\cdot) \quad as \quad n \to \infty$$

P^**-almost surely in the Skorohod topology on* $D[0,1]$*, where* W *is a Brownian bridge with the covariance structure*

$$\sigma(s,t) = \sum_{i \in \mathbb{Z}} Cov\left(\mathbf{I}(X_0 \le s), \mathbf{I}(X_i \le t)\right).$$

Moreover, if the distribution of X_0 *is continuous,* $P^*(W(t) \in C[0,1]) = 1$.

We now present the main theorem in this chapter, namely the empirical central limit theorem for the moving block bootstrap. It will be then followed by an application of the theorem in accordance with Remark 2.2.5. The proof of the theorem will appear in the next section.

Theorem 3.2.2 *Let* X_n*,* $n \in \mathbb{Z}$*, be a stationary causal linear sequence satisfying Assumptions 3.1.1.1-3.1.1.3 and suppose that* $\{l_n\}_{n\in\mathbb{N}}$ *and* $\{k_n\}_{n\in\mathbb{N}}$ *are any sequences as in the statement of Theorem 3.2.1. Then, as* $n \to \infty$,

$$W_n^{(b)}(\cdot) = n^{\frac{1}{2}} \left[F_n^{(b)}(\cdot) - F_n(\cdot) \right] \xrightarrow{\mathbf{D}} W(\cdot)$$

P^*-almost surely in the space $D(\mathbb{R})$ endowed with Skorohod topology, where $W(\cdot)$ is a Gaussian process with zero mean and covariance

$$\sigma(x,y) = \sum_{i\in\mathbb{Z}} Cov\left(\mathbf{I}(X_0 \le x), \mathbf{I}(X_i \le y)\right).$$

Moreover, since the distribution of X_0 is continuous, $P^*(W(x) \in C(\mathbb{R})) = 1$.

Application 3.2.3 As in Remark 2.2.5, we want to test

$$\begin{aligned} H_0^{(\theta)} &: \{\exists F \quad \text{such that} \quad \mathbf{X}_n \in \mathbf{\Psi}_n(F)\} \\ H_1^{(\theta)} &: \{\exists F^{(1)} \neq F^{(2)} \text{ such that } \mathbf{X}_n \in \mathbf{\Psi}_n(\theta, F^{(1)}, F^{(2)}\}, \end{aligned}$$

where $\theta \in (0,1)$ is known and $\mathbf{X}_n = (X_1, X_2, \dots, X_n)$.

Let (l_{n1}, k_{n1}) and (l_{n2}, k_{n2}) be two pairs of sequences of natural numbers as in Theorem 3.2.1 such that $[n\theta] = l_{n1}k_{n1}$ and $n - [n\theta] = l_{n2}k_{n2}$ and suppose that Assumptions 2.1.1 and 3.1.1 are satisfied. We bootstrap two samples separately by applying the MBB techniques on the sequences Y_{ni} and Z_{ni} defined as follows:

$$Y_{ni} := \begin{cases} X_i & \text{if} \quad 1 \le i \le [n\theta] \\ X_{i-[n\theta]} & \text{if} \quad [n\theta]+1 \le i \le [n\theta] + l_{n1} - 1 \end{cases}$$

and

$$Z_{ni} := \begin{cases} X_{i+[n\theta]} & \text{if} \quad 1 \le i \le n - [n\theta] \\ X_{i+2[n\theta]-n} & \text{if} \quad n - [n\theta] + 1 \le i \le n - [n\theta] + l_{n2} - 1. \end{cases}$$

We denote the bootstrapped empirical processes based on the sequences Y_{ni} and Z_{ni} by $W_{[n\theta]}^{(b)}(x)$ and $W_{n-[n\theta]}^{(*,b)}(x)$, respectively.

The testing procedure is based on $V_n^{(2,\theta,b)}$, the bootstrapped counterpart of the process described earlier in Remark 2.2.5, and defined as follows:

$$\begin{aligned} V_n^{(2,\theta,b)}(x) &:= \frac{[n\theta](n-[n\theta])}{n^{\frac{3}{2}}} \left(F_{[n\theta]}^{(b)}(x) - F_{[n\theta]}(x)\right) \\ &- \frac{[n\theta](n-[n\theta])}{n^{\frac{3}{2}}} \left(F_{n-[n\theta]}^{(*,b)}(x) - F_{n-[n\theta]}^{*}(x)\right), \end{aligned}$$

where $F^{(b)}_{[n\theta]}(x)$ and $F^{(*,b)}_{n-[n\theta]}(x)$ are the bootstrapped empirical distributions based respectively on Y_{ni} and Z_{ni}.

Write

$$\begin{aligned} V_n^{(2,\theta,b)}(x) &= \frac{\sqrt{[n\theta]}(n-[n\theta])}{n^{\frac{3}{2}}} W^{(b)}_{[n\theta]}(x) - \frac{[n\theta]\sqrt{n-[n\theta]}}{n^{\frac{3}{2}}} W^{(*,b)}_{n-[n\theta]}(x) \\ &\xrightarrow{\mathbf{D}} \sqrt{\theta}(1-\theta)W^{(b,1)}(x) - \theta\sqrt{1-\theta}W^{(b,2)}(x) \quad \text{(a.s)}, \end{aligned}$$

where $W^{(b,1)}(x)$ and $W^{(b,2)}(x)$ are respectively the independent limiting distributions of $W^{(b)}_{[n\theta]}(x)$ and $W^{(*,b)}_{n-[n\theta]}(x)$ using Theorem 3.2.2. Under $H_0^{(\theta)}$, the limiting distribution of $V_n^{(2,\theta,b)}(\cdot)$ is the same as the distribution of $L_{W^{(1)},\theta}(\cdot)$ defined before in equation (2.2.11). Under $H_1^{(\theta)}$, the limit is a mean zero Gaussian process with covariance function

$$\begin{aligned} \sigma_b(x,y) &= \theta(1-\theta)^2 \sum_{i\in\mathbf{Z}} Cov\left(\mathbf{I}(Y_0 \le x), \mathbf{I}(Y_i \le y)\right) \\ &+ \theta^2(1-\theta) \sum_{i\in\mathbf{Z}} Cov\left(\mathbf{I}(Z_0 \le x), \mathbf{I}(Z_i \le y)\right). \end{aligned}$$

To test the pair $\left(H_0^{(\theta)}, H_1^{(\theta)}\right)$, we will be using the bootstrapped counterparts of the statistics seen earlier in Remark 2.2.5

$$T^{(b)}_{3,\theta} = \sup_{x\in\mathbb{R}} \left|V_n^{(2,\theta,b)}(x)\right|$$

or

$$T^{(b)}_{4,\theta} = \int_{\mathbb{R}} \left|V_n^{(2,\theta,b)}(x)\right|^2 dF_n(x).$$

The continuous mapping theorem yields weak convergence of both test statistics. More precisely, we get the following conditional limiting distributions under both $H_0^{(\theta)}$ and $H_1^{(\theta)}$:

$$\begin{aligned} T^{(b)}_{3,\theta} &\xrightarrow{d} \sup_{x\in\mathbb{R}} \left|\sqrt{\theta}(1-\theta)W^{(b,1)}(x) - \theta\sqrt{1-\theta}W^{(b,2)}(x)\right| \quad \text{(a.s)} \\ T^{(b)}_{4,\theta} &\xrightarrow{d} \int_{\mathbb{R}} \left|\sqrt{\theta}(1-\theta)W^{(b,1)}(x) - \theta\sqrt{1-\theta}W^{(b,2)}(x)\right|^2 dH(x) \quad \text{(a.s)}, \end{aligned}$$

where $H = \theta F^{(1)} + (1-\theta)F^{(2)}$.

As expected, we remark that the test and bootstrapped statistics converge respectively under $H_0^{(\theta)}$ to the same limits as in Proposition 2.2.6. We also note that under $H_1^{(\theta)}$, while the test statistics go to infinity, the bootstrapped statistics converge weakly to finite limits which allow us to tabulate suitable critical values by constructing repeated moving block bootstrap samples.

3.3 Proof of Theorem 3.2.2

In order to establish the validity of the Theorem 3.2.2, we shall prove that its conditions imply those of Theorem 3.2.1. In the sequel, C denotes a generic constant which is independent of any other parameter involved in the analysis and may change its value from line to line. We require some preliminary results.

Lemma 3.3.1 *Assume Assumptions 1.1.1.1, 3.1.1.2 and 3.1.1.3 hold. Then, there exists a sequence* $b_j \geq 0$, $j \in \mathbb{Z}$, *such that* $\sum_{j\in\mathbb{Z}} b_j < \infty$ *and*

$$|Cov(R_0(x), R_j(y))| \leq b_j.$$

Proof:

For $j > 0$

$$\begin{aligned} |Cov(R_0(x), R_j(y))| &= \left| E\left[\sum_{h\geq 0}\sum_{h'\geq 0} U_{0,h}(x)U_{j,h'}(y)\right]\right| \\ &= \left| E\left[\sum_{h\geq 0} U_{0,h}(x)U_{j,j+h}(y)\right]\right| \\ &\leq \sum_{h\geq 0} E\left[|U_{0,h}(x)U_{j,j+h}(y|)|\right] \\ &\leq \sum_{h\geq 0} E^{\frac{1}{2}}[U_{0,h}^2(x)]E^{\frac{1}{2}}[U_{j,j+h}^2(y)], \end{aligned}$$

where the second line follows from (1.2.1).

From Lemma 1.2.2 and since $|U_{0,h}(x)| \leq 1$, for $j > h_0$

$$\begin{aligned}
\sum_{h=0}^{h_0} E^{\frac{1}{2}}[U_{0,h}^2(x)]E^{\frac{1}{2}}[U_{j,j+h}^2(y)] &\leq \sum_{h=0}^{h_0} E^{\frac{1}{2}}[U_{j,j+h}^2(y)] \\
&\leq \sum_{h=0}^{h_0} E^{\frac{1}{2}}\left[C^2|a_{j+h}|^{2\gamma}(1+|\xi_{-h}|^{\gamma})^2\right] \\
&\leq C\sum_{h=0}^{h_0}|a_{j+h}|^{\gamma} \\
&= b_j'.
\end{aligned}$$

Using again Lemma 1.2.2 for $j > h_0$

$$\begin{aligned}
\sum_{h\geq h_0+1} E^{\frac{1}{2}}[U_{0,h}^2(x)]E^{\frac{1}{2}}[U_{j,j+h}^2(y)] &\leq C\sum_{h\geq h_0+1}|a_h|^{\gamma}|a_{j+h}|^{\gamma} \\
&= b_j''.
\end{aligned}$$

By Assumption 1.1.1.1, we can see that

$$\begin{aligned}
\sum_{j>h_0}(b_j'+b_j'') &\leq C\left[\sum_{j\geq 0}\sum_{h=0}^{h_0}|a_{j+h}|^{\gamma}+\sum_{j\geq 0}\sum_{h\geq h_0+1}|a_h|^{\gamma}|a_{j+h}|^{\gamma}\right] \\
&= C\left[\sum_{h=0}^{h_0}\sum_{j\geq 0}|a_{j+h}|^{\gamma}+\sum_{h\geq h_0+1}|a_h|^{\gamma}\sum_{j\geq 0}|a_{j+h}|^{\gamma}\right] \\
&\leq C\left[\sum_{h=0}^{h_0}\sum_{j\geq 0}|a_{j+h}|^{\gamma}+\sum_{h\geq 0}|a_h|^{\gamma}\right] \\
&< \infty.
\end{aligned}$$

Thus, for $j > 0$

$$|Cov(R_0(x), R_j(y))| \leq b_j,$$

where $b_j = b_j' + b_j''$ for $j > h_0$ and $b_j = 1$ for $0 \leq j \leq h_0$, since we know that $|R_j(x)| = |\mathbf{I}(X_j \leq x) - F(x)| \leq 1$.

Using stationarity leads to a similar result for $j < 0$ and completes the proof of Lemma 3.3.1.

□

Proposition 3.3.2 *Suppose Assumptions 3.1.1 hold. Then, equation (3.2.2) is in force.*

Proof: We first note that Assumption 3.1.1.1 ensures that

$$\begin{aligned} m\sum_{j\geq m}|a_j|^\gamma &\leq \sum_{j\geq m} j|a_j|^\gamma \\ &\leq C. \end{aligned}$$

In order to use Lemma 3.3.1, we need to consider two cases depending on the position of m with respect to h_0.

• $m > h_0$

In this case, we have for $\delta \leq 1$,

$$\begin{aligned} & m^\delta \sum_{j\geq m} |Cov(\mathbf{I}(x < X_0 \leq y), \mathbf{I}(x < X_j \leq y))| \\ &= m^\delta \sum_{j\geq m} |Cov(R_0(y,x), R_j(y,x))| \\ &\leq Cm^\delta \sum_{j\geq m} b_j \\ &= Cm^\delta \sum_{j\geq m}\left(\sum_{h=0}^{h_0}|a_{j+h}|^\gamma + \sum_{h\geq h_0+1}|a_h|^\gamma|a_{j+h}|^\gamma\right) \\ &\leq C\left(\sum_{h=0}^{h_0} m^\delta\sum_{j\geq m}|a_j|^\gamma + \sum_{h\geq h_0+1}|a_h|^\gamma m^\delta \sum_{j\geq m}|a_j|^\gamma\right) \\ &\leq C\left(h_0 + \sum_{h\geq h_0+1}|a_h|^\gamma\right) \\ &\leq C. \end{aligned} \tag{3.3.1}$$

• $m \leq h_0$

Using again Lemma 3.3.1, we get in this case for $\delta \leq 1$,

$$\begin{aligned} & m^\delta \sum_{j\geq m} |Cov(\mathbf{I}(x < X_0 \leq y), \mathbf{I}(x < X_j \leq y))| \\ &\leq Cm^\delta \sum_{j\geq m} b_j \end{aligned}$$

$$
\begin{aligned}
&= Cm^{\delta}\left(\sum_{j=m}^{h_0} b_j + \sum_{j\geq h_0+1} b_j\right) \\
&\leq Cm^{\delta}\left[h_0 + \sum_{j\geq h_0+1}\left(\sum_{h=0}^{h_0}|a_{j+h}|^{\gamma} + \sum_{h\geq h_0+1}|a_h|^{\gamma}|a_{j+h}|^{\gamma}\right)\right] \\
&\leq C\left[h_0^{\delta+1} + m^{\delta}\left(\sum_{h=0}^{h_0}\sum_{j\geq m}|a_j|^{\gamma} + \sum_{h\geq h_0+1}|a_h|^{\gamma}\sum_{j\geq m}|a_j|^{\gamma}\right)\right] \\
&\leq C\left(h_0^{\delta+1} + h_0 + \sum_{h\geq h_0+1}|a_h|^{\gamma}\right) \\
&\leq C. \qquad (3.3.2)
\end{aligned}
$$

From (3.3.1) and (3.3.2), we can deduce that for $\delta \leq 1$

$$\sum_{j\geq m}|Cov(\mathbf{I}(x < X_0 \leq y), \mathbf{I}(x < X_j \leq y))| \leq Cm^{-\delta},$$

and equation (3.2.2) is in force. This completes the proof of Proposition 3.3.2.

□

Lemma 3.3.3 *Under Assumptions 1.1.1.1, 3.1.1.2 and 3.1.1.3, we have*

$$E\left|E\left[R_i(x,y)|\mathcal{F}_j\right]\right| \leq CA_{i-j}(\gamma) \quad \textit{for} \quad i-j > h_0.$$

Proof: As in Lemma 1.2.4, put

$$X_j = X_j^h + \widetilde{X}_j^h \quad \text{where} \quad X_j^h = \sum_{k=0}^{h} a_k\xi_{j-k} \quad \text{and} \quad \widetilde{X}_j^h = \sum_{k>h} a_k\xi_{j-k}.$$

Also, denote by F_h and $\widetilde{F}_h$ the distribution functions of X_j^h and $\widetilde{X}_j^h$ respectively. Because of independence of the ξ_k's, the distribution function of X_j can be seen as the convolution of F_h and $\widetilde{F}_h$, namely $F = F_h * \widetilde{F}_h$.

Remark now that for $j \geq 1$,

$$
\begin{aligned}
E[R_j(x)|\mathcal{F}_0] &= E[\mathbf{I}(X_j \leq x) - F(x)|\mathcal{F}_0] \\
&= E\left[\mathbf{I}\left(X_j^{j-1} \leq x - \widetilde{X}_j^{j-1}\right) - F(x)\middle|\,\mathcal{F}_0\right]
\end{aligned}
$$

$$
\begin{aligned}
&= F_{j-1}\left(x-\widetilde{X}_j^{j-1}\right)-\int_{\mathbb{R}} F_{j-1}(x-u)d\widetilde{F}_{j-1}(u) \\
&= \int_{\mathbb{R}} F_{j-1}\left(x-\widetilde{X}_j^{j-1}\right)-F_{j-1}(x-u)d\widetilde{F}_{j-1}(u).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
|E[R_j(x,y)|\mathcal{F}_0]| &= |E[R_j(x)|\mathcal{F}_0]-E[R_j(y)|\mathcal{F}_0]| \\
&\leq \int_{\mathbb{R}} \left|F_{j-1}\left(x-\widetilde{X}_j^{j-1}\right)-F_{j-1}(x-u)\right| d\widetilde{F}_{j-1}(u) \\
&+ \int_{\mathbb{R}} \left|F_{j-1}\left(y-\widetilde{X}_j^{j-1}\right)-F_{j-1}(y-u)\right| d\widetilde{F}_{j-1}(u).
\end{aligned}
$$

By Assumption 3.1.1.2, for $j > h_0$, F_{j-1} is differentiable with a bounded density (See Comments 1.1.2).

Thus, for $j > h_0$ and since the integrand is bounded by 1,

$$
\begin{aligned}
|E[R_j(x,y)|\mathcal{F}_0]| &\leq \int_{\mathbb{R}} C\left|\widetilde{X}_j^{j-1}-u\right| \wedge 1 \; d\widetilde{F}_{j-1}(u) \\
&\leq C\int_{\mathbb{R}} \left|\widetilde{X}_j^{j-1}-u\right|^{\gamma} d\widetilde{F}_{j-1}(u) \\
&\leq C\int_{\mathbb{R}} \left(\left|\widetilde{X}_j^{j-1}\right|^{\gamma}+|u|^{\gamma}\right) d\widetilde{F}_{j-1}(u) \\
&= C\left(\left|\widetilde{X}_j^{j-1}\right|^{\gamma}+E\left[\left|\widetilde{X}_j^{j-1}\right|^{\gamma}\right]\right).
\end{aligned}
$$

Consequently, one gets

$$
\begin{aligned}
E|E[R_j(x,y)|\mathcal{F}_0]| &\leq CE\left[\left|\widetilde{X}_j^{j-1}\right|^{\gamma}\right] \\
&= CE\left[\left|\sum_{k\geq j} a_k\xi_{j-k}\right|^{\gamma}\right] \\
&\leq C\left(E\left[\sum_{k\geq j} |a_k|^{\gamma}\,|\xi_{j-k}|^{\gamma}\right]\right) \\
&\leq C\sum_{k\geq j}|a_k|^{\gamma} \\
&= CA_j(\gamma),
\end{aligned}
$$

where the third and the fourth lines follow since $\gamma \leq 1$ and from Assumption 3.1.1.3 respectively. Therefore, by stationarity, we have for $i - j > h_0$

$$E\left|E\left[R_i(x,y)|\mathcal{F}_j\right]\right| \leq CA_{i-j}(\gamma).$$

This completes the proof of Lemma 3.3.3.

□

Corollary 3.3.4 *Under Assumptions 1.1.1.1, 3.1.1.2 and 3.1.1.3, we have*

$$|Cov\left(R_i(x,y), R_j(x,y)\right)| \leq CA_{i-j}(\gamma) \quad \textit{for} \quad i - j > h_0.$$

Proof: We have

$$\begin{aligned} |Cov\left(R_i(x,y), R_j(x,y)\right)| &= \left|E\left[R_i(x,y)R_j(x,y)\right]\right| \\ &= \left|E\left[R_j(x,y)E\left[R_i(x,y)|\mathcal{F}_j\right]\right]\right| \\ &\leq E\left|R_j(x,y)E\left[R_i(x,y)|\mathcal{F}_j\right]\right| \\ &\leq CA_{i-j}(\gamma), \end{aligned}$$

where the last line follows from Lemma 3.3.3 and remarking that $|R_j(x,y)| \leq 1$.

□

Lemma 3.3.5 *Assume Assumptions 1.1.1.1, 3.1.1.2 and 3.1.1.3 hold. Then, for* $i \wedge h > j + h_0$

$$E\left|E\left[\mathbf{I}_{i,h} - E\left[\mathbf{I}_{i,h}\right]|\mathcal{F}_j\right]\right| \leq C\left(A_{i-j}(\gamma) + A_{h-j}(\gamma)\right),$$

where $\mathbf{I}_{i,h} = \mathbf{I}(x < X_i \leq y)\mathbf{I}(x < X_h \leq y)$.

Proof: For $u, v > 0$, let $F_{u,v}$ denote the joint distribution of

$$\left(X_i^u, X_{i+v}^{u+v}\right) = \left(\sum_{k=0}^{u} a_k \xi_{i-k}, \sum_{r=0}^{u+v} a_r \xi_{i+v-r}\right).$$

We also define $\widetilde{F}_{u,v}$ as the joint distribution of

$$\left(\widetilde{X}_i^u, \widetilde{X}_{i+v}^{u+v}\right) = \left(\sum_{k\geq u+1} a_k \xi_{i-k}, \sum_{r \geq u+v+1} a_r \xi_{i+v-r}\right).$$

Hence, by independence and stationarity, the joint distribution of (X_i, X_{i+v}) is defined to be $\overline{F}_v = F_{u,v} * \widetilde{F}_{u,v}$.

Remark first that for $u \geq h_0$ and $v \geq 0$, we have

$$\begin{aligned}
& |F_{u,v}(x,y) - F_{u,v}(x',y')| \\
\leq\ & |F_{u,v}(x,y) - F_{u,v}(x',y)| + |F_{u,v}(x',y) - F_{u,v}(x',y')| \\
=\ & |P\left(X_i^u \leq x, X_{i+v}^{u+v} \leq y\right) - P\left(X_i^u \leq x', X_{i+v}^{u+v} \leq y\right)| \\
+\ & |P\left(X_i^u \leq x', X_{i+v}^{u+v} \leq y\right) - P\left(X_i^u \leq x', X_{i+v}^{u+v} \leq y'\right)| \\
=\ & P\left(x' < X_i^u \leq x, X_{i+v}^{u+v} \leq y\right) \vee P\left(x < X_i^u \leq x', X_{i+v}^{u+v} \leq y\right) \\
+\ & P\left(X_i^u \leq x', y' < X_{i+v}^{u+v} \leq y\right) \vee P\left(X_i^u \leq x', y < X_{i+v}^{u+v} \leq y'\right) \\
\leq\ & P\left(x' < X_i^u \leq x\right) \vee P\left(x < X_i^u \leq x'\right) \\
+\ & P\left(y' < X_{i+v}^{u+v} \leq y\right) \vee P\left(y < X_{i+v}^{u+v} \leq y'\right) \\
=\ & |F_u(x) - F_u(x')| + |F_{u+v}(y) - F_{u+v}(y')| \\
\leq\ & C\left(|x - x'| + |y - y'|\right),
\end{aligned}$$

where the last line follows from the mean value theorem and Assumption 3.1.1.2.

Suppose now for instance that $h_0 < i \leq h$ and let $u = i - 1$ and $v = h - i$, so that $i + v = h$ and $u + v = h - 1$.

Then

$$\begin{aligned}
& |E\left[\mathbf{I}\left(X_i \leq x\right)\mathbf{I}\left(X_h \leq y\right) - E\left[\mathbf{I}\left(X_i \leq x\right)\mathbf{I}\left(X_h \leq y\right)\right]|\mathcal{F}_0\right]| \\
=\ & \left|E\left[\mathbf{I}\left(X_i^{i-1} \leq x - \widetilde{X}_i^{i-1}\right)\mathbf{I}\left(X_h^{h-1} \leq y - \widetilde{X}_h^{h-1}\right)\right.\right. \\
& \left.\left. -E\left[\mathbf{I}\left(X_i^{i-1} \leq x - \widetilde{X}_i^{i-1}\right)\mathbf{I}\left(X_h^{h-1} \leq y - \widetilde{X}_h^{h-1}\right)\right]\right|\mathcal{F}_0\right]\Big| \\
=\ & \left|\iint_{\mathbb{R}^2} F_{i-1,h-i}\left(x - \widetilde{X}_i^{i-1}, y - \widetilde{X}_h^{h-1}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& \left. -F_{i-1,h-i}(x-s,y-t)d\widetilde{F}_{i-1,h-1}(s,t)\right| \\
\leq\ & \iint_{\mathbb{R}^2} C\left(\left|\widetilde{X}_i^{i-1}-s\right|+\left|\widetilde{X}_h^{h-1}-t\right|\right)\wedge 1\ d\widetilde{F}_{i-1,h-1}(s,t) \\
\leq\ & \iint_{\mathbb{R}^2} \left(C\left|\widetilde{X}_i^{i-1}-s\right|\wedge 1\right)+\left(C\left|\widetilde{X}_h^{h-1}-t\right|\wedge 1\right)d\widetilde{F}_{i-1,h-1}(s,t) \\
\leq\ & C\iint_{\mathbb{R}^2} \left|\widetilde{X}_i^{i-1}-s\right|^{\gamma}+\left|\widetilde{X}_h^{h-1}-t\right|^{\gamma} d\widetilde{F}_{i-1,h-1}(s,t).
\end{aligned}
$$

One can verify in a similar way as in Lemma 3.3.3 that

$$
\begin{aligned}
& E\left|E\left[\mathbf{I}\left(X_i\leq x\right)\mathbf{I}\left(X_h\leq y\right)-E\left[\mathbf{I}\left(X_i\leq x\right)\mathbf{I}\left(X_h\leq y\right)\right]|\mathcal{F}_0\right]\right| \\
\leq\ & CE\left[\left|\widetilde{X}_i^{i-1}\right|^{\gamma}+\left|\widetilde{X}_h^{h-1}\right|^{\gamma}\right] \\
\leq\ & C\left(A_i(\gamma)+A_h(\gamma)\right).
\end{aligned}
$$

Hence, by stationarity, for $i\wedge h>j+h_0$

$$
\begin{aligned}
& E\left|E\left[\mathbf{I}\left(X_i\leq x\right)\mathbf{I}\left(X_h\leq y\right)-E\left[\mathbf{I}\left(X_i\leq x\right)\mathbf{I}\left(X_h\leq y\right)\right]|\mathcal{F}_j\right]\right| \\
\leq\ & C\left(A_{i-j}(\gamma)+A_{h-j}(\gamma)\right).
\end{aligned}
$$

Therefore,

$$
E\left|E\left[\mathbf{I}_{i,h}-E\left[\mathbf{I}_{i,h}\right]|\mathcal{F}_j\right]\right|\leq C\left(A_{i-j}(\gamma)+A_{h-j}(\gamma)\right)\quad\text{for}\quad i\wedge h>j+h_0.
$$

This completes the proof of Lemma 3.3.5.

□

Lemma 3.3.6 *Assume Assumptions 1.1.1.1, 3.1.1.2 and 3.1.1.3 hold. Then, for* $i-j>h_0$

$$
E\left|E\left[Y_{li}^2(x,y)-E\left[Y_{li}^2(x,y)\right]|\mathcal{F}_j\right]\right|\leq Cl^2A_{i-j}(\gamma).
$$

Proof: Recall first that

$$
Y_{li}(x,y)\ =\ \sum_{k=i}^{i+l-1}\left(\mathbf{I}(x<X_k\leq y)-(F(y)-F(x))\right)
$$

$$= \sum_{k=i}^{i+l-1} R_k(y,x).$$

We reintroduce the notation used in the preceding lemma:

$$\begin{aligned}
\mathbf{I}_{i,h} &= \mathbf{I}(x < X_i \leq y)\mathbf{I}(x < X_h \leq y) \\
\mathbf{I}_{i} &= \mathbf{I}(x < X_i \leq y) \\
\mathbf{I}_{h} &= \mathbf{I}(x < X_h \leq y),
\end{aligned}$$

and

$$F(x,y) = F(x) - F(y).$$

Then

$$\begin{aligned}
E\left[Y_{li}^2(x,y)|\mathcal{F}_j\right] &= E\left[\sum_{k=i}^{i+l-1}\sum_{h=i}^{i+l-1}\big(\mathbf{I}_{k,h} - F(y,x)\left(\mathbf{I}_k + \mathbf{I}_h\right) + F^2(y,x)\big)\,|\mathcal{F}_j\right] \\
= \quad E&\left[\sum_{k=i}^{i+l-1}\mathbf{I}_k + 2\sum_{k=i}^{i+l-2}\sum_{h=k+1}^{i+l-1}\mathbf{I}_{k,h} - 2lF(y,x)\sum_{k=i}^{i+l-1}\mathbf{I}_k|\mathcal{F}_j\right] + l^2F^2(y,x).
\end{aligned}$$

Hence, we get

$$E\left[Y_{li}^2(x,y)\right] = lF(y,x) + 2\sum_{k=i}^{i+l-2}\sum_{h=k+1}^{i+l-1} E\left[\mathbf{I}_{k,h}\right] - l^2F^2(y,x).$$

Therefore,

$$\begin{aligned}
&E\left[Y_{li}^2(x,y) - E\left[Y_{li}^2(x,y)\right]|\mathcal{F}_j\right] \\
&= \sum_{k=i}^{i+l-1} E\left[R_k(y,x)|\mathcal{F}_j\right] + 2\sum_{k=i}^{i+l-2}\sum_{h=k+1}^{i+l-1} E\left[\mathbf{I}_{k,h} - E\left[\mathbf{I}_{h,k}\right]|\mathcal{F}_j\right] \\
&- \quad 2lF(y,x)\sum_{k=i}^{i+l-1} E\left[R_k(y,x)|\mathcal{F}_j\right] \\
&= \quad (1 - 2lF(y,x))\sum_{k=i}^{i+l-1} E\left[R_k(y,x)|\mathcal{F}_j\right] \\
&+ \quad 2\sum_{k=i}^{i+l-2}\sum_{h=k+1}^{i+l-1} E\left[\mathbf{I}_{k,h} - E\left[\mathbf{I}_{h,k}\right]|\mathcal{F}_j\right].
\end{aligned}$$

Combining this equality with Lemma 3.3.3, Lemma 3.3.5 and the fact that the sequence $\{A_i(\gamma)\}_{i\in\mathbb{N}}$ is non-increasing, we get

$$\begin{aligned}
& E\left|E\left[Y_{li}^2(x,y) - E\left[Y_{li}^2(x,y)\right]|\mathcal{F}_j\right]\right| \\
\leq\ & Cl\sum_{k=i}^{i+l-1} A_{k-j}(\gamma) + C\sum_{k=i}^{i+l-2}\sum_{h=k+1}^{i+l-1}\left(A_{k-j}(\gamma)+A_{h-j}(\gamma)\right) \\
\leq\ & Cl^2 A_{i-j}(\gamma).
\end{aligned}$$

This completes the proof of Lemma 3.3.6.

□

Proposition 3.3.7 *Suppose Assumptions 3.1.1 hold. Then, equation (3.2.3) is in force.*

Proof: Remark first that since $|R_j(y,x)| \leq 1$, then

$$|Y_{li}(x,y)| = \left|\sum_{j=i}^{i+l-1} R_j(y,x)\right| \leq l. \tag{3.3.3}$$

On the other hand, by stationarity and Lemma 3.3.1 we get

$$\begin{aligned}
E[Y_{li}^2(x,y)] &= E[Y_{l1}^2(x,y)] \\
&= \sum_{j=1}^{l}\sum_{k=1}^{l} E[R_j(y,x)R_k(y,x)] \\
&= \sum_{j=1}^{l} E[R_j^2(y,x)] + 2\sum_{j=1}^{l-1}\sum_{k=j+1}^{l} E[R_j(y,x)R_k(y,x)] \\
&\leq l + C\sum_{j=1}^{l-1}\sum_{k=j+1}^{l} b_{k-j} \\
&\leq l + C\sum_{j=1}^{l}\sum_{k\geq j+1} b_{k-j} \\
&\leq Cl.
\end{aligned}\tag{3.3.4}$$

Write

$$
\begin{aligned}
& Var\left(\sum_{i=1}^{m} Y_{li}^2(x,y)\right) \\
= & \sum_{i=1}^{m} Var\left(Y_{li}^2(x,y)\right) + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} Cov\left(Y_{li}^2(x,y), Y_{lj}^2(x,y)\right) \\
\leq & \sum_{i=1}^{m} E\left[Y_{li}^4(x,y)\right] + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} Cov\left(Y_{li}^2(x,y), Y_{lj}^2(x,y)\right) \\
\leq & \sum_{i=1}^{m} E\left[Y_{li}^4(x,y)\right] + 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{i+l+h_0} Cov\left(Y_{li}^2(x,y), Y_{lj}^2(x,y)\right) \\
& +2\sum_{i=1}^{m-1}\sum_{j=i+l+h_0+1}^{m} Cov\left(Y_{li}^2(x,y), Y_{lj}^2(x,y)\right) \\
= & I + II + III.
\end{aligned}
$$

Note now, using equations (3.3.3) and (3.3.4), that

$$
\begin{aligned}
I &= \sum_{i=1}^{m} E\left[Y_{li}^4(x,y)\right] \\
&= \sum_{i=1}^{m} E\left[Y_{li}^2(x,y) Y_{li}^2(x,y)\right] \\
&\leq l^2 \sum_{i=1}^{m} E\left[Y_{li}^2(x,y)\right] \\
&\leq Cml^3,
\end{aligned}
$$

and

$$
\begin{aligned}
II &= 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{i+l+h_0} Cov\left(Y_{li}^2(x,y), Y_{lj}^2(x,y)\right) \\
&\leq 2\sum_{i=1}^{m-1}\sum_{j=i+1}^{i+l+h_0} E\left[Y_{li}^2(x,y) Y_{lj}^2(x,y)\right] \\
&\leq Cml^4.
\end{aligned}
$$

Finally, we make use of Lemma 3.3.6 and Assumption 3.1.1.1, and noting that $Y_{li}(x,y)$ is $\mathcal{F}_{i+l-1}$-measurable to get

$$\begin{aligned}
III &= 2\sum_{i=1}^{m-1}\sum_{j=i+l+h_0+1}^{m} Cov\left(Y_{li}^2(x,y), Y_{lj}^2(x,y)\right) \\
&\leq 2\sum_{i=1}^{m-1}\sum_{j=i+l+h_0+1}^{m} E\left[Y_{li}^2(x,y)E\left[Y_{lj}^2(x,y) - E\left[Y_{lj}^2(x,y)\right]|\mathcal{F}_{i+l}\right]\right] \\
&\leq Cl^2\sum_{i=1}^{m-1}\sum_{j=i+l+h_0+1}^{m} E\left[\left|E\left[Y_{lj}^2(x,y) - E\left[Y_{lj}^2(x,y)\right]|\mathcal{F}_{i+l}\right]\right|\right] \\
&\leq Cl^4\sum_{i=1}^{m-1}\sum_{j=i+l+h_0+1}^{m} A_{j-i-l}(\gamma) \\
&\leq Cl^4\sum_{i=1}^{m-1}\sum_{k\geq 0} A_k(\gamma) \\
&\leq Cml^4
\end{aligned}$$

Combine now I, II and III to complete the proof of Proposition 3.3.7.

□

Equipped with the necessary tools, we are now ready to prove Theorem 3.2.2, the functional central limit theorem for the bootstrapped empirical process.

Proof of Theorem 3.2.2

We first notice that the limiting process has the same covariance structure as the limit for the original empirical process.

Remark that our demonstration of Proposition 3.3.2 and Proposition 3.3.7 involved the original variables X_i. However, if we define $U_i = F(X_i)$ then

- The U_i's are uniformly distributed over $[0, 1]$.
- The U_i's satisfy equations (3.2.2) and (3.2.3),

since for $s = F(x)$ and $t = F(y)$

$$Cov\left(\mathbf{I}(s < U_0 \leq t), \mathbf{I}(s < U_i \leq t)\right) = Cov\left(\mathbf{I}\left(x < X_0 \leq y\right), \mathbf{I}\left(x < X_i \leq y\right)\right)$$

and

$$
\begin{aligned}
Y_{li}(s,t) &= \sum_{j=i}^{i+l-1} \mathbf{I}(s < U_i \leq t) - (t-s) \\
&= \sum_{j=i}^{i+l-1} \mathbf{I}(x < X_i \leq y) - (F(y) - F(x)).
\end{aligned}
$$

Therefore, the conclusion of Theorem 3.2.1 for the bootstrapped empirical process $\overline{W_n^{(b)}(\cdot)}$ based on U_i holds. The result can now be derived directly for $\overline{W_n^{(b)}(F(x))} = W_n^{(b)}(x)$ (See [5] and [47] for more details).

■

Chapter 4

Functional Central Limit Theorems for the Sequential Bootstrapped Empirical Process

In this chapter, we derive the limiting distribution of a sequential bootstrapped empirical process. We first provide all the necessary definitions and notation that will be needed to introduce our construction of the sequential bootstrapped empirical process. Our first result, Theorem 4.1.1, is a sequential version of Theorem 3.2.1. In Theorem 4.2.2, we examine the convergence of the sequential bootstrapped empirical processes in the case where a change-point is involved in the analysis. It will be seen that the bootstrap remains valid under a sequence of converging alternatives, and in Section 4.3 this will be applied to our test statistics . The proofs of the theorems will appear in the last section.

4.1 FCLT for the sequential bootstrapped empirical process

Recall the triangular array defined in Section 3.1

$$X_{ni} := \begin{cases} X_i & \text{if} \quad 1 \le i \le n \\ X_{i-n} & \text{if} \quad n+1 \le i \le n+l-1. \end{cases}$$

Define now the i^{th} block empirical distribution to be

$$F_{l,i}(x) = \frac{1}{l} \sum_{j=i}^{i+l-1} \mathbf{I}(X_{nj} \le x), \text{ and } F_n(x) = F_{n,1}(x).$$

Using the definition (3.1.2) of the bootstrapped empirical distribution and the fact that $\sum_{i=1}^{n} \mathbf{I}(I_{nj} = i) = 1$, we can rewrite the bootstrapped empirical process defined earlier in (3.1.1) as follows

$$\begin{aligned} W_n^{(b)}(x) &= \sqrt{n}\left[F_n^{(b)}(x) - F_n(x)\right] \\ &= \sqrt{n}\left[\frac{1}{k}\sum_{j=1}^{k} F_{l,I_{nj}}(x) - F_n(x)\right] \\ &= \frac{\sqrt{n}}{k}\sum_{j=1}^{k}\left[\sum_{i=1}^{n} \mathbf{I}(I_{nj} = i)(F_{l,i}(x) - F_n(x))\right]. \end{aligned} \quad (4.1.1)$$

Contrary to some authors, as in [32], who proposed a sequential bootstrapped empirical model based on the sample size n, we choose to use the number of blocks k. This ensures that the terms in our partial sum are independent and identically distributed conditionally on the observations. Hence, the sequential bootstrapped empirical process is defined, for $(x, s) \in (\mathbb{R} \times [0, 1])$ and $n = lk$, as

$$\begin{aligned} W_n^{(b)}(x, s) &= \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\left[\sum_{i=1}^{n} \mathbf{I}(I_{nj} = i)(F_{l,i}(x) - F_n(x))\right] \\ &= \frac{l[ks]}{\sqrt{n}}\left(F_{l[ks]}^{(b)}(x) - F_n(x)\right). \end{aligned} \quad (4.1.2)$$

We are now in position to state the theorem that establishes weak convergence of the sequential bootstrapped empirical process, conditionally on the observations, to a Gaussian process almost surely. In fact, the following result is a sequential version of Theorem 3.2.1:

Theorem 4.1.1 *Let $\{X_n\}_{n\in\mathbb{Z}}$, be a stationary sequence of random variables. Let l_n, k_n be any sequences of natural numbers as in the statement of Theorem 3.2.1 and assume that conditions (3.2.2) and (3.2.3) are satisfied. Then, as $n \to \infty$,*

$$W_n^{(b)}(\cdot,\cdot) \stackrel{\mathbf{D}}{\to} W(\cdot,\cdot)$$

P^*-almost surely in the space $D(\mathbb{R}\times[0,1])$ endowed with Skorohod topology, where $W(\cdot,\cdot)$ is a Gaussian process with zero mean and covariance

$$Cov\left(W(x,s),W(y,t)\right) = (s\wedge t)\sum_{i\in\mathbb{Z}} Cov\left(\mathbf{I}(X_0\le x),\mathbf{I}(X_i\le y)\right). \tag{4.1.3}$$

Since the distribution of X_0 is continuous, then $P^*(W(\cdot,\cdot)\in C(\mathbb{R}\times[0,1])) = 1$.

We remark that the limiting process has the same covariance structure as the limit of the original sequential empirical process given in Theorem 1.2.1.

It was shown in Proposition 3.3.2 and Proposition 3.3.7 that Assumptions 3.1.1 imply Peligrad's conditions (3.2.2) and (3.2.3). Therefore, Theorem 4.1.2 is an immediate consequence of the theorem above.

Theorem 4.1.2 *Let X_n, $n\in\mathbb{Z}$, be a stationary causal sequence satisfying Assumptions 3.1.1.1-3.1.1.3 and suppose that $\{l_n\}_{n\in\mathbb{N}}$ and $\{k_n\}_{n\in\mathbb{N}}$ are any sequences as in the statement of Theorem 3.2.1. Then, as $n\to\infty$,*

$$W_n^{(b)}(\cdot,\cdot) \stackrel{\mathbf{D}}{\to} W(\cdot,\cdot)$$

P^*-almost surely in the Skorohod topology on $D(\mathbb{R}\times[0,1])$, where $W(\cdot,\cdot)$ is a Gaussian process with zero mean and covariance structure given by (4.1.3).
Moreover, since the distribution of X_0 is continuous, $P^*(W(t)\in C(\mathbb{R}\times[0,1])) = 1$.

4.2 FCLT for the sequential bootstrapped empirical process with a change-point

Let us first recall the model introduced in Chapter 2:

$$X_i = \begin{cases} Y_i & \text{for } \ i \le [n\theta_n] \\ Z_i & \text{otherwise,} \end{cases}$$

where

$$Y_i = \sum_{j=0} a_j^{(1)} \xi_{i-j} \quad \text{and} \quad Z_i = \sum_{j=0} a_j^{(2)} (\xi_{i-j} + b)$$

are two stationary causal linear processes. b is a constant, $0 \le \theta_n \le 1$ such that $\lim_{n\to\infty} \theta_n = \theta$. We will proceed under the following assumptions:

Assumptions 4.2.1

1. Let $\{a_j^{(\ell)}, j \in \mathbb{Z}\}$ be a sequence of non-random weights, infinitely many of which are non-zero, satisfying

$$\sum_{j\ge 0} j|a_j^{(\ell)}|^\gamma < \infty, \ \ell = 1, 2 \quad \text{for} \quad \gamma \in (0, 1].$$

2. There exist constants $C < \infty$ and $\Delta > 0$ such that for all $u \in \mathbb{R}$

$$|E \exp(iu\xi_0)| \le \frac{C}{(1 + |u|)^\Delta}.$$

3. $E[|\xi_0|^{2\gamma}] < \infty$.

In order to ensure that the sequential bootstrapped empirical process converges, we must assume that we have a converging alternative: i.e. $\theta_n \to 0$ or 1. In fact, it is needed to control the conditional variance of the sequential bootstrapped empirical process.

Two cases emerge depending on the rate of convergence of θ_n towards 0 or 1. It will be seen that the assumptions considered here for the converging alternatives are simple and seem more natural than the one used by Inoue in [32].

Theorem 4.2.2 *Suppose that Assumptions 4.2.1.1-4.2.1.3 hold and that* $\{l_n\}_{n\in\mathbb{N}}$ *and* $\{k_n\}_{n\in\mathbb{N}}$ *are any sequences as in the statement of Theorem 3.2.1. Then, as* $n \to \infty$,

$$W_n^{(b)}(\cdot,\cdot) \xrightarrow{\mathbf{D}} W(\cdot,\cdot)$$

P^**-almost surely in the space* $D(\mathbb{R}\times[0,1])$ *endowed with Skorohod topology, where* $W(\cdot,\cdot)$ *is a Gaussian process with zero mean and covariance*

$$\begin{aligned} &Cov\left(W(x,s), W(y,t)\right) \\ &= \begin{cases} (s\wedge t)\displaystyle\sum_{i\in\mathbb{Z}} Cov\left(\mathbf{I}(Z_0\le x), \mathbf{I}(Z_i \le y)\right) & \text{if} \quad l_n\theta_n \ll n^{-c} \\ \\ (s\wedge t)\displaystyle\sum_{i\in\mathbb{Z}} Cov\left(\mathbf{I}(Y_0\le x), \mathbf{I}(Y_i \le y)\right) & \text{if} \quad l_n(1-\theta_n) \ll n^{-c}, \end{cases}\end{aligned}$$

for some positive constant c.

Comment: We will see in the proof of Theorem 4.1.1 that almost sure convergence of the conditional covariances of $W_n^{(b)}$ is key to the proof of P^*-almost sure convergence of the finite dimensional distributions. In particular, this entails almost sure convergence of $\frac{1}{k}\sum_{i=1}^{n}(F_{l,i}(x) - F_n(x))^2$ (see Corollary 4.4.2). When there is a change-point θ_n, an error of order $\theta_n(1-\theta_n)l_n(F^{(1)}(x) - F^{(2)}(x))^2$ occurs in the conditional variances, necessitating the assumption of a converging alternative.

4.3 Testing for the sequential empirical process with a change-point

In practical situations, we are more interested in testing for a change when no prior knowledge of F is available. We will be assuming a converging alternative.
We recall the pair of hypotheses described earlier in Section 2.2.2:

$$H_0 \quad : \quad \{\exists F \quad \text{such that} \quad \mathbf{X}_n \in \mathbf{\Psi}_n(F)\}$$

$$H_1 \quad : \quad \{\exists \theta_n \in (0,1)\ \exists F^{(1)} \neq F^{(2)} \text{ such that } \mathbf{X}_n \in \mathbf{\Psi}_n(\theta_n, F^{(1)}, F^{(2)}\},$$

where $\mathbf{X}_n = (X_1, X_2, \ldots, X_n)$ and $\theta_n \in (0,1)$ such that $\theta_n \to 0$ or 1 as $n \to \infty$. Let l_n and k_n be any sequences of natural numbers as in Theorem 3.2.1 such that $n = l_n k_n$ and recall the bootstrap sample of size n described earlier in Section 3.1

$$\left(X_1^{(b)}, \ldots, X_n^{(b)}\right) = \left(X_{nI_{n1}}, \ldots, X_{n,I_{n1}+l-1}, \ldots, X_{nI_{nk}}, \ldots, X_{n,I_{nk}+l-1}\right),$$

where $I_{n1}, I_{n2}, \ldots, I_{nk}$ are independent and identically distributed random variables each having uniform distribution on $\{1, 2, \ldots, n\}$.

The testing procedure is based on $V^{(2,b)}(\cdot,\cdot)$, the bootstrapped counterpart of the process described earlier in Section 2.2.2, and defined as follows:

$$V_n^{(2,b)}(x,s) = \frac{l[ks](k-[ks])}{k\sqrt{n}}\left(F_{l[ks]}^{(b)}(x) - F_{lk-l[ks]}^{(*,b)}(x)\right),$$

where $F_{l[ks]}^{(b)}(x)$ and $F_{lk-l[ks]}^{(*,b)}(x)$ are the bootstrapped empirical distributions based respectively on $X_1^{(b)}, \ldots, X_{l[ks]}^{(b)}$ and $X_{l[ks]+1}^{(b)}, \ldots, X_n^{(b)}$.

We can observe that

$$\begin{aligned}
&V_n^{(2,b)}(x,s)\\
&= \frac{l(k-[ks])}{k\sqrt{n}} \sum_{j=1}^{[ks]} \frac{1}{l} \sum_{i=I_{nj}}^{I_{nj}+l-1} \mathbf{I}(X_{ni} \leq x) - \frac{l[ks]}{k\sqrt{n}} \sum_{j=[ks]+1}^{k} \frac{1}{l} \sum_{i=I_{nj}}^{I_{nj}+l-1} \mathbf{I}(X_{ni} \leq x)\\
&= \frac{l}{k\sqrt{n}} \left(k \sum_{j=1}^{[ks]} \frac{1}{l} \sum_{i=I_{nj}}^{I_{nj}+l-1} \mathbf{I}(X_{ni} \leq x) - [ks] \sum_{j=1}^{k} \frac{1}{l} \sum_{i=I_{nj}}^{I_{nj}+l-1} \mathbf{I}(X_{ni} \leq x) \right)\\
&= \frac{l[ks]}{\sqrt{n}} \left(F_{l[ks]}^{(b)}(x) - F_n^{(b)}(x) \right)\\
&= W_n^{(b)}(x,s) - \frac{[ks]}{k} W_n^{(b)}(x,1).
\end{aligned}$$

We define a bootstrapped version of the Kolmogorov-Smirnov and Cramér-Von Mises statistics used previously in Section 2.2.2 by

- Bootstrapped Kolmogorov-Smirnov statistic:

$$T_3^{(b)} = \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|V_n^{(2,b)}(x,s)\right|.$$

- Bootstrapped Cramér-Von Mises statistic:

$$T_4^{(b)} = \int_0^1\int_{\mathbb{R}} \left|V_n^{(2,b)}(x,s)\right|^2 dF_n(x)ds.$$

In the following, we assume that the conditions of Theorem 4.1.2 or Theorem 4.2.2 are satisfied depending on what is needed. By applying the continuous mapping theorem to Theorem 4.1.2 and Theorem 4.2.2, one obtains the following limiting distributions of $T_3^{(b)}$ and $T_4^{(b)}$.

Proposition 4.3.1 *Under the null hypothesis, we get a.s*

$$\begin{aligned} T_3^{(b)} &\xrightarrow{d} \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W^{(3)}(x,s) - sW^{(3)}(x,1)\right| \\ T_4^{(b)} &\xrightarrow{d} \int_0^1\int_{\mathbb{R}} \left|W^{(3)}(x,s) - sW^{(3)}(x,1)\right|^2 dF(x)ds, \end{aligned}$$

where $W^{(3)}(x,s)$ is the limiting distribution in Theorem 4.1.2.

Proposition 4.3.2 *Under the alternative hypothesis and if $l_n\theta_n \ll n^{-c}$ or $l_n(1-\theta_n) \ll n^{-c}$ for some $c > 0$, we get a.s*

$$\begin{aligned} T_3^{(b)} &\xrightarrow{d} \sup_{(x,s)\in\mathbb{R}\times[0,1]} \left|W^{(4)}(x,s) - sW^{(4)}(x,1)\right| \\ T_4^{(b)} &\xrightarrow{d} \int_0^1\int_{\mathbb{R}} \left|W^{(4)}(x,s) - sW^{(4)}(x,1)\right|^2 dH(x)ds, \end{aligned}$$

where $W^{(4)}(x,s)$ is the limiting distribution in Theorem 4.2.2 and

$$H = \begin{cases} F^{(2)} & \text{if} \quad l_n\theta_n \ll n^{-c} \\ F^{(1)} & \text{if} \quad l_n(1-\theta_n) \ll n^{-c}. \end{cases}$$

Comment: We remark that under the conditions of Proposition 2.2.3 and Proposition 4.3.1, the test statistics T_3 and T_4 and their bootstrapped counterparts, $T_3^{(b)}$ and $T_4^{(b)}$, converge respectively under H_0 to the same limits. On the other hand, provided that the conditions of Proposition 2.2.4 and Proposition 4.3.2 are satisfied, we note that under H_1 the test statistics based on the original variables go to infinity, whereas the bootstrapped statistics converge weakly to finite limits. This will allow us to tabulate critical values by constructing repeated moving block bootstrap samples. More precisely, the testing approach will be consistent if

$$\begin{cases} \theta_n \to 0 \\ \sqrt{n}\theta_n \to \infty \\ l_n\theta_n \ll n^{-c}, \end{cases} \quad \text{or} \quad \begin{cases} \theta_n \to 1 \\ \sqrt{n}(1-\theta_n) \to \infty \\ l_n(1-\theta_n) \ll n^{-c}, \end{cases} \tag{4.3.1}$$

for some constant $c > 0$.

This will certainly be true, for instance, if $n^{h-1/2} \ll \theta_n \ll n^{-1/3}$ for some $0 < h < 1/6$ in the first case or analogously $n^{h-1/2} \ll 1-\theta_n \ll n^{-1/3}$ in the second case. Moreover, the proposed tests will still have some power when θ_n doesn't converge to 0 or 1 since the test statistics diverge faster than their corresponding bootstrapped versions.

4.4 Proofs

As mentioned before in Section 4.1, since Assumptions 3.1.1 (4.2.1) imply conditions (3.2.2) and (3.2.3) of Peligrad [47], we will only be concerned with the proof of Theorem 4.1.1 and Theorem 4.2.2.

For notational convenience, the dependence of the length and the number of blocks, l_n and k_n, on n will be suppressed henceforth when no ambiguity arises.

4.4.1 Proof of Theorem 4.1.1

In what follows, we fix a realization of the stochastic process, $\{x_i\}$. Hence, the resampling mechanism becomes the unique source of randomness. We also consider the triangular array $\{x_{ni}\}$ defined similarly as in Section 3.1

$$x_{ni} := \begin{cases} x_i & \text{if } \ 1 \le i \le n \\ x_{i-n} & \text{if } \ n+1 \le i \le n+l-1. \end{cases}$$

Some definitions will be needed in the sequel.

The i^{th} block sample mean:

$$\overline{x}_{li} = \frac{1}{l} \sum_{j=i}^{i+l-1} x_{nj},$$

and the sample mean:

$$\overline{x}_n = \frac{1}{n} \sum_{j=1}^{n} x_j = \frac{1}{n} \sum_{i=1}^{n} \overline{x}_{li}.$$

The bootstrapped sample mean is then defined as

$$\overline{x}_n^{(b)} = \frac{1}{n} \sum_{i=1}^{n} x_i^{(b)},$$

where $(x_1^{(b)}, \ldots, x_n^{(b)}) = (x_{nI_{n1}}, \ldots, x_{n,I_{n1}+l-1}, \ldots, x_{nI_{nk}}, \ldots, x_{n,I_{nk}+l-1})$ was defined previously in Section 3.1.

We shall also use the notation:

$$\begin{aligned} f_{li}(x) &= \frac{1}{l} \sum_{j=i}^{i+l-1} \mathbf{I}(x_{nj} \le x), & f_n(x) &= f_{n1}(x) \\ f_{li}(x,y) &= f_{li}(x) - f_{li}(y), & f_n(x,y) &= f_{n1}(x,y). \end{aligned}$$

Define now, for $x \in \mathbb{R}$, $Z_n^{(b)}(x)$ to be a sample-based version of the bootstrapped empirical process by replacing X_{ni} by x_{ni} in (4.1.1). Hence,

$$\begin{aligned} Z_n^{(b)}(x) &= \sqrt{n}(f_n^{(b)}(x) - f_n(x)) \\ &= \frac{\sqrt{n}}{k} \sum_{j=1}^{k} \left[\sum_{i=1}^{n} \mathbf{I}(I_{nj} = i)(f_{li}(x) - f_n(x)) \right], \end{aligned} \tag{4.4.1}$$

where $f_n^{(b)}(x)$ is similarly obtained by replacing X_{ni} by x_{ni} in (3.1.2).

We also define a sample-based version of the sequential bootstrapped empirical process, by replacing X_{ni} by x_{ni} in (4.1.2), for $(x, s) \in (\mathbb{R} \times [0, 1])$ as

$$Z_n^{(b)}(x,s) = \frac{\sqrt{n}}{k} \sum_{j=1}^{[ks]} \left[\sum_{i=1}^{n} \mathbf{I}(I_{nj} = i)(f_{li}(x) - f_n(x)) \right]. \tag{4.4.2}$$

We should note again that the terms in the partial sum are independent and identically distributed.

We now prove sequential versions of the results in Section 3 of [47].

Proposition 4.4.1 *Let $\{x_i, i \geq 1\}$ be a bounded sequence of real numbers. Let k and l be integers such that $n = kl$ and*

$$\frac{l^2}{n} \to 0 \quad as \quad n \to \infty. \tag{4.4.3}$$

For each n, let $\{I_{n1}, I_{n2}, \ldots, I_{nk}\}$ be i.i.d uniform on $\{1, 2, \ldots, n\}$ and assume that

$$V_n = \frac{1}{k} \sum_{i=1}^{n} (\overline{x}_{li} - \overline{x}_n)^2 \to \sigma^2 > 0 \quad as \quad n \to \infty. \tag{4.4.4}$$

Then, for $0 \leq s < t \leq 1$

$$\frac{\sqrt{n}}{k} \sum_{j=[ks]+1}^{[kt]} \sum_{i=1}^{n} \mathbf{I}(I_{nj} = i) (\overline{x}_{li} - \overline{x}_n) \xrightarrow{d} \mathcal{N}(0, (t-s)\sigma^2) \quad as \quad n \to \infty. \tag{4.4.5}$$

Proof: To avoid trivialities, assume that k is large enough that $[ks] < [kt]$ and define for $j = [ks] + 1, \ldots, [kt]$

$$D_{nj} = \frac{\sqrt{n}}{k} \sum_{i=1}^{n} \mathbf{I}(I_{nj} = i) (\overline{x}_{li} - \overline{x}_n), \quad s \in [0, 1).$$

In order to establish the weak convergence in (4.4.5), we shall prove a Lindeberg central limit for the independent $\{D_{nj}, [ks] + 1 \leq j \leq [kt]\}$. In fact, we have

$$E[D_{nj}] \quad = \quad \frac{\sqrt{n}}{k} \sum_{i=1}^{n} \frac{1}{n} (\overline{x}_{li} - \overline{x}_n)$$

$$\begin{aligned} &= \frac{\sqrt{n}}{k}\left(\frac{1}{n}\sum_{i=1}^{n}\overline{x}_{li} - \overline{x}_n\right) \\ &= 0. \end{aligned}$$

Using the fact that $\{\mathbf{I}(I_{nj} = i), 1 \le i \le n\}$ are disjoint, we also get

$$\begin{aligned} Var(D_{nj}) &= E[D_{nj}^2] \\ &= \frac{n}{k^2}E\left[\left(\sum_{i=1}^{n}\mathbf{I}(I_{nj} = i)\left(\overline{x}_{li} - \overline{x}_n\right)\right)^2\right] \\ &= \frac{n}{k^2}\sum_{i=1}^{n}E\left[\mathbf{I}(I_{nj} = i)\left(\overline{x}_{li} - \overline{x}_n\right)^2\right] \\ &= \frac{1}{k^2}\sum_{i=1}^{n}\left(\overline{x}_{li} - \overline{x}_n\right)^2 \\ &= \frac{1}{k}V_n. \end{aligned} \tag{4.4.6}$$

Thus,

$$\sum_{j=[ks]+1}^{[kt]} Var(D_{nj}) = \frac{[kt]-[ks]}{k}V_n.$$

Remark now that for $[ks]+1 \le j \le [kt]$

$$\begin{aligned} |D_{nj}| &\le \frac{\sqrt{n}}{k}\max_{1\le i\le n}|\overline{x}_{li} - \overline{x}_n| \\ &\le C\frac{\sqrt{n}}{k}, \end{aligned} \tag{4.4.7}$$

where the last line follows since the sequence $\{x_i, i \ge 1\}$ is bounded.

Hence, by (4.4.3), (4.4.6) and (4.4.7) we get

$$\begin{aligned} &\frac{k}{([kt]-[ks])V_n}\sum_{j=[ks]+1}^{[kt]}\int_{|D_{nj}|>\varepsilon\sqrt{\frac{[kt]-[ks]}{k}V_n}} D_{nj}^2 dP \\ &\le C\frac{n}{k([kt]-[ks])V_n}\sum_{j=[ks]+1}^{[kt]} P\left(|D_{nj}| > \varepsilon\sqrt{\frac{([kt]-[ks])V_n}{k}}\right) \\ &\le C\frac{n}{k([kt]-[ks])]V_n}\sum_{j=[ks]+1}^{[kt]} E\left[D_{nj}^2\right]\frac{k}{\varepsilon^2([kt]-[ks])V_n} \end{aligned}$$

$$\begin{aligned} &= C\frac{n}{k([kt]-[ks])V_n\varepsilon^2} \\ &\le C\frac{n}{k^2(t-s)V_n\varepsilon^2} \\ &= C\frac{l^2}{n(t-s)V_n\varepsilon^2} \\ &\to 0 \quad \text{as} \quad n\to\infty. \end{aligned}$$

We used here the fact that if $[kt]-[ks]>0$, then

$$\begin{aligned} kt-ks &\le [kt]-[ks]+1 \\ &\le 2([kt]-[ks]). \end{aligned}$$

Therefore, using the Lindeberg central limit theorem, we get

$$\sqrt{\frac{k}{([kt]-[ks])V_n}}\sum_{j=[ks]+1}^{[kt]} D_{nj} \xrightarrow{d} \mathcal{N}(0,1) \qquad \text{as} \quad n\to\infty.$$

One can see now by (4.4.4) that

$$\frac{\sqrt{n}}{k}\sum_{j=[ks]+1}^{[kt]}\sum_{i=1}^{n}\mathbf{I}(I_{nj}=i)\,(\overline{x}_{li}-\overline{x}_n) \xrightarrow{d} \mathcal{N}(0,(t-s)\sigma^2) \qquad \text{as} \quad n\to\infty.$$

This completes the proof of (4.4.5) and that of Proposition 4.4.1.

□

Corollary 4.4.2 *Let $\{x_i, i\ge 1\}$ and $\{I_{ni}, 1\le i\le k\}$ be as in Proposition 4.4.1 and assume that*

$$\frac{1}{k}\sum_{i=1}^{n}(f_{li}(x)-f_n(x))^2 \to \sigma^2(x)>0 \quad \text{as} \quad n\to\infty.$$

If (4.4.3) holds, then for $0\le s\le 1$

$$Z_n^{(b)}(x,s) \xrightarrow{d} \mathcal{N}(0,s\sigma^2(x)) \quad \text{as} \quad n\to\infty.$$

Proof: Denote $y_i=\mathbf{I}(x_i\le x)$ and remark that

$$\frac{1}{k}\sum_{i=1}^{n}\left(\overline{y}_{li}(x)-\overline{y}_n(x)\right)^2 \quad = \quad \frac{1}{k}\sum_{i=1}^{n}\left(f_{li}(x)-f_n(x)\right)^2$$

$$\to \quad \sigma^2(x) \quad \text{as} \quad n \to \infty.$$

If we apply Proposition 4.4.1 with x_i replaced by y_i, then (4.4.5) holds with x_i replaced by y_i. This completes the proof of Corollary 4.4.2.

□

The following proposition proves the convergence of finite dimensional distributions $Z_n^{(b)}(\cdot,\cdot)$. We shall first define for x, y in $\mathbb{R}$

$$V_n(x,y) = \frac{1}{k}\sum_{i=1}^{n}(f_{li}(x,y) - f_n(x,y))^2. \tag{4.4.8}$$

Proposition 4.4.3 *Assume all the conditions of Corollary 4.4.2 are satisfied and that $0 \le x_i \le 1$ for every $1 \le i \le n$. Assume also for every x, y in $[0,1]$*

$$\lim_{n\to\infty} V_n(x,y) \quad \textit{exists.} \tag{4.4.9}$$

Then, for every x, y in $[0,1]$

$$\lim_{n\to\infty}\frac{1}{k}\sum_{i=1}^{n}(f_{li}(x) - f_n(x))(f_{li}(y) - f_n(y)) = \sigma(x,y) \tag{4.4.10}$$

exists and for every $(z_1,\ldots,z_p,s_1,\ldots,s_p) \in [0,1]^{2p}$

$$\left(Z_n^{(b)}(z_1,s_1),\ldots,Z_n^{(b)}(z_p,s_p)\right) \xrightarrow{d} \left(\mathcal{N}_{z_1,s_1},\ldots,\mathcal{N}_{z_p,s_p}\right) \quad \textit{as} \quad n\to\infty, \tag{4.4.11}$$

where $\{\mathcal{N}_{z_i,s_i}\}_{1\le i\le p}$ are zero-mean Gaussian with covariance function

$$\sigma_{ij} = (s_i \wedge s_j)\sigma(z_i,z_j).$$

Proof: By (4.4.1), we have the following representation

$$Z_n^{(b)}(x) = \sum_{j=1}^{k} E_{nj}(x),$$

where

$$E_{nj}(x) = \frac{\sqrt{n}}{k}\sum_{i=1}^{n}\mathbf{I}(I_{nj} = i)(f_{li}(x) - f_n(x)).$$

We know that

$$\begin{aligned} Var\left(Z_n^{(b)}(x) - Z_n^{(b)}(y)\right) &= Var\left(Z_n^{(b)}(x)\right) + Var\left(Z_n^{(b)}(y)\right) \\ &- 2Cov\left(Z_n^{(b)}(x), Z_n^{(b)}(y)\right). \end{aligned}$$

Using the fact that the sets $\{\mathbf{I}(I_{nj} = i), 1 \leq i \leq n\}$ are disjoint and the independence of $\{I_{nj}, 1 \leq j \leq k\}$, we get

$$\begin{aligned} Var\left(Z_n^{(b)}(x)\right) &= \frac{1}{k}\sum_{i=1}^{n}(f_{li}(x) - f_n(x))^2 \\ Var\left(Z_n^{(b)}(y)\right) &= \frac{1}{k}\sum_{i=1}^{n}(f_{li}(y) - f_n(y))^2 \\ Var\left(Z_n^{(b)}(x) - Z_n^{(b)}(y)\right) &= V_n(x, y). \end{aligned}$$

On the other hand, we can see that

$$V_n(x, y) = V_n(x, 0) + V_n(y, 0) - \frac{2}{k}\sum_{i=1}^{n}(f_{li}(x) - f_n(x))(f_{li}(y) - f_n(y)).$$

Since $\lim_{n\to\infty} V_n(x, y)$ exists for every x, y in $[0, 1]$, then the limit in (4.4.10) exists too. We can also deduce from what preceded that

$$Cov\left(Z_n^{(b)}(x), Z_n^{(b)}(y)\right) = \frac{1}{k}\sum_{i=1}^{n}(f_{li}(x) - f_n(x))(f_{li}(y) - f_n(y)). \tag{4.4.12}$$

Using now the representation in (4.4.2), we can see that for $0 \leq s < t \leq 1$

$$\begin{aligned} Cov\left(Z_n^{(b)}(x, s), Z_n^{(b)}(y, t)\right) &= \frac{[ks]}{k^2}\sum_{i=1}^{n}(f_{li}(x) - f_n(x))(f_{li}(y) - f_n(y)) \\ &\to s\sigma(x, y) \quad \text{as} \quad n \to \infty, \end{aligned}$$

where the last line follows from the fact that $s - \dfrac{l}{n} \leq \dfrac{[ks]}{k} \leq s$ and the convergence in (4.4.10).

Suppose now, for instance, that $0 = s_0 \leq s_1 < s_2 < \ldots < s_p \leq 1$ and let $\alpha_1, \alpha_2, \ldots, \alpha_p$ be real numbers. We will use the Cramér-Wold theorem and prove that

$$\sum_{u=1}^{p} \alpha_u Z_n^{(b)}(z_u, s_u) \xrightarrow{d} \sum_{u=1}^{p} \alpha_u \mathcal{N}_{z_u, s_u} \quad \text{as} \quad n \to \infty.$$

Similarly to the proof of Proposition 4.4.1, we have the following representation

$$\begin{aligned}
\sum_{u=1}^{p} \alpha_u Z_n^{(b)}(z_u, s_u) &= \frac{\sqrt{n}}{k} \sum_{j=1}^{[ks_1]} \sum_{i=1}^{n} \sum_{u=1}^{p} \alpha_u \mathbf{I}(I_{nj} = i)(f_{li}(z_u) - f_n(z_u)) \\
&+ \frac{\sqrt{n}}{k} \sum_{j=[ks_1]+1}^{[ks_2]} \sum_{i=1}^{n} \sum_{u=2}^{p} \alpha_u \mathbf{I}(I_{nj} = i)(f_{li}(z_u) - f_n(z_u)) \\
&+ \ldots + \frac{\sqrt{n}}{k} \sum_{j=[ks_{p-1}]+1}^{[ks_p]} \sum_{i=1}^{n} \alpha_p \mathbf{I}(I_{nj} = i)(f_{li}(z_p) - f_n(z_p)) \\
&= \sum_{u=0}^{p-1} \frac{\sqrt{n}}{k} \sum_{j=[ks_u]+1}^{[ks_{u+1}]} \sum_{i=1}^{n} \sum_{v=u+1}^{p} \alpha_v \mathbf{I}(I_{nj} = i)(f_{li}(z_v) - f_n(z_v)). \qquad (4.4.13)
\end{aligned}$$

Denote

$$y_i^u = \sum_{v=u+1}^{p} \alpha_v \mathbf{I}(x_i \le z_v)$$

and apply Proposition 4.4.1 with x_i replaced by y_i^u to get, as $n \to \infty$

$$\frac{\sqrt{n}}{k} \sum_{j=[ks_u]+1}^{[ks_{u+1}]} \sum_{i=1}^{n} \sum_{v=u+1}^{p} \alpha_v \mathbf{I}(I_{nj} = i)(f_{li}(z_v) - f_n(z_v)) \xrightarrow{d} \mathcal{N}(0, (s_{u+1} - s_u)\sigma_u^2),$$

provided the existence of

$$\sigma_u^2 = \lim_{n\to\infty} \frac{1}{k} \sum_{i=1}^{n} \left(\overline{y^u}_{li} - \overline{y^u}_n\right)^2.$$

Using (4.4.10), we get

$$\begin{aligned}
\frac{1}{k} \sum_{i=1}^{n} \left(\overline{y^u}_{li} - \overline{y^u}_n\right)^2 &= \frac{1}{k} \sum_{i=1}^{n} \left(\sum_{v=u+1}^{p} \alpha_v \left(f_{li}(z_v) - f_n(z_v)\right) \right)^2 \\
&= \frac{1}{k} \sum_{i=1}^{n} \sum_{v=u+1}^{p} \alpha_v^2 \left(f_{li}(z_v) - f_n(z_v)\right)^2 \\
&+ \frac{2}{k} \sum_{i=1}^{n} \sum_{v=u+1}^{p-1} \sum_{v'=v+1}^{p} \alpha_v \alpha_{v'} \left(f_{li}(z_v) - f_n(z_v)\right)\left(f_{li}(z_{v'}) - f_n(z_{v'})\right) \\
&\to \sum_{v=u+1}^{p} \alpha_v^2 \sigma(z_v, z_v) + 2 \sum_{v=u+1}^{p-1} \sum_{v'=v+1}^{p} \alpha_v \alpha_{v'} \sigma(z_v, z_{v'}).
\end{aligned}$$

Remark now that the $u-$indexed sums in (4.4.13) are independent and

$$\begin{aligned}
&\sum_{u=0}^{p-1}(s_{u+1}-s_u)\sigma_u^2 \\
&= \sum_{u=0}^{p-1}(s_{u+1}-s_u)\left[\sum_{v=u+1}^{p}\alpha_v^2\sigma(z_v,z_v)+2\sum_{v=u+1}^{p-1}\sum_{v'=v+1}^{p}\alpha_v\alpha_{v'}\sigma(z_v,z_{v'})\right] \\
&= \sum_{v=1}^{p}\sum_{u=0}^{v-1}(s_{u+1}-s_u)\alpha_v^2\sigma(z_v,z_v) \\
&\quad +2\sum_{v=1}^{p-1}\sum_{v'=v+1}^{p}\sum_{u=0}^{v-1}(s_{u+1}-s_u)\alpha_v\alpha_{v'}\sigma(z_v,z_{v'}) \\
&= \sum_{v=1}^{p}s_v\alpha_v^2\sigma(z_v,z_v)+2\sum_{v=1}^{p-1}\sum_{v'=v+1}^{p}s_v\alpha_v\alpha_{v'}\sigma(z_v,z_{v'}) \\
&= Var\left(\sum_{u=1}^{p}\alpha_u\mathcal{N}_{z_u,s_u}\right).
\end{aligned}$$

This completes the proof of this proposition.

□

Proposition 4.4.4 *Let $\{x_i, 1\le i\le n\}$ and $\{I_{ni}, 1\le i\le n\}$ as in Proposition 4.4.1, $0\le x_i\le 1$ for every $0\le i\le n$. Let k and l be integers such that $n=kl$ and*

$$l\le C_1 n^{\frac{1}{2}-a}, \quad \textit{for some} \quad 0<a<\frac{1}{2} \quad \textit{and} \quad C_1>0. \tag{4.4.14}$$

Assume there are constants $C_2>0$, $0<b<1$ and $c>0$ such that, for every x and y in $[0,1]$ and every $n\ge 1$, we have

$$V_n(x,y)=\frac{1}{k}\sum_{i=1}^{n}(f_{li}(x,y)-f_n(x,y))^2\le C_2(|x-y|^b+n^{-c}). \tag{4.4.15}$$

Then $Z_n^{(b)}(x,s)$ defined by (4.4.2) is tight, that is, for every $\varepsilon,\eta>0$ there exists δ, $0<\delta<1$ and N_0 such that for every $n\ge N_0$,

$$P\left(\sup_{\substack{|x-y|<\delta\\|s-t|<\delta}}\left|Z_n^{(b)}(x,s)-Z_n^{(b)}(y,t)\right|\ge\varepsilon\right)\le\eta, \tag{4.4.16}$$

and consequently, if Y is taken as a limiting distribution on a subsequence, $P(Y \in C([0,1]^2)) = 1$.

For computational clarity, we will identify all of the constants involved in the following proof.

Proof: The tightness of the sequence $Z_n^{(b)}(x,s)$ will be proven by closely following the approach used by Naik-Nimbalkar and Rajarshi in [45] and using a restricted chaining argument given in Theorem VII.26 in [49] applied with the semimetric $d((x,s),(y,t)) = C_3 \max(|x-y|^{\frac{b}{2}}, |s-t|^{\frac{b}{2}})$, where x, y, s, t in $[0,1]$, $0 < b < 1$ and $C_3 > 0$.

By (4.4.2), we have the following representation

$$Z_n^{(b)}(x,s) = \sum_{j=1}^{[ks]} E_{nj}(x),$$

where

$$E_{nj}(x) = \frac{\sqrt{n}}{k} \sum_{i=1}^{n} \mathbf{I}(I_{nj} = i)(f_{li}(x) - f_n(x)).$$

Suppose for instance that $s \le t$, so we can now obtain by virtue of the independence of the E_{nj}'s and inequality (4.4.15) that

$$\begin{aligned}
& Var\left(Z_n^{(b)}(x,s) - Z_n^{(b)}(y,t)\right) \\
= \quad & \sum_{j=1}^{[ks]} Var\left(E_{nj}(x,y)\right) + \sum_{j=[ks]+1}^{[kt]} Var\left(E_{nj}(y)\right) \\
= \quad & \frac{[ks]}{k} V_n(x,y) + \frac{[kt]-[ks]}{k} V_n(y,0) \\
\le \quad & V_n(x,y) + \left(t - s + \frac{1}{k}\right) V_n(y,0) \\
\le \quad & C_2(|x-y|^b + n^{-c}) + C_2\left(|t-s| + \frac{1}{k}\right)(|y|^b + n^{-c}) \\
\le \quad & C_2(|x-y|^b + n^{-c}) + C_2\left(|t-s|^b + \frac{1}{k}\right)(1 + n^{-c}) \\
\le \quad & C_2'\left(|x-y|^b + |t-s|^b + \frac{1}{k} + n^{-c}\right)
\end{aligned}$$

$$= \quad D_n(x,y,s,t).$$

We also have by (4.4.14)

$$|E_{nj}(x,y)| \le \frac{2\sqrt{n}}{k} \le 2C_1 n^{-a} \quad \text{and} \quad |E_{nj}(y)| \le \frac{\sqrt{n}}{k} \le C_1 n^{-a}.$$

Therefore, by Bennett's inequality (see [49], page 192), we have for every $\eta > 0$

$$\begin{aligned} & P\left(\left|Z_n^{(b)}(x,s) - Z_n^{(b)}(y,t)\right| > \eta\right) \\ \le \;& 2\exp\left(-\frac{1}{2}\frac{\eta^2}{D_n(x,y,s,t)} B\left(\frac{2C_1 n^{-a}\eta}{D_n(x,y,s,t)}\right)\right), \end{aligned} \tag{4.4.17}$$

where $B(\lambda) = 2\lambda^{-2}[(1+\lambda)\log(1+\lambda) - \lambda]$ for $\lambda > 0$.

Remark now that for any $\delta > 0$

$$d((x,s),(y,t)) \le \delta \quad \Rightarrow \quad \max(|x-y|, |s-t|) \le \left(\frac{\delta}{C_3}\right)^{\frac{2}{b}}.$$

Hence, the covering number, which is the smallest m for which there exist points $(x_{\alpha 1}, s_{\alpha 1}), \ldots, (x_{\alpha m}, s_{\alpha m})$ with $\min_i d((x,s),(x_{\alpha i}, s_{\alpha i})) \le \delta$ for every (x,s) in $[0,1]^2$, is found to be in this case

$$N(\delta) = N(\delta, d, [0,1]^2) = \left(\left[\frac{C_3}{\delta}\right]^{\frac{2}{b}} + 1\right)^2.$$

Let us denote the nearest member of the α-net of $[0,1]^2$ to (x,s) with respect to the semimetric d by (x_α, s_α). If $d((x,s),(y,t)) \le \delta$ and $n^{-r} \le \delta^2$, where $r = \min(\frac{1}{2}+a, c)$, then $D_n(x,y,s,t) \le C_4\delta^2$ for some $C_4 > 0$ by (4.4.14).

It will be shown later that

i) For every $\lambda \in (0,1)$, every $\eta > 0$ and $\delta > 0$ such that $\delta^2 \ge n^{-r}$ and $\dfrac{\delta^2}{\eta} \ge \dfrac{2C_1 n^{-a}}{C_4 B^{-1}(\lambda)}$, we have

$$P\left(\left|Z_n^{(b)}(x,s) - Z_n^{(b)}(y,t)\right| \ge \eta\right) \le 2\exp\left(-\frac{\frac{1}{2}\eta^2\lambda}{C_4\delta^2}\right), \tag{4.4.18}$$

provided that $d((x,s),(y,t)) \le \delta$.

ii) For any $\nu > 0$

$$\limsup_{n\to\infty} P\left[\sup_{(x,s)\in[0,1]^2}\left|Z_n^{(b)}(x,s) - Z_n^{(b)}(x_\alpha,s_\alpha)\right| \geq \nu\right] = 0 \text{ a.s,} \tag{4.4.19}$$

where $\alpha^2 = \alpha_n^2 = n^{-r} + \dfrac{2C_1 n^{-a}}{C_4 B^{-1}(\lambda)}$.

To apply Theorem VII.26 of Pollard [49], it remains to show that the associated covering integral with respect to the δ-net of the semimetric d is finite for any $0 < \delta \leq 1$. By definition, the covering integral is

$$\begin{aligned}
J(\delta) = J(\delta, d, [0,1]^2) &= \int_0^\delta \left(2\log\left(\frac{N(u)^2}{u}\right)\right)^{\frac{1}{2}} du \\
&\leq \int_0^\delta \left(2\log\left(\frac{\left(\left(\frac{C}{u}\right)^{\frac{2}{b}}+1\right)^4}{u}\right)\right)^{\frac{1}{2}} du \\
&\leq \int_0^\delta \left(2\log\left(\frac{C}{u^{\frac{8}{b}+1}}\right)\right)^{\frac{1}{2}} du \\
&\leq \int_0^\delta (2\log C)^{\frac{1}{2}} + \left(2\log\left(\frac{1}{u^{\frac{8}{b}+1}}\right)\right)^{\frac{1}{2}} du \\
&\leq C\delta + \left(\frac{16}{b}+2\right)^{\frac{1}{2}} \int_0^\delta \left(\log\left(\frac{1}{u}\right)\right)^{\frac{1}{2}} du \\
&\leq C\delta + C\int_0^\delta \log\left(\frac{1}{u}\right)\left(\log\left(\frac{1}{u}\right)\right)^{-\frac{1}{2}} du \\
&\leq C\delta + C\left(\log\left(\frac{1}{\delta}\right)\right)^{-\frac{1}{2}} \int_0^\delta \log\left(\frac{1}{u}\right) du \\
&\leq C\delta\left(\log\left(\frac{1}{\delta}\right)\right)^{\frac{1}{2}} \quad \text{for } \delta \text{ small enough.}
\end{aligned}$$

Now, taking $\lambda = 1/4$, $D = 2\sqrt{C_4}$ and $\alpha^2 = n^{-r} + 2C_1 n^{-a}/C_4 B^{-1}(1/4)$ in Theorem VII.26 of Pollard [49], inequality (4.4.16) holds.

To complete the proof of the proposition, we shall prove inequalities (4.4.18) and (4.4.19).

If $d((x,s),(y,t)) \leq \delta$ and $n^{-r} \leq \delta^2$, then $D_n(x,y,s,t) \leq C_4\delta^2$ as mentioned before. Since $\lambda B(\lambda)$ is an increasing function, we have by (4.4.17)

$$P\left(\left|Z_n^{(b)}(x,s) - Z_n^{(b)}(x,s)\right| > \eta\right) \leq 2\exp\left(-\frac{1}{2}\frac{\eta^2}{C_4\delta^2}B\left(\frac{2C_1n^{-a}\eta}{C_4\delta^2}\right)\right).$$

Further, if $\dfrac{\delta^2}{\eta} \geq \dfrac{2C_1n^{-a}}{C_4B^{-1}(\lambda)}$, then inequality (4.4.18) holds since $B(0^+) = 1$ and $B(\cdot)$ is a continuous, decreasing function.

We shall verify now (4.4.19). Let us recall the representation (4.4.2)

$$Z_n^{(b)}(x,s) = \sum_{j=1}^{[ks]} E_{nj}(x),$$

where

$$E_{nj}(x) = \frac{\sqrt{n}}{k}\sum_{i=1}^{n}\mathbf{I}(I_{nj} = i)(f_{li}(x) - f_n(x)).$$

Let x and y be any points in $[0,1]$. Then, arguing as in [45]

$$\begin{aligned}
f_n(x) = f_n(y) \;&\Rightarrow\; \mathbf{I}(x_j \leq x) = \mathbf{I}(x_j \leq y) \quad \text{for } j = 1,\ldots,n \\
&\Rightarrow\; \mathbf{I}(x_{nj} \leq x) = \mathbf{I}(x_{nj} \leq y) \quad \text{for } j = 1,\ldots,n+l-1 \\
&\Rightarrow\; Z_n^{(b)}(x,s) = Z_n^{(b)}(y,s) \quad \text{for } s \in [0,1].
\end{aligned}$$

Since f_n and $[ks]$ assume $(n+1)$ and $(k+1)$ different values respectively, $Z_n^{(b)}(x,s) - Z_n^{(b)}(y,t)$ assumes at most $(n+1)^2(k+1)^2$ values as (x,s) and (y,t) vary in $[0,1]^2$. Therefore,

$$\begin{aligned}
P&\left[\sup_{(x,s)\in[0,1]^2}|Z_n^{(b)}(x,s) - Z_n^{(b)}(x_\alpha,s_\alpha)| > \nu\right] \\
&\leq (n+1)^2(k+1)^2\sup_{(x,s)\in[0,1]^2}P\left[|Z_n^{(b)}(x,s) - Z_n^{(b)}(x_\alpha,s_\alpha)| > \nu\right].
\end{aligned}$$

Suppose for instance that $s \leq s_\alpha$, then by the kind of computations seen before, we get

$$Var\left(Z_n^{(b)}(x,s) - Z_n^{(b)}(x_\alpha,s_\alpha)\right) \leq D_n(x,x_\alpha,s,s_\alpha).$$

Bernstein's inequality (see [49], page 193) leads to the following:

$$P\left[|Z_n^{(b)}(x,s) - Z_n^{(b)}(x_\alpha, s_\alpha)| > \nu\right] \leq 2\exp\left(-\frac{\frac{1}{2}\nu^2}{D_n(x, x_\alpha, s, s_\alpha) + \frac{2C_1 n^{-a}}{3}\nu}\right).$$

The definition of α and the condition (4.4.14) imply that

$$D_n(x, x_\alpha, s, s_\alpha) + \frac{2C_1 n^{-a}}{3}\nu \leq C_5 n^{-p},$$

for some constants $C_5 > 0$ and $p > 0$. Hence,

$$P\left[\sup_{(x,s)\in[0,1]^2} |Z_n^{(b)}(x,s) - Z_n^{(b)}(x_\alpha, s_\alpha)| > \nu\right] \leq 2(n+1)^4 \exp(-C_6 n^p).$$

This completes the proof of inequality (4.4.19) and that of Proposition 4.4.4.

□

Back now to the proof of Theorem 4.1.1. We will be using all the results presented earlier in this section by considering the sequence of the random variables $\{X_i\}$ instead of the sequence of the numbers $\{x_i\}$. For instance, by fixing ω and denoting $x_i = X_i(\omega)$, one can see the previous propositions as a behaviour of a fixed trajectory $X_1(\omega), X_2(\omega), \ldots$.

We remark that the limiting process has the same covariance structure as the limit of the original sequential empirical process given in Theorem 1.2.1.

Define the variables $U_i = F(X_i)$ and consider the bootstrapped empirical process $W_n^{(b)}(\cdot)$ based on the U_i's, similarly to the one previously presented in (3.1.1).
We also define

$$V_{n1}(x,y) = Var\left(W_n^{(b)}(x) - W_n^{(b)}(y)\right).$$

Let us now replace P with P^* and $x_i = U_i(\omega)$ in Propositions 4.4.1, 4.4.3 and 4.4.4. In this context, the conditions (4.4.9) and (4.4.15) translate to

$$\lim_{n\to\infty} V_{n1}(x,y) \quad \text{exists a.s.,} \tag{4.4.20}$$

and for each x and y in $[0,1]$

$$V_{n1}(x,y) \leq C(|x-y|^b + n^{-c}) \quad \text{a.s.,} \tag{4.4.21}$$

for some $C > 0$, depending only on trajectory, $0 < b < 1$ and $c > 0$.

In order to prove the P^*-almost sure convergence of the sequential bootstrapped empirical process based on the U_i's, $\overline{W_n^{(b)}(\cdot,\cdot)}$, one must verify that the conditions (4.4.20) and (4.4.21) hold also almost surely. This has been proven by Peligrad in [47] (Proposition 4.1 and Proof of Theorem 2.2). The conclusion of Theorem 4.1.2 can now be derived in a routine manner, as in [5], for the sequential bootstrapped empirical process $\overline{W_n^{(b)}(F(x),s)} = W_n^{(b)}(x,s)$.

□

4.4.2 Proof of Theorem 4.2.2

Recall the change-point model introduced in Chapter 2

$$X_i = \begin{cases} Y_i & \text{for } \; i \leq [n\theta_n] \\ Z_i & \text{otherwise,} \end{cases}$$

where $0 \leq \theta_n \leq 1$ and $\lim_{n\to\infty} \theta_n = \theta$.

Let l_n and k_n be any sequences satisfying equation (4.4.14) such that $n = l_n k_n$. Similarly to the previous section, we fix a realization of each of the stochastic processes X_i, Y_i and Z_i, $\{x_i\}$, $\{y_i\}$ and $\{z_i\}$. We will again assume that the sequences x_i, y_i and z_i are defined on $[0,1]$ and define the following triangular arrays

$$x_{ni} := \begin{cases} x_i & \text{if } \; 1 \leq i \leq n \\ x_{i-n} & \text{if } \; n+1 \leq i \leq n+l-1 \end{cases}$$

and

$$y_{ni} := \begin{cases} y_i & \text{if } \; 1 \leq i \leq n \\ y_{i-n} & \text{if } \; n+1 \leq i \leq n+l-1 \end{cases}$$

and

$$z_{ni} := \begin{cases} z_i & \text{if } \; 1 \leq i \leq n \\ z_{i-n} & \text{if } \; n+1 \leq i \leq n+l-1. \end{cases}$$

Recall the i^{th} block sample empirical distribution defined in the previous section

$$f_{li}^X(x) = \frac{1}{l}\sum_{j=i}^{i+l-1} \mathbf{I}(x_{nj} \le x) \quad \text{and} \quad f_n^X(x) = f_{n1}^X(x),$$

and define similarly f_{li}^Y and f_{li}^Z to be the i^{th} block sample empirical distributions based on the arrays $\{y_{ni}\}$ and $\{z_{ni}\}$ respectively.

We further define f_{li}^{Y,θ_n} and f_{li}^{Z,θ_n} by

$$f_{li}^{Y,\theta_n}(x) = \frac{1}{l}\sum_{j=i}^{i+l-1} \mathbf{I}(y_{nj}^* \le x) \quad \text{and} \quad f_{li}^{Z,\theta_n}(x) = \frac{1}{l}\sum_{j=i}^{i+l-1} \mathbf{I}(z_{nj}^* \le x),$$

where

$$y_{ni}^* = \begin{cases} y_i & \text{if} \quad 1 \le i \le [n\theta_n] \\ y_{i-[n\theta_n]} & \text{if} \quad [n\theta_n]+1 \le i \le [n\theta_n]+l-1 \end{cases}$$

and

$$z_{ni}^* = \begin{cases} z_i & \text{if} \quad 1 \le i \le [n\theta_n] \\ z_{i-[n\theta_n]} & \text{if} \quad [n\theta_n]+1 \le i \le [n\theta_n]+l-1. \end{cases}$$

Since $f_{[n\theta_n]}^T(x) = f_{[n\theta_n]}^{T,\theta_n}(x)$ for $T = Y, Z$, one easily remarks that

$$f_n^X(x) = \frac{[n\theta_n]}{n} f_{[n\theta_n]}^Y(x) + \frac{n-[n\theta_n]}{n} f_{n-[n\theta_n],[n\theta_n]+1}^Z(x) \tag{4.4.22}$$

and

$$f_n^Z(x) = \frac{[n\theta_n]}{n} f_{[n\theta_n]}^Z(x) + \frac{n-[n\theta_n]}{n} f_{n-[n\theta_n],[n\theta_n]+1}^Z(x), \tag{4.4.23}$$

where

$$f_{n-[n\theta_n],[n\theta_n]+1}^Z(x) = \frac{1}{n-[n\theta_n]} \sum_{j=[n\theta_n]+1}^{n} \mathbf{I}(z_{nj} \le x).$$

We shall define for x, s in $[0,1]$

$$A_n(x,s) = \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{n} \mathbf{I}(I_{nj}=i)\left(\left(f_{li}^X(x) - f_n^X(x)\right) - \left(f_{li}^Z(x) - f_n^Z(x)\right)\right),$$

and

$$B_n(x,s) = \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{n}\mathbf{I}(I_{nj}=i)\left(\left(f_{li}^X(x) - f_n^X(x)\right) - \left(f_{li}^Y(x) - f_n^Y(x)\right)\right).$$

Proposition 4.4.5 *Let* $\{y_i, 1 \le i \le n\}$ *and* $\{z_i, 1 \le i \le n\}$ *be any sequences of real numbers satisfying the conditions of Proposition 4.4.3 and Proposition 4.4.4.*

1. *If* $l_n\theta_n \ll n^{-c}$ *for some positive constant* c*, then*

$$\sup_{(x,s)\in[0,1]^2}\frac{\sqrt{n}}{k}\left|A_n(x,s)\right| \xrightarrow{P} 0, \tag{4.4.24}$$

and consequently, the sample-based version of the sequential bootstrapped empirical processes based on $\{x_{ni}\}$ *and* $\{z_{ni}\}$ *converge to the same limiting distribution in* $D([0,1]\times[0,1])$*.*

2. *If* $l_n(1-\theta_n) \ll n^{-c}$ *for some positive constant* c*, then*

$$\sup_{(x,s)\in[0,1]^2}\frac{\sqrt{n}}{k}\left|B_n(x,s)\right| \xrightarrow{P} 0, \tag{4.4.25}$$

and consequently, the sample-based version of the sequential bootstrapped empirical processes based on $\{x_{ni}\}$ *and* $\{y_{ni}\}$ *converge to the same limiting distribution in* $D([0,1]\times[0,1])$*.*

Proof: We only focus on the first case (4.4.24) since the proofs of both cases are similar and require the same arguments.

Using (4.4.22) and (4.4.23) and the fact that $\sum_{i=1}^{n}\mathbf{I}\left(I_{nj}=i\right) = 1$, we have for $(x,s)\in[0,1]^2$

$$\begin{aligned} A_n(x,s) &= \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{n}\mathbf{I}(I_{nj}=i)\left(f_{li}^X(x) - f_{li}^Z(x)\right) \\ &- \quad \frac{[ks][n\theta_n]}{k\sqrt{n}}\left(f_{[n\theta_n]}^Y(x) - f_{[n\theta_n]}^Z(x)\right) \\ &= \quad \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(\left(f_{li}^X(x) - f_{li}^{Y,\theta_n}(x)\right) + \left(f_{li}^{Z,\theta_n}(x) - f_{li}^Z(x)\right)\right) \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(f_{li}^{Y,\theta_n}(x)-f_{li}^{Z,\theta_n}(x)\right)\\
&+ \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=[n\theta_n]+1}^{n}\mathbf{I}(I_{nj}=i)\left(f_{li}^{X}(x)-f_{li}^{Z}(x)\right)\\
&- \frac{[ks][n\theta_n]}{k\sqrt{n}}\left(f_{[n\theta_n]}^{Y}(x)-f_{[n\theta_n]}^{Z}(x)\right)\\
&= \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(f_{li}^{X}(x)-f_{li}^{Y,\theta_n}(x)\right) \qquad (4.4.26)\\
&+ \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(f_{li}^{Z,\theta_n}(x)-f_{li}^{Z}(x)\right) \qquad (4.4.27)\\
&+ \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(f_{li}^{Y,\theta_n}(x)-f_{[n\theta_n]}^{Y}(x)\right) \qquad (4.4.28)\\
&- \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(f_{li}^{Z,\theta_n}(x)-f_{[n\theta_n]}^{Z}(x)\right) \qquad (4.4.29)\\
&+ \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]}\left(\mathbf{I}(I_{nj}=i)-\frac{1}{n}\right)\left(f_{[n\theta_n]}^{Y}(x)-f_{[n\theta_n]}^{Z}(x)\right) \qquad (4.4.30)\\
&+ \frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=[n\theta_n]+1}^{n}\mathbf{I}(I_{nj}=i)\left(f_{li}^{X}(x)-f_{li}^{Z}(x)\right). \qquad (4.4.31)
\end{aligned}
$$

In what follows, we denote the last six terms (4.4.26)-(4.4.31) by A_{ni}, $i=1,\ldots,6$. We will show that for $i=1,\ldots,6$

$$\sup_{(x,s)\in[0,1]^2}|A_{ni}(x,s)| \xrightarrow{P} 0 \quad \text{as} \quad n\to\infty.$$

We first note that for x, s in $[0,1]$

$$A_{n1}(x,s)=\frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=[n\theta_n]-l+2}^{[n\theta_n]}\mathbf{I}(I_{nj}=i)\left(f_{li}^{X}(x)-f_{li}^{Y,\theta_n}(x)\right).$$

Hence

$$\sup_{(x,s)\in[0,1]^2}|A_{n1}(x,s)|\le\frac{\sqrt{n}}{k}\sum_{j=1}^{k}\sum_{i=[n\theta_n]-l+2}^{[n\theta_n]}\mathbf{I}(I_{nj}=i).$$

Since $E[\mathbf{I}(I_{nj} = i)] = \dfrac{1}{n}$, then

$$E \sup_{(x,s)\in[0,1]^2} |A_{n1}(x,s)| \le \frac{l}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

Similarly, it can be shown that

$$E \sup_{(x,s)\in[0,1]^2} |A_{n2}(x,s)| \le \frac{l}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

Consider now (4.4.31) and remark that for x, s in $[0,1]$

$$A_{n6}(x,s) = \frac{\sqrt{n}}{k} \sum_{j=1}^{[ks]} \sum_{i=n-l+2}^{n} \mathbf{I}(I_{nj} = i) \left(f_{li}^X(x) - f_{li}^Z(x) \right).$$

Thus,

$$\sup_{(x,s)\in[0,1]^2} |A_{n6}(x,s)| \le \frac{\sqrt{n}}{k} \sum_{j=1}^{k} \sum_{i=n-l+2}^{n} \mathbf{I}(I_{nj} = i).$$

Therefore,

$$E \sup_{(x,s)\in[0,1]^2} |A_{n6}(x,s)| \le \frac{l}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.$$

The remainder of the proof will require a different approach. Let us consider for instance (4.4.30) and write $A_{n5}(x,s) = A_{n5}^{(1)}(x,s) - A_{n5}^{(2)}(x,s)$, where

$$\begin{aligned} A_{n5}^{(1)}(x,s) &= \frac{\sqrt{n}}{k} \sum_{j=1}^{[ks]} \sum_{i=1}^{[n\theta_n]} \left(\mathbf{I}(I_{nj} = i) - \frac{1}{n} \right) f_{[n\theta_n]}^Y(x) \\ A_{n5}^{(2)}(x,s) &= \frac{\sqrt{n}}{k} \sum_{j=1}^{[ks]} \sum_{i=1}^{[n\theta_n]} \left(\mathbf{I}(I_{nj} = i) - \frac{1}{n} \right) f_{[n\theta_n]}^Z(x). \end{aligned}$$

For x, s in $[0,1]$, we have

$$\left| A_{n5}^{(1)}(x,s) \right| \le \frac{\sqrt{n}}{k} \left| \sum_{j=1}^{[ks]} \sum_{i=1}^{[n\theta_n]} \left(\mathbf{I}(I_{nj} = i) - \frac{1}{n} \right) \right|.$$

By independence of $\{\mathbf{I}_{nj}\}_j$ and the fact that the sets $\{\mathbf{I}(I_{nj} = i), 1 \le i \le n\}$ are disjoint, we obtain

$$E \left(A_{n5}^{(1)}(x,s) \right)^2 \le \frac{n}{k^2} E \left(\sum_{j=1}^{[ks]} \sum_{i=1}^{[n\theta_n]} \left(\mathbf{I}(I_{nj} = i) - \frac{1}{n} \right) \right)^2$$

$$\leq \frac{n}{k^2}\sum_{j=1}^{[ks]}\sum_{i,i'=1}^{[n\theta_n]} E\left(\left(\mathbf{I}(I_{nj}=i)-\frac{1}{n}\right)\left(\mathbf{I}(I_{nj}=i')-\frac{1}{n}\right)\right)$$
$$\leq \frac{n[ks]}{k^2}\left(\frac{[n\theta_n]}{n}-\frac{[n\theta_n]^2}{n^2}\right)$$
$$\leq l\theta_n \to 0 \quad \text{as} \quad n\to\infty.$$

Write now

$$A_{n5}^{(1)}(x,s)=\sum_{j=1}^{[ks]} B_{nj}^{(1,5)}(x),$$

where, for each $1\leq j\leq [ks]$

$$B_{nj}^{(1,5)}(x)=\frac{\sqrt{n}}{k}\sum_{i=1}^{[n\theta_n]}\left(\mathbf{I}(I_{nj}=i)-\frac{1}{n}\right) f_{[n\theta_n]}^{Y}(x)$$

is a triangular array of independent random variables.

It is easy to see that

$$Var\left(B_{nj}^{(1,5)}(x,y)\right) = Var\left(\frac{\sqrt{n}}{k}\sum_{i=1}^{[n\theta_n]}\left(\mathbf{I}(I_{nj}=i)-\frac{1}{n}\right) f_{[n\theta_n]}^{Y}(x,y)\right)$$
$$= \frac{n}{k^2}\sum_{i=1}^{[n\theta_n]}\sum_{i'=1}^{[n\theta_n]} E\left(\mathbf{I}(I_{nj}=i)-\frac{1}{n}\right)\left(\mathbf{I}(I_{nj}=i')-\frac{1}{n}\right)\left(f_{[n\theta_n]}^{Y}(x,y)\right)^2$$
$$= \frac{n}{k^2}\left(\frac{[n\theta_n]}{n}-\frac{[n\theta_n]^2}{n^2}\right)\left(f_{[n\theta_n]}^{Y}(x,y)\right)^2$$
$$\leq \frac{l\theta_n}{k}.$$

Similarly, we can also see that

$$Var\left(B_{nj}^{(1,5)}(y)\right)\leq \frac{l\theta_n}{k}.$$

Therefore, for $s\leq t$

$$Var\left(A_{n5}^{(1)}(x,s)-A_{n5}^{(1)}(y,t)\right) = \sum_{j=1}^{[ks]} Var B_{nj}^{(1,5)}(x,y)+\sum_{j=[ks]+1}^{[kt]} Var B_{nj}^{(1,5)}(y)$$
$$\leq [ks]\frac{l\theta_n}{k}+([kt]-[ks])\frac{l\theta_n}{k}$$

$$\begin{aligned} &\leq l\theta_n \\ &\leq Cn^{-c}. \end{aligned}$$

We also have using equation (4.4.14)

$$|B_{nj}^{(1,5)}(x,y)| \leq 2\frac{\sqrt{n}}{k} \leq Cn^{-a} \quad \text{and} \quad |B_{nj}^{(1,5)}(y)| \leq Cn^{-a}.$$

Following the same procedure as in Proposition 4.4.4 and Naik-Nimbalkar and Rajarshi [45], we can show tightness of the sequence $A_{n5}^{(1)}(\cdot,\cdot)$. Moreover, using the same arguments presented above, one can prove that the sequence $A_{n5}^{(2)}(\cdot,\cdot)$ is tight.

Therefore,

$$\sup_{(x,s)\in[0,1]^2} |A_{n5}(x,s)| \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

Return now to (4.4.28). Once again, we make use of the independence of the random variables $\{I_{nj}\}_j$ and the fact that the sets $\{\mathbf{I}(I_{nj} = i), 1 \leq i \leq n\}$ are disjoint to get for x, s in $[0,1]$

$$\begin{aligned} E\left(A_{n3}(x,s)\right)^2 &= E\left[\left(\frac{\sqrt{n}}{k}\sum_{j=1}^{[ks]}\sum_{i=1}^{[n\theta_n]} \mathbf{I}(I_{nj} = i)\left(f_{li}^{Y,\theta_n}(x) - f_{[n\theta_n]}^{Y}(x)\right)\right)^2\right] \\ &= \frac{[ks]}{k^2}\sum_{i=1}^{[n\theta_n]}\left(f_{li}^{Y,\theta_n}(x) - f_{[n\theta_n]}^{Y}(x)\right)^2 \\ &\leq l\theta_n \to 0 \quad \text{as} \quad n \to \infty, \end{aligned}$$

where the second line follows since $\sum_{i=1}^{[n\theta_n]} f_{li}^{Y,\theta_n}(x) = [n\theta_n] f_{[n\theta_n]}^{Y}(x)$.

Similarly to the previous term, write

$$A_{n3}(x,s) = \sum_{j=1}^{[ks]} B_{nj}^{(3)}(x),$$

where, for each $1 \leq j \leq [ks]$

$$B_{nj}^{(3)}(x) = \frac{\sqrt{n}}{k}\sum_{i=1}^{[n\theta_n]} \mathbf{I}(I_{nj} = i)\left(f_{li}^{Y,\theta_n}(x) - f_{[n\theta_n]}^{Y}(x)\right)$$

is a triangular array of independent random variables.

We remark that

$$\begin{aligned} Var\left(B_{nj}^{(3)}(x)\right) &= \frac{1}{k^2}\sum_{i=1}^{[n\theta_n]}\left(f_{li}^{Y,\theta_n}(x) - f_{[n\theta_n]}^{Y}(x)\right)^2 \\ &\leq \frac{l\theta_n}{k}, \end{aligned}$$

and

$$\begin{aligned} Var\left(B_{nj}^{(3)}(x,y)\right) &= \frac{1}{k^2}\sum_{i=1}^{[n\theta_n]}\left(f_{li}^{Y,\theta_n}(x,y) - f_{[n\theta_n]}^{Y}(x,y)\right)^2 \\ &\leq \frac{l\theta_n}{k}. \end{aligned}$$

Hence, we get for $s \leq t$

$$\begin{aligned} Var\left(A_{n3}(x,s) - A_{n3}(y,t)\right) &= \sum_{j=1}^{[ks]} VarB_{nj}^{(3)}(x,y) + \sum_{j=[ks]+1}^{[kt]} VarB_{nj}^{(3)}(y) \\ &\leq [ks]\frac{l\theta_n}{k} + ([kt]-[ks])\frac{l\theta_n}{k} \\ &\leq l\theta_n \\ &\leq Cn^{-c}. \end{aligned}$$

Equation (4.4.14) leads to

$$|B_{nj}^{(3)}(x,y)| \leq \frac{\sqrt{n}}{k} \leq Cn^{-a} \quad \text{and} \quad |B_{nj}^{(3)}(y)| \leq Cn^{-a}.$$

Using again the same techniques as in Proposition 4.4.4 and Naik-Nimbalkar and Rajarshi [45], we can show tightness of the sequence $A_{n3}(\cdot,\cdot)$. Thus,

$$\sup_{(x,s)\in[0,1]^2} |A_{n3}(x,s)| \xrightarrow{P} 0 \quad \text{as} \quad n \to \infty.$$

Lastly, one only needs to note that the terms (4.4.28) and (4.4.29) are similar to complete the proof of Proposition 4.4.5.

□

Since Assumptions 4.2.1 imply Peligrad's conditions (3.2.2) and (3.2.3) for both sequences (Y_i) and (Z_i), the remainder of the proof of Theorem 4.2.2 can be produced similarly to the one of Theorem 4.1.1 with the use of the above proposition.

■

Chapter 5

Applications

In this chapter, we investigate the performance of the proposed tests in a set of Monte Carlo experiments. We will be examining the finite-sample accuracy of our testing procedure and the power performance of the test statistics as well. The first section will be dedicated to implementing examples in accordance with Application 3.2.3 where a candidate change-point is available. Examples with an unknown change-point will be discussed in the second section.

5.1 Application 1

Consider the following stationary processes for $i \in \mathbb{Z}$ and $b \in \mathbb{R}$

$$Y_i = \sum_{j \geq 0} a_j^{(1)} \xi_{i-j} \quad \text{and} \quad Z_i = \sum_{j \geq 0} a_j^{(2)} (\xi_{i-j} + b),$$

where ξ_j are independent and identically distributed random variables.

Recall now the model presented earlier:

$$X_i = \begin{cases} Y_i & \text{if} \quad 1 \leq i \leq [n\theta] \\ Z_i & \text{if} \quad [n\theta] < i \leq n, \end{cases}$$

where $\theta \in (0, 1)$ is a candidate change-point.

We will be proceeding under the following assumptions:

1. For $\ell = 1, 2$, the sequence $\left\{|a_j^{(\ell)}|\right\}_{j\in\mathbb{N}}$ is non-increasing such that

$$\sum_{j\geq 0} |a_j^{(\ell)}|^{\frac{\gamma}{2}} < \infty \quad \text{for some} \quad \gamma \in (0, 1].$$

2. There exist constants $C < \infty$ and $\Delta \in \left(\frac{2}{3}, 1\right]$ such that for all $u \in \mathbb{R}$

$$|E \exp(iu\xi_0)| \leq \frac{C}{(1 + |u|)^{\Delta}}.$$

3. $E[|\xi_0|^{2\gamma}] < \infty$.

Therefore, both Assumptions 2.1.1 and Assumptions 3.1.1 are satisfied.

Let (l_1, k_1) and (l_2, k_2) be two pairs of sequences of natural numbers as in Theorem 3.2.1 such that $[n\theta] = l_1 k_1$ and $n - [n\theta] = l_2 k_2$. Henceforth, we assume that the innovations $\xi_i \sim \mathcal{N}(0, 1)$. The parameters used throughout this analysis, to produce the figures of this section, are: $n = 10000$ for the sample size, $\theta = 0.5$ for the location of the change-point, $l_1 = 5$, $l_2 = 8$, $k_1 = 1000$ and $k_2 = 625$ for the length and the number of blocks considered before and after the occurrence of the change, $B = 500$ for the number of bootstrap replications.

The simulations will be made at a nominal level of significance $\alpha = 5\%$ and each case will be performed $p = 400$ times for the power analysis. We consider here both test statistics, the Kolmogorov-Smirnov (K.S) and Cramér-Von Mises (C.V.M), in order to compare their power performances.

5.1.1 Example 1

The goal here is to examine the power of the testing approach that detects a change in the coefficients, the mean and/or the variance of an AR(1) process. [1]

[1] The detailed algorithm, programmed in R, is provided in the appendix.

• Change in the coefficients:

In this case, we consider

$$\begin{cases} Y_i = \rho_1 Y_{i-1} + \xi_i \\ Z_i = \rho_2 Z_{i-1} + \xi_i. \end{cases}$$

Under the null hypothesis, $\rho_1 = \rho_2 = 0.5$. Furthermore, ρ_2 varies from 0.1 to 0.9 under the alternatives. The analogous negative coefficients will be also considered. The size and the power of the tests in both subcases are illustrated in Figure 5.1 and Figure 5.2 for both the K.S and C.V.M statistics.

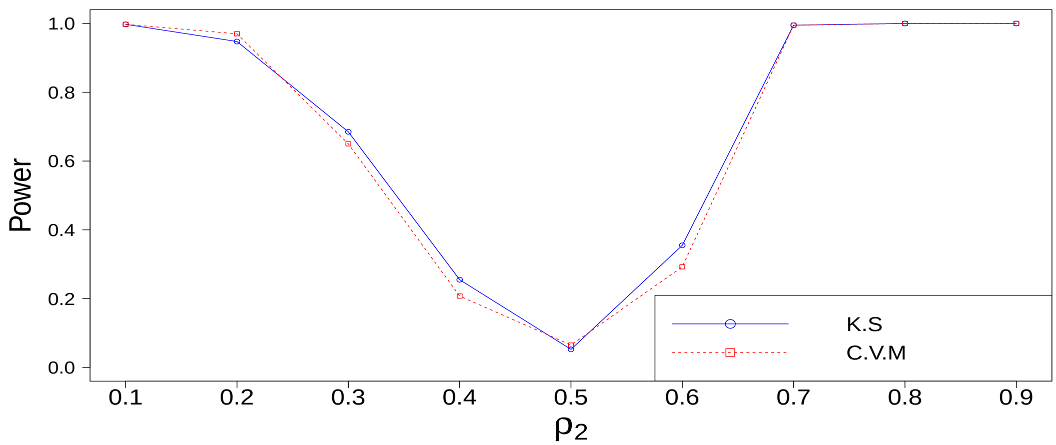

Figure 5.1: *Detection of a change in the positive coefficients of an AR(1) model*

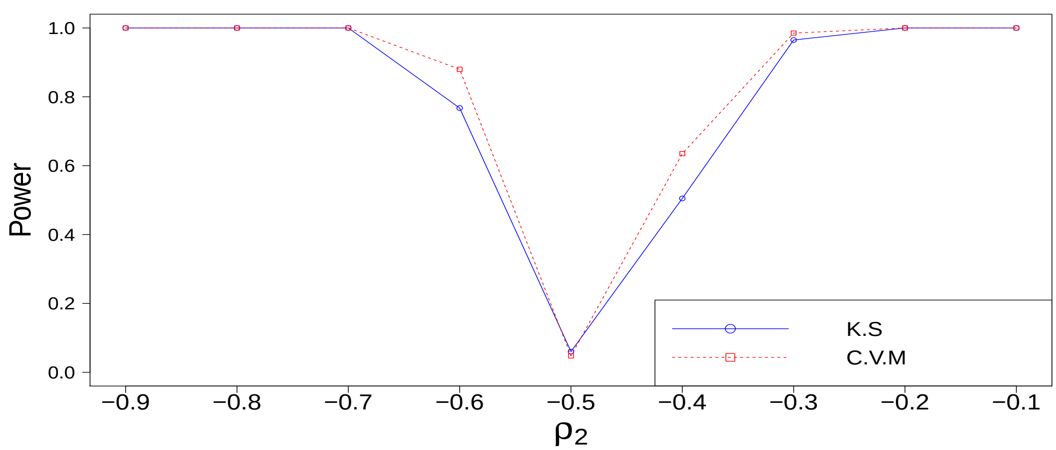

Figure 5.2: *Detection of a change in the negative coefficients of an AR(1) model*

• Change in the mean of the innovations:

In this case, we consider the following model with $\rho = 0.5$:

$$\begin{cases} Y_i = \rho Y_{i-1} + \xi_i \\ Z_i = \rho Z_{i-1} + \xi_i + \mu. \end{cases}$$

Under the null hypothesis, we consider $\mu = 0$. The mean of the innovations is chosen under the alternatives as follows: $\mu = -0.2, -0.15, -0.05, -0.1, 0.05, 0.1, 0.15, 0.2$. The performance of the K.S and C.V.M tests for this case are illustrated in the following figure.

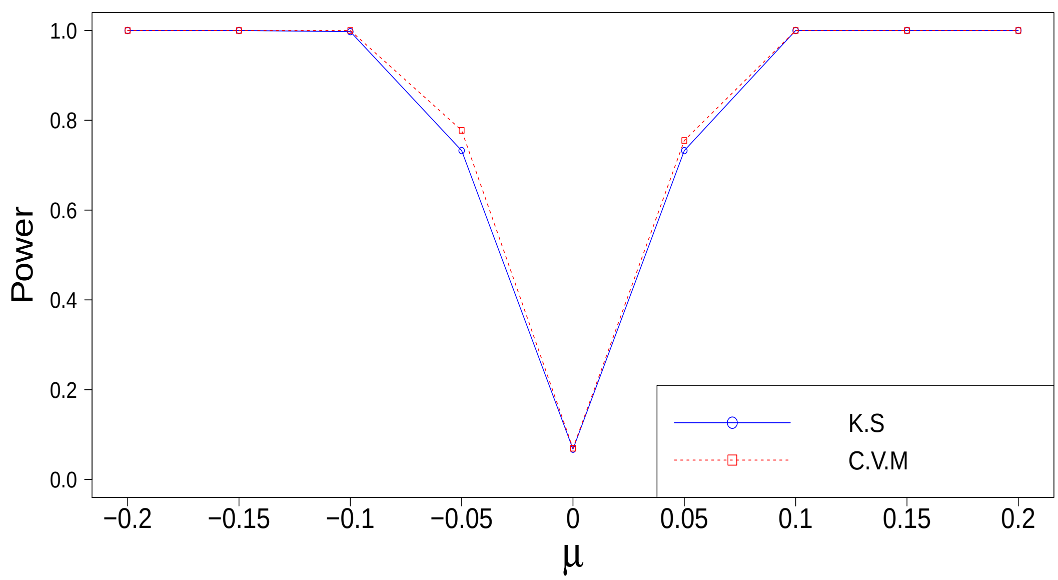

Figure 5.3: *Detection of a change in the mean of the innovations of an AR(1) model*

• Change in the variance of the innovations:

In this case, we consider the following model with $\rho = 0.5$:

$$\begin{cases} Y_i = \rho Y_{i-1} + \xi_i \\ Z_i = \rho Z_{i-1} + \sigma \xi_i. \end{cases}$$

We consider $\sigma = 1$ under the null hypothesis and $\sigma = 0.6, 0.7, 0.8, 0.9, 1.1, 1.2, 1.3, 1.4$ under the alternatives. The empirical size and the power performance of the tests are illustrated in the following figure.

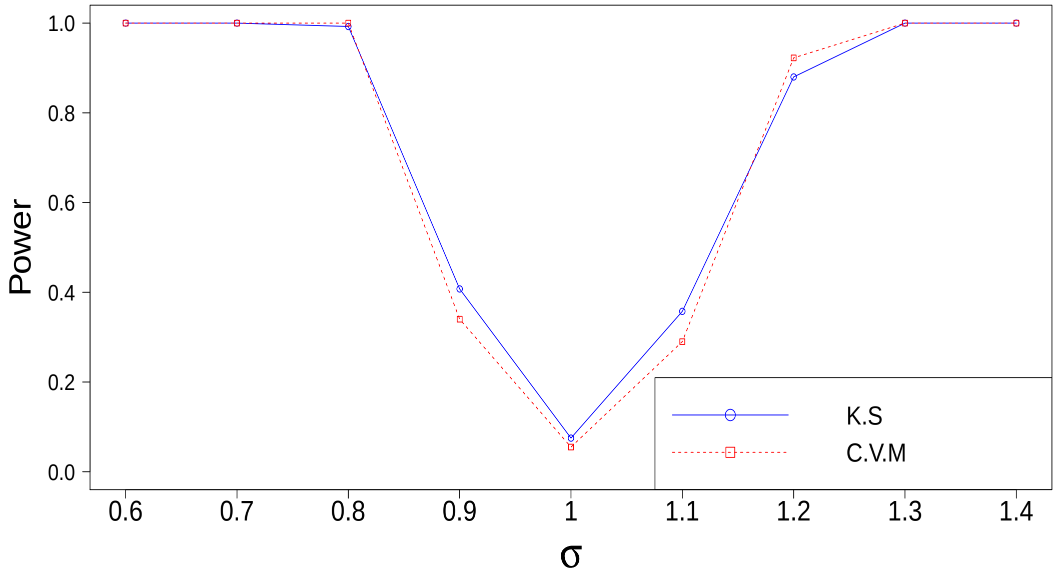

Figure 5.4: *Detection of a change in the variance of the innovations of an AR(1) model*

We were able to deal with the change-point problem in a unified fashion. In all cases, we can see that the rejection rate under the null hypothesis is close to the nominal level of significance $\alpha = 0.05$. We also observe that our testing method is performing very well under the alternatives: the power is monotone in all cases. Lastly, we note that contrary to what is frequently observed, the Cramér-Von Mises statistic does not consistently outperform the Kolmogorov-Smirnov statistic. In fact, the performance of the test statistics is very similar and often indistinguishable.

5.1.2 Example 2

Our change-point model assumes a change from one stationary process to another at the time of the change. We now investigate the behaviour of our test statistics if the assumption of stationarity after the change-point is violated. In particular, consider the following model:

$$X_i = \begin{cases} \rho_1 X_{i-1} + \xi_i & \text{if} \quad 1 \leq i \leq [n\theta] \\ \rho_2 X_{i-1} + \sigma\xi_i + \mu & \text{if} \quad [n\theta] < i \leq n. \end{cases}$$

Despite the fact that the stationarity of the process is lost after $[n\theta]$, we will see that our testing approach still allows us to detect simultaneously any change in the coefficients of the process, the mean or the variance of the innovations.
For comparison, we will be using the same parameters for each case as in the previous example. We begin with a change from ρ_1 to ρ_2.

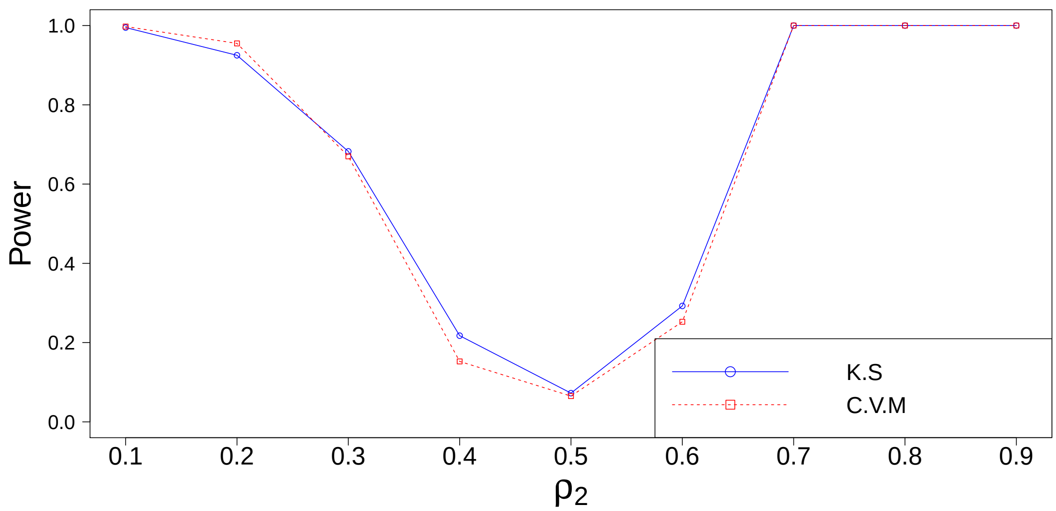

Figure 5.5: *Detection of a change from* ρ_1*=0.5*

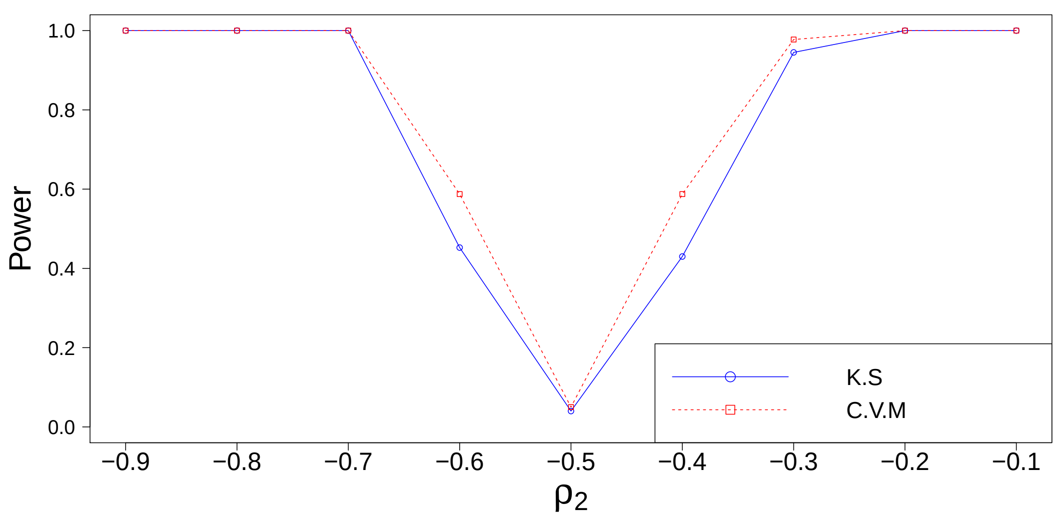

Figure 5.6: *Detection of a change from* ρ_1*=-0.5*

The following figures show the empirical size and the power performance of the tests when there is a change in the mean or the variance of the innovations.

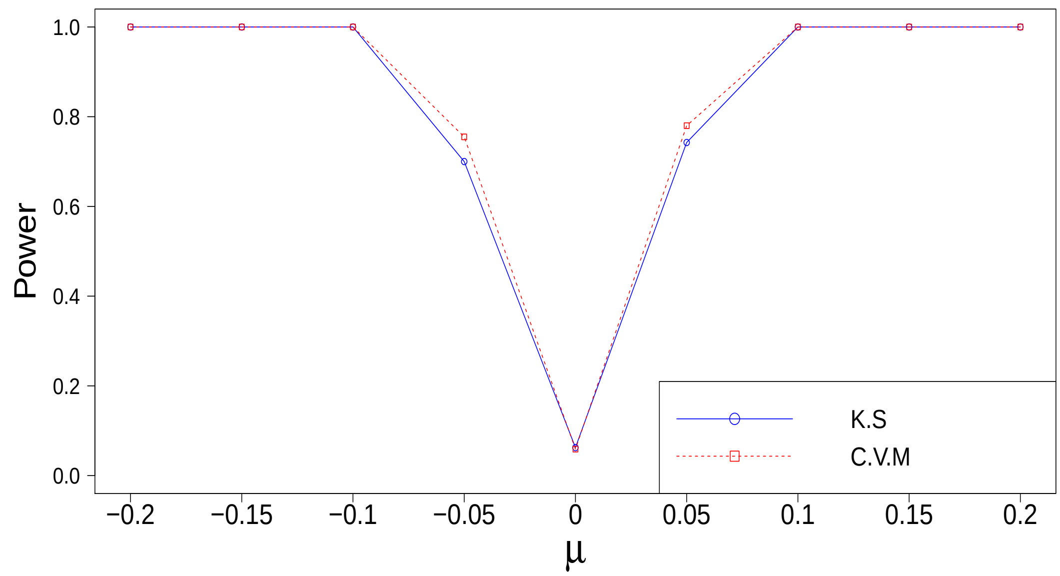

Figure 5.7: *Detection of a change from* $\mu_1 = 0$

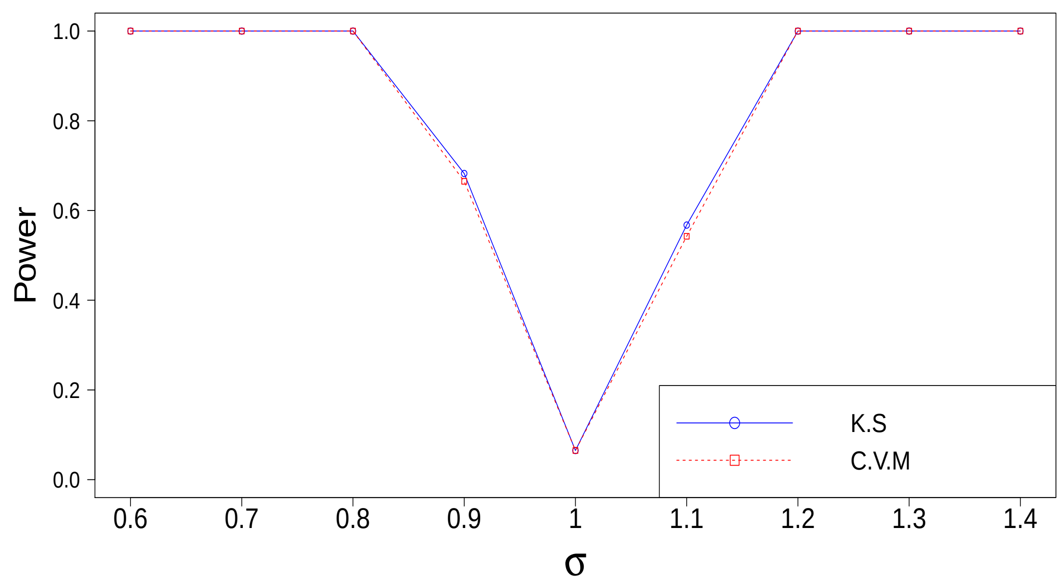

Figure 5.8: *Detection of a change from* $\sigma_1 = 1$

The general pattern is that when there is a change in location or scale of the innovations, the power of the tests is similar to the case where stationarity was satisfied. On the other hand, when the AR(1) coefficients change, the tests are less powerful than in the original example. This is likely due to the time taken for the post-change process to converge to stationarity. In general, although the assumption of stationarity is not satisfied, the tests still perform well. Finally, we observe that the sample size $n = 10000$, while quite large, is appropriate for many types of financial data. The empirical power obtained for smaller sample sizes may be found in the tables in the Appendix.

5.2 Application 2

In this section, we will make use of the results of Chapter 2 and Chapter 4. We first recall the model that will be considered here

$$X_i = \begin{cases} Y_i & \text{if} \quad 1 \le i \le [n\theta_n] \\ Z_i & \text{if} \quad [n\theta_n] < i \le n, \end{cases}$$

where Y_i and Z_i are the two linear processes as in Section 5.1 and $\theta_n \in (0,1)$.
In this case, we assume the following conditions:

1. For $\ell = 1, 2$, the sequence $\left\{|a_j^{(\ell)}|\right\}_{j\in\mathbb{N}}$ is non-increasing such that
$$\sum_{j\ge 0} |a_j^{(\ell)}|^{\frac{\gamma}{2}} < \infty \quad \text{for some} \quad \gamma \in (0,1].$$

2. There exist constants $C < \infty$ and $\Delta \in \left(\frac{2}{3}, 1\right]$ such that for all $u \in \mathbb{R}$
$$|E\exp(iu\xi_0)| \le \frac{C}{(1+|u|)^{\Delta}}.$$

3. $E[|\xi_0|^{2\gamma}] < \infty$.

4. $n^{h-1/2} \ll \theta_n \ll n^{-1/3}$ or $n^{h-1/2} \ll 1 - \theta_n \ll n^{-1/3}$ for some $0 < h < 1/6$.

This will ensure that both Assumptions 2.1.1 and 4.2.1 are satisfied. It will also make the testing approach consistent as was explained at the end of Section 4.3.

We also consider the sequences of natural numbers, l and k, as in Theorem 3.2.1 such that $n = lk$. Once again, we assume that the innovations ξ_i are i.i.d $\mathcal{N}(0,1)$.

In the following set of simulations, by symmetry it is enough to consider an early change in the observation period. We choose a number of bootstrap replicas $B = 500$. The simulations will be made at a nominal level of significance $\alpha = 5\%$ and each case will be performed $p = 400$ times in order to get the empirical power.

5.2.1 Example 3

In this example, we will be investigating the size and the power performance of the K.S and C.V.M statistics in detecting a change in the coefficients, the mean and/or the variance of an AR(1) process.

• Change in the coefficients:

We choose here a sample size $n = 12000$, block length $l = 3$, number of blocks $k = 4000$ and a change point at $\theta_n = 0.05$. We begin with a change in the positive coefficients from ρ_1 to ρ_2 where we assume that $\rho_1 = 0.75$ and ρ_2 varies as in the previous examples from 0.1 to 0.9. The size and the power of the test are illustrated in the following figure.

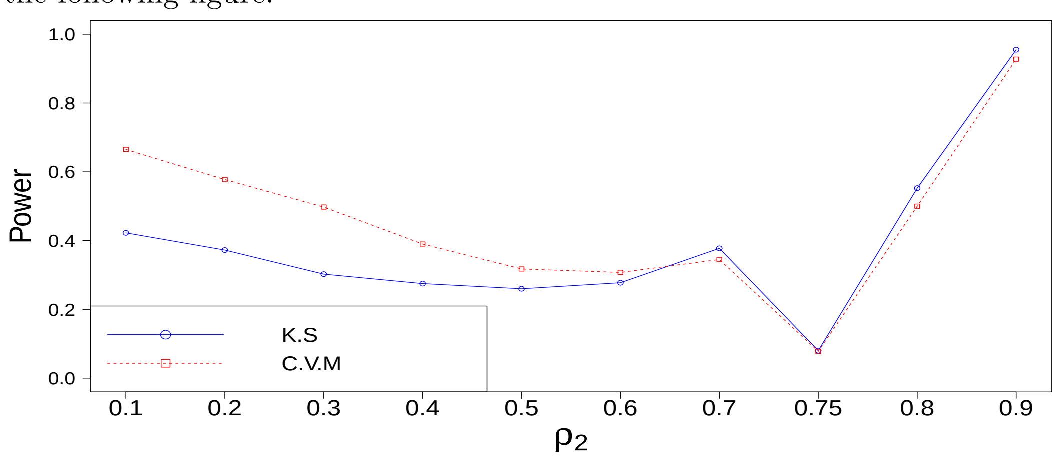

Figure 5.9: *Detection of an early change in coefficients of an AR(1) model*

The case for the analogous negative coefficients is also considered where $\rho_1 = -0.75$ and $\rho_2 = -0.9, -0.8, -0.75, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1$.

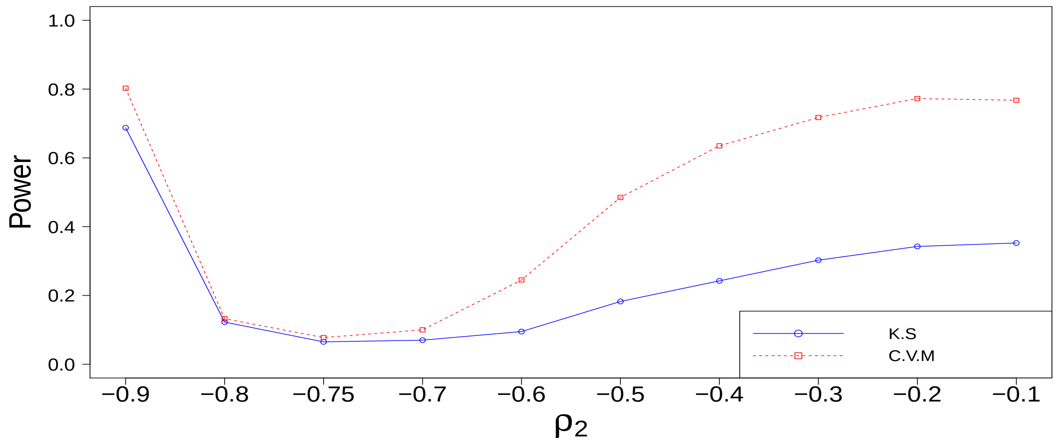

Figure 5.10: *Detection of an early change in coefficients of an AR(1) model*

For the two next cases, we will consider a sample size $n = 10000$, length block $l = 5$, number of blocks $k = 2000$ and a change-point at $\theta_n = 0.05$.

- Change in the mean:

The following figure shows the empirical size and the power performance of the tests when there is a change in the mean of the innovations. Here, we consider $\mu_1 = \mu_2 = 0$ under the null hypothesis. On the other hand, μ_2 will take the following values under the alternative: $\mu_2 = -0.4, -0.3, -0.2, -0.1, 0.1, 0.2, 0.3, 0.4$.

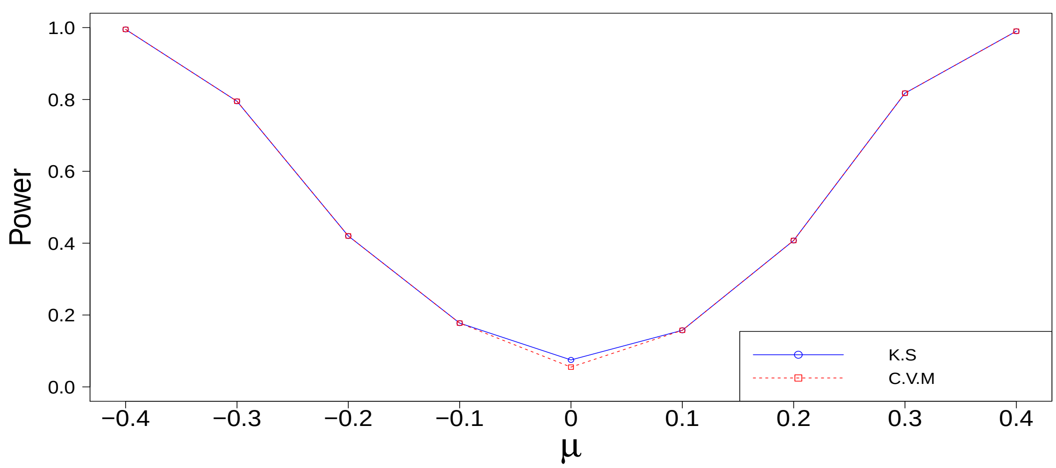

Figure 5.11: *Detection of an early change in the mean of the innovations of an AR(1) model*

• Change in the variance:

Figure 5.12 deals with a change in the variance of the innovations of the autoregressive model. In this case, $\sigma_1 = 1$ and $\sigma_2 = 0.2, 0.4, 0.6, 0.8, 1, 1.2, 1.4, 1.6, 1.8$.

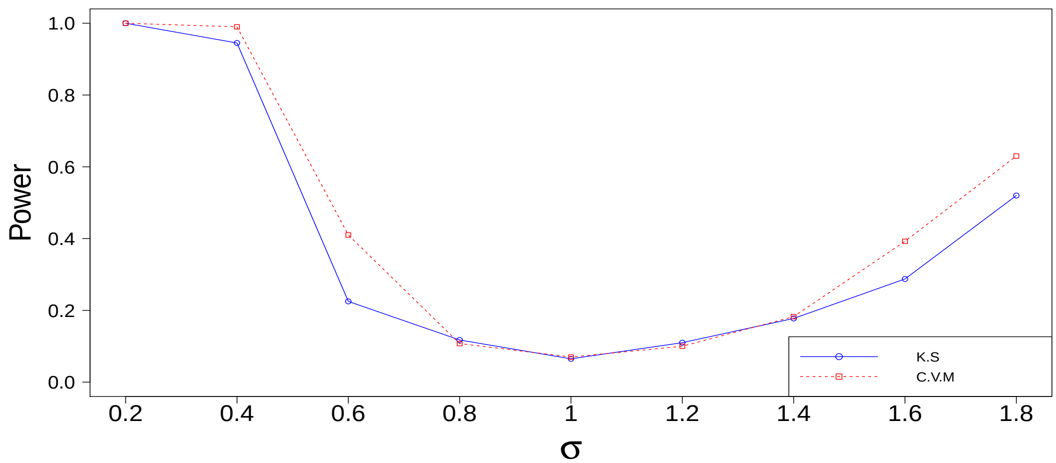

Figure 5.12: *Detection of an early change in the variance of the innovations of an AR(1) model*

The simulations presented in this example show that the testing method performs well especially for the case of a change in the mean of the innovations. For the other cases, we observe that the tests are asymmetric with respect to the change-point locations. It can also be seen that the C.V.M performs notably better than the K.S in the mentioned cases. As expected, the tests presented in this example are less powerful than the ones presented earlier in the first example where a candidate change-point was available. Moreover, we observed from many more simulations, not presented here, that the choice of the block length is very crucial in optimizing the empirical size and the power of the tests.

5.2.2 Example 4

In this example, we will investigate the performance of the tests statistics when the assumption of converging alternatives is violated. More precisely, we will reproduce the simulations presented in the preceding example with $\theta = 0.5$.

We begin with a change in the coefficients (positive and negative coefficients) of the autoregressive model.

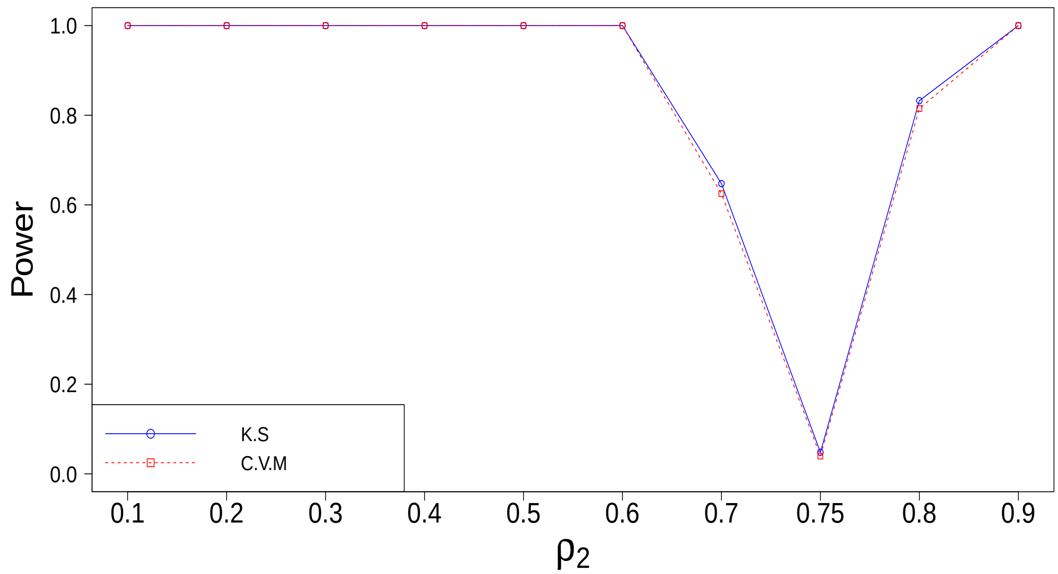

Figure 5.13: *Detection of a change from* $\rho_1 = 0.75$

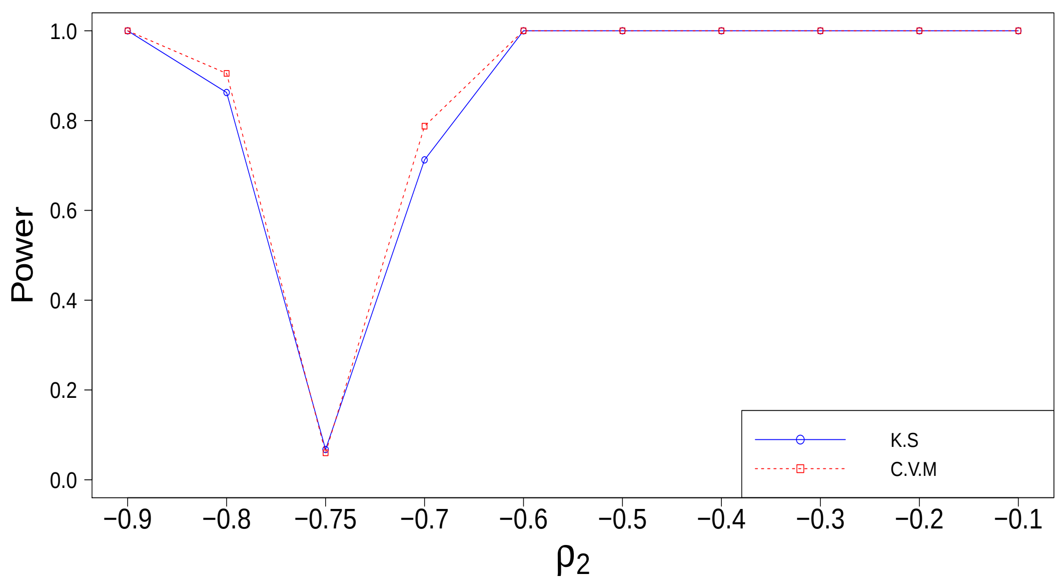

Figure 5.14: *Detection of a change from* $\rho_1 = -0.75$

Figure 5.15 and Figure 5.16 show the empirical size and the power performance of the tests when there is a change in the mean or the variance of the innovations.

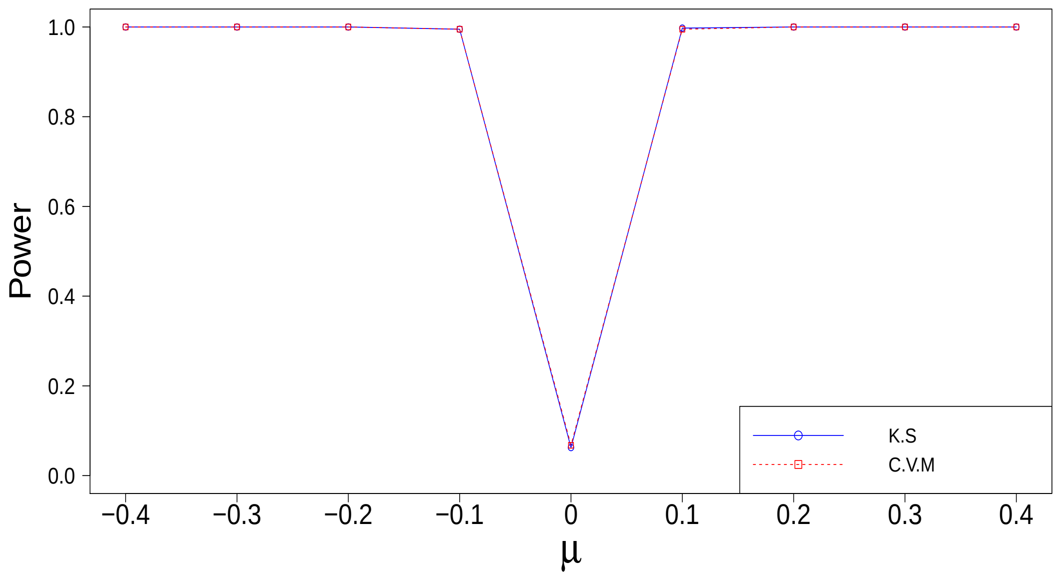

Figure 5.15: *Detection of a change from* $\mu_1 = 0$

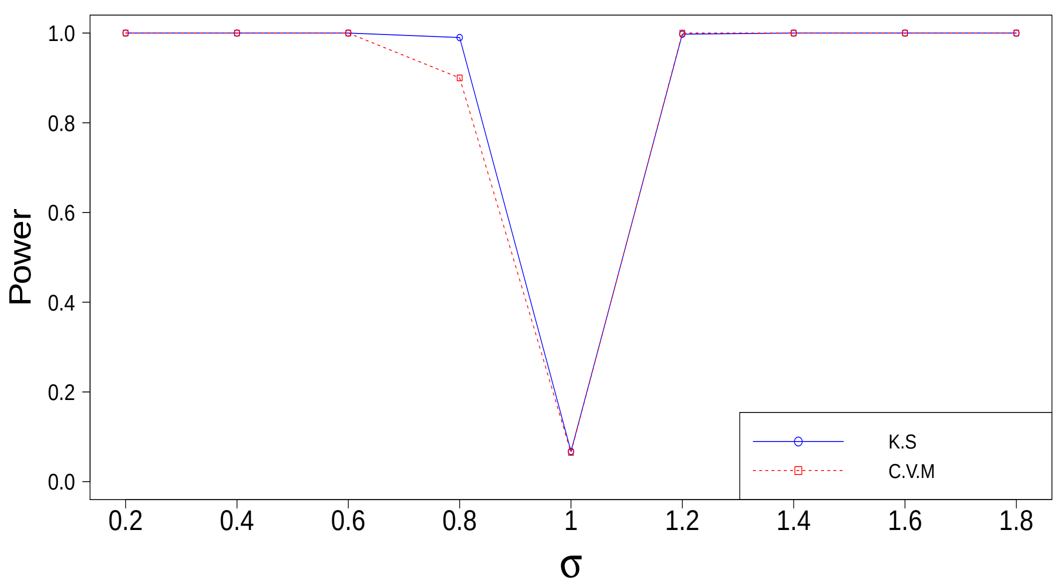

Figure 5.16: *Detection of a change from* $\sigma_1 = 1$

We were puzzled with simulations of this example that demonstrate more power under the alternatives, accurate empirical size under the null, symmetric tests and elimination of the superiority of the C.V.M. This is likely due to the important information gathered before and after the change-point. This observation make us believe that the detection problem can be addressed, with a different approach, when a change-point is not close to 0 nor 1. This also explains the results of the preceding example since the effect of the pre-change parameters dominates during the "adjustment period" after the change and has an influence on the accuracy of the estimation of the empirical distributions.

Our testing method is very general and does only require a minimal information about the pre-change and post-change distributions. It should not replace existing parametric tests but rather complement them. However, when the correct parametrization is unknown, our tests can be seen as a useful pretest. As pointed before, the choice of the block length is very important in practice and must be analysed carefully. Here, we did not try to fine-tune the testing method in order to obtain optimum performance. We leave it for future research.

■

Conclusion

As indicated in the introduction, this thesis proposes a retrospective testing method to detect a change in the marginal distribution of a causal linear process. The detection approach described here allows us to test simultaneously for changes in the coefficients, in the location and/or scale of the innovations under reasonable and minimal conditions. More precisely, the use of mixing or association assumptions was avoided by opting for a combination of conditions on the moments of the innovations and the summability of the coefficients. These simple conditions justify a functional central limit theorem for the sequential empirical process associated with a stationary causal linear model. This task was completed in Chapter 1. A second invariance principle, in the change-point framework, for the sequential empirical process was proved in Chapter 2. We have also derived in the latter chapter the limiting distributions of the test statistics used throughout this work; namely, the Kolmogorov-Smirnov and Cramér-Von Mises statistics, under the null and the alternative hypotheses.

As in similar studies, the covariance of the Gaussian limit in the invariance principles for the the sequential empirical process presents a level of complexity that makes critical values very difficult to evaluate. To deal with this problem, we made use of moving block bootstrap techniques that imitate the data generating process without imposing any particular linear model. Weak convergence of the associated bootstrapped empirical process was established under conditions similar to the previous ones; in particular mixing is not assumed. This objective has been accomplished in Chapter 3.

Chapter 4 provides an extension of the preceding result to functional central limit theorems for the sequential bootstrapped empirical process under the null hypothesis and under converging alternatives. Furthermore, we derived the limiting distributions of the bootstrapped version of the test statistics with a view to tabulating appropriate critical values for the testing procedure. In Chapter 5, simulations demonstrate the performance of the test statistics for various illustrative examples.

This thesis addressed the problem of change-point detection in the marginal distribution for a wide range of linear processes where mixing conditions are not satisfied or very difficult to verify. The goal was to produce a testing approach that is theoretically well founded in a unified framework that can be easily implemented by practitioners in the time series environment. Furthermore, it suggests new research questions that may require a different methodology. Indeed, the research contained in this thesis can be extended in various directions. Some of these are:

- From the simulations illustrated in Chapter 5 and many more not presented here, it was clear to us that further investigation is needed on the connection between the parameters (n, θ, l, k) in the sequential model in order to produce better results in finite-sample situations.

- The assumption of converging alternatives, which was of great use in solving the change-point problem, does not adequately address the problem when the change-point is not close to 0 nor 1. This problem needs to be approached in a different manner with different tools.

- The change-point problem can be approached from a sequential perspective where the decision is made on line with the observations. This is another problem which can be considered in this area.

- The testing procedure proposed in this thesis can be extended to spatial processes in higher dimensions with a view of detecting a change-point or a change-set in the marginal distribution of a stationary causal linear field.

■

Appendix

In this appendix, we include the algorithms and R codes for the simulations. To supplement the results presented in Chapter 5, we also include further tables of empirical power for various sample sizes n.

Example 1

• **<u>Case 1</u>**: Detection of a change in the coefficients of an $AR(1)$ model.

Step 1a: Generate the data and the test outcomes under H_0

```
data0<-function(n,r){
sd1<-1/(sqrt(1-r^2));X<-c()
X[1]<-r*rnorm(1,0,sd1)+rnorm(1)
for(i in 2:n){X[i]<-r*X[i-1]+rnorm(1)}
X}

test0<-function(n){
X<-data0(n,r)
N<-floor(n*theta)
x1<-X[1:N];x2<-X[(N+1):n]
FN1<-ecdf(x1);FN2<-ecdf(x2);
V1<-N*(n-N)*(FN1(X)-FN2(X))/(n^(3/2));
V2<-V1^2;
```

```
T1<-max(abs(V1));T2<-mean(V2);
y1<-c(x1,x1[1:(l1-1)]);y2<-c(x2,x2[1:(l2-1)]);
m1<-c(1:N);m2<-c(1:(n-N));
Tb<-vapply(1:b, FUN.VALUE=numeric(2),function(i){
x1b<-as.vector(sapply(sample(m1, k1, rep=TRUE),
function(p1)y1[p1:(p1+l1-1)]))
x2b<-as.vector(sapply(sample(m2, k2, rep=TRUE),
function(p2)y2[p2:(p2+l2-1)]))
FN1b<-ecdf(x1b);FN2b<-ecdf(x2b);
Vb1 <- N*(n-N)*(FN1b(X)-FN1(X)-FN2b(X)+FN2(X))/(n^(3/2))
Vb2<-Vb1^2;
c(max(abs(Vb1)),mean(Vb2))})
res0<-c(0,0);
if (T1>quantile(Tb[1,],0.95)){res0[1]<-1}
if (T2>quantile(Tb[2,],0.95)){res0[2]<-1}
res0}
```

<u>Step 1b</u>: Generate the data and the test outcomes under H_1

```
data1<-function(n,r1,r2,theta){
z<-rnorm(n);
a<-vapply(1:n, FUN.VALUE=numeric(2), function(i){
c((r1)^(i-1)*z[i],(r2)^(i-1)*z[i])})
Y<-c();Z<-c();xi<-rnorm(n);N<-floor(n*theta);
Y[1]<-(r1)*sum(a[1,])+xi[1];Z[1]<-(r2)*sum(a[2,])+xi[1];
for(i in 2:N){Y[i]<-(r1)*Y[i-1]+xi[i]};
for(i in 2:n){Z[i]<-(r2)*Z[i-1]+xi[i]}
X<-c(Y,Z[(N+1):n])
```

```
X}
test1<-function(n){
X<-data1(n,r1,r2,theta)
N<-floor(n*theta)
x1<-X[1:N];x2<-X[(N+1):n]
FN1<-ecdf(x1);FN2<-ecdf(x2);
V1<-N*(n-N)*(FN1(X)-FN2(X))/(n∧(3/2));
V2<-V1∧2;
T1<-max(abs(V1));T2<-mean(V2);
y1<-c(x1,x1[1:(l1-1)]);y2<-c(x2,x2[1:(l2-1)]);
m1<-c(1:N);m2<-c(1:(n-N));
Tb<-vapply(1:b, FUN.VALUE=numeric(2),function(i){
x1b<-as.vector(sapply(sample(m1, k1, rep=TRUE),
function(p1)y1[p1:(p1+l1-1)]))
x2b<-as.vector(sapply(sample(m2, k2, rep=TRUE),
function(p2)y2[p2:(p2+l2-1)]))
FN1b<-ecdf(x1b);FN2b<-ecdf(x2b);
Vb1 <- N*(n-N)*(FN1b(X)-FN1(X)-FN2b(X)+FN2(X))/(n∧(3/2))
Vb2<-Vb1∧2;
c(max(abs(Vb1)),mean(Vb2))})
res1<-c(0,0);
if (T1>quantile(Tb[1,],0.95)){res1[1]<-1}
if (T2>quantile(Tb[2,],0.95)){res1[2]<-1}
res1}
library(parallel)
```

```
pp<-matrix(,nrow=2,ncol=9)
```

Step 2: Compute the empirical size of the test

```
theta<-0.5;r<-0.5;l1<-5;k1<-1000;l2<-8;k2<-625;b<-500;
cl5<- makeCluster(detectCores())
clusterSetRNGStream(cl5)
clusterExport(cl5,c("r","theta","l1","k1","l2","k2","b"))
clusterExport(cl5,"data0")
pp5<-parSapply(cl5,rep(10000,400),test0)
(pp[,5]<-c(mean(pp5[1,]),mean(pp5[2,])))
stopCluster(cl5)
```

Step 3: Compute the empirical power of the test for r2=0.1

```
theta<-0.5;r1<-0.5;r2<-0.1;l1<-5;k1<-1000;l2<-8;
k2<-625;b<-500;
cl1<- makeCluster(detectCores())
clusterSetRNGStream(cl1)
clusterExport(cl1,c("r1","r2","theta","l1","k1","l2","k2","b"))
clusterExport(cl1,"data1")
pp1<-parSapply(cl1,rep(10000,400),test1)
(pp[,1]<-c(mean(pp1[1,]),mean(pp1[2,])))
stopCluster(cl1)
```

Step 4: Repeat Step 3 for r1=0.2, 0.3, 0.4, 0.6, 0.7, 0.8, 0.9.

Step 5: Repeat Step 2 and Step 3 for different values of n and the corresponding l_1, k_1, l_2 and k_2.

Step 6: Repeat Step 2, Step 3 and Step 4 for the analogous negative coefficients of the AR(1) model.

Table 5.1: Empirical power for a change in positive coefficients (Example 1)

		ρ_2								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$n = 1000$, $l_1 = l_2 = 4$, $k_1 = k_2 = 125$, $B = 500$	K.S	0.2275	0.1925	0.165	0.145	0.08	0.18	0.3475	0.765	1
	C.V.M	0.1825	0.165	0.1475	0.1425	0.105	0.1675	0.31	0.71	1
$n = 5000$, $l_1 = l_2 = 5$, $k_1 = k_2 = 500$, $B = 500$	K.S	0.7475	0.5525	0.3125	0.1725	0.0825	0.2175	0.9125	1	1
	C.V.M	0.785	0.5425	0.285	0.1425	0.0875	0.195	0.9175	1	1
$n = 10000$, $l_1 = 5, k_1 = 1000$, $l_2 = 8, k_2 = 625$, $B = 500$	K.S	0.9975	0.9475	0.685	0.255	0.0525	0.355	0.995	1	1
	C.V.M	0.9975	0.97	0.650	0.2075	0.065	0.2925	0.995	1	1

Table 5.2: Empirical power for a change in negative coefficients (Example 1)

		ρ_2								
		-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
$n = 1000$, $l_1 = l_2 = 4$, $k_1 = k_2 = 125$, $B = 500$	K.S	1	0.96	0.56	0.115	0.0475	0.0725	0.14	0.1875	0.2075
	C.V.M	1	0.9975	0.695	0.135	0.0325	0.085	0.1675	0.23	0.2275
$n = 5000$, $l_1 = l_2 = 5$, $k_1 = k_2 = 500$, $B = 500$	K.S	1	1	1	0.455	0.05	0.1975	0.6125	0.8575	0.9275
	C.V.M	1	1	1	0.5875	0.05	0.2725	0.7725	0.9375	0.975
$n = 10000$, $l_1 = 5, k_1 = 1000$, $l_2 = 8, k_2 = 625$, $B = 500$	K.S	1	1	1	0.7675	0.06	0.505	0.965	1	1
	C.V.M	1	1	1	0.88	0.0475	0.635	0.985	1	1

• **Case 2**: Detection of a change in the mean of the innovations of an $AR(1)$ model.

Step 1a: Generate the data under H_1

data2<–function(n,r,mu,sigma,theta){

```
sd1<-1/(sqrt(1-r∧2));
Y<-c();
Y[1]<-r*rnorm(1,0,sd1)+rnorm(1);
for(i in 2:n){Y[i]<-r*Y[i-1]+rnorm(1)}
Z<-(sigma*Y)+mu/(1-r);
N<-floor(n*theta);
X<-c(Y[1:N],Z[(N+1):n])
X}
```

Step 1b: Generate the test outcomes under H_1

```
test2<-function(n){
X<-data2(n,r,mu,sigma,theta)
N<-floor(n*theta);
x1<-X[1:N];x2<-X[(N+1):n]
FN1<-ecdf(x1);FN2<-ecdf(x2);
V1<-N*(n-N)*(FN1(X)-FN2(X))/(n∧(3/2));
V2<-V1∧2;
T1<-max(abs(V1));T2<-mean(V2);
y1<-c(x1,x1[1:(l1-1)]);y2<-c(x2,x2[1:(l2-1)]);
m1<-c(1:N);m2<-c(1:(n-N));
Tb<-vapply(1:b, FUN.VALUE=numeric(2),function(i){
x1b<-as.vector(sapply(sample(m1, k1, rep=TRUE),
function(p1)y1[p1:(p1+l1-1)]))
x2b<-as.vector(sapply(sample(m2, k2, rep=TRUE),
function(p2)y2[p2:(p2+l2-1)]))
FN1b<-ecdf(x1b);FN2b<-ecdf(x2b);
Vb1 <- N*(n-N)*(FN1b(X)-FN1(X)-FN2b(X)+FN2(X))/(n∧(3/2))
Vb2<-Vb1∧2;
```

```
c(max(abs(Vb1)),mean(Vb2))})
res1<-c(0,0);
if (T1>quantile(Tb[1,],0.95)){res1[1]<-1}
if (T2>quantile(Tb[2,],0.95)){res1[2]<-1}
res1}
library(parallel)
cm<-matrix(,nrow=2,ncol=9)
```

Step 2: Repeat Step 2 from Case 1 to compute the empirical size of the test

Step 3: Compute the empirical power of the test for mu=0.1

```
theta<-0.5;r<-0.5;mu<-0.1;sigma<-1;l1<-5;k1<-1000;
l2<-8;k2<-625;b<-500;
cl1<- makeCluster(detectCores())
clusterSetRNGStream(cl1)
clusterExport(cl1,c("r","mu","sigma","theta","l1","k1","l2",
                    "k2","b"))
clusterExport(cl1,"data2")}
cm1<-parSapply(cl1,rep(10000,400),test2)
(cm[,1]<-c(mean(cm[1,]),mean(cm[2,])))
stopCluster(cl1)
```

Step 4: Repeat Step 3 for mu=-0.3, -0.25, -0.2, -0.1, 0.2, 0.25, 0.3.

Step 5: Repeat Step 2, Step 3 and Step 4 for different values of n and the corresponding l_1, k_1, l_2 and k_2.

Table 5.3: Empirical power for a change in the mean (Example 1)

		μ								
		-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2
$n = 1000$ $l_1 = l_2 = 4$ $k_1 = k_2 = 125$ $B = 500$	K.S	0.9025	0.6625	0.46	0.19	0.1125	0.2175	0.4025	0.745	0.94
	C.V.M	0.9175	0.71	0.5	0.2025	0.1375	0.2175	0.4575	0.785	0.95
$n = 5000$ $l_1 = l_2 = 5$ $k_1 = k_2 = 500$ $B = 500$	K.S	1	1	0.9525	0.4975	0.0875	0.5375	0.96	1	1
	C.V.M	1	1	0.9675	0.53	0.085	0.5825	0.975	1	1
$n = 10000$ $l_1 = 5, k_1 = 1000$ $l_2 = 8, k_2 = 625$ $B = 500$	K.S	1	1	0.9975	0.7325	0.0675	0.7325	1	1	1
	C.V.M	1	1	1	0.7775	0.0695	0.755	1	1	1

• **Case 3**: Detection of a change in the variance of the innovations of an $AR(1)$ model.

Step 1: cm<–matrix(,nrow=2,ncol=9)

Step 2: Repeat Step 2 from Case 1 to compute the empirical size of the test

Step 3: Compute the empirical power of the test for sigma=0.6

```
theta<–0.5;r<–0.5;mu<–0;sigma<–0.6;l1<–5;k1<–1000;
l2<–8;k2<–625;b<–500;
cl1 <– makeCluster(detectCores())
clusterSetRNGStream(cl1)
clusterExport(cl1,c(”r”,”mu”,”sigma”,”theta”,”l1”,”k1”,”l2”,
                ”k2”,”b”))
clusterExport(cl1,”data2”)}
cv1<–parSapply(cl1,rep(10000,400),test2)
(cv[,1]<–c(mean(cv1[1,]),mean(cv1[2,])))
```

stopCluster(cl1)

Step 4: Repeat Step 3 for sigma=0.7, 0.8, 0.9, 1.1, 1.2, 1.3, 1.4.

Step 5: Repeat Step 2, Step 3 and Step 4 for different values of n and the corresponding l_1, k_1, l_2 and k_2.

Table 5.4: Empirical power for a change in the variance (Example 1)

		σ								
		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$n=1000$, $l_1=l_2=4$, $k_1=k_2=125$, $B=500$	K.S	1	0.845	0.3925	0.1775	0.1025	0.1325	0.275	0.505	0.735
	C.V.M	1	0.86	0.31	0.1675	0.1175	0.125	0.2225	0.4375	0.765
$n=5000$, $l_1=l_2=5$, $k_1=k_2=500$, $B=500$	K.S	1	1	0.9925	0.4075	0.1175	0.3575	0.88	1	1
	C.V.M	1	1	1	0.34	0.115	0.29	0.9225	1	1
$n=10000$, $l_1=5, k_1=1000$, $l_2=8, k_2=625$, $B=500$	K.S	1	1	1	0.695	0.0575	0.5575	1	1	1
	C.V.M	1	1	1	0.6625	0.0625	0.5475	1	1	1

Example 2

It is enough to replace the functions "data1" and "data2" in the codes of the preceding example respectively by the functions "data11" and "data22" described below.

• **Case 1**: Detection of a change in the coefficients.

```
data11<–function(n,theta,r1,r2){
sd1<–1/(sqrt(1-(r1)∧2));
Y<–c();Z<–c();N<–floor(n*theta)
Y[1]<–(r1)*rnorm(1,0,sd1)+rnorm(1);
for(i in 2:N){Y[i]<–(r1)*Y[i-1]+rnorm(1)}
Z[1]<–(r2)*Y[N]+rnorm(1)
for(i in 2:(n-N)){Z[i]<–(r2)*Z[i-1]+rnorm(1)}
X<–c(Y,Z);
X}
```

• **Case 2 & 3**: Detection of a change in the mean or the variance.

```
data22<–function(n,theta,r,mu,sigma){
sd1<–1/(sqrt(1-r∧2));
Y<–c();Z<–c();N<–floor(n*theta)
Y[1]<–r*rnorm(1,0,sd1)+rnorm(1);
for(i in 2:N){Y[i]<–r*Y[i-1]+rnorm(1)}
Z[1]<–r*Y[N]+(mu+sigma*rnorm(1))
for(i in 2:(n-N)){Z[i]<–(r2)*Z[i-1]+(mu+sigma*rnorm(1))}
X<–c(Y,Z);
X}
```

Table 5.5: Empirical power for a change in positive coefficients(Example 2)

		ρ_2								
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$n = 1000$, $l_1 = l_2 = 4$, $k_1 = k_2 = 125$, $B = 500$	K.S	0.205	0.19	0.13	0.1175	0.1025	0.17	0.31	0.7575	1
	C.V.M	0.1675	0.1825	0.115	0.1125	0.105	0.165	0.275	0.7475	1
$n = 5000$, $l_1 = l_2 = 5$, $k_1 = k_2 = 500$, $B = 500$	K.S	0.685	0.555	0.3475	0.1375	0.09	0.2475	0.8875	1	1
	C.V.M	0.74	0.575	0.2925	0.115	0.09	0.2075	0.9	1	1
$n = 10000$, $l_1 = 5, k_1 = 1000$, $l_2 = 8, k_2 = 625$, $B = 500$	K.S	0.995	0.925	0.6825	0.2175	0.07	0.2925	1	1	1
	C.V.M	0.9975	0.955	0.67	0.1525	0.065	0.2525	1	1	1

Table 5.6: Empirical power for a change in negative coefficients (Example 2)

		ρ_2								
		-0.9	-0.8	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
$n = 1000$, $l_1 = l_2 = 4$, $k_1 = k_2 = 125$, $B = 500$	K.S	1	0.9725	0.5025	0.115	0.042	0.0625	0.14	0.175	0.255
	C.V.M	1	0.995	0.6175	0.1375	0.0425	0.0825	0.16	0.235	0.3325
$n = 5000$, $l_1 = l_2 = 5$, $k_1 = k_2 = 500$, $B = 500$	K.S	1	1	1	0.4175	0.06	0.2475	0.56	0.8725	0.94
	C.V.M	1	1	1	0.5575	0.0475	0.3375	0.7425	0.955	0.9825
$n = 10000$, $l_1 = 5, k_1 = 1000$, $l_2 = 8, k_2 = 625$, $B = 500$	K.S	1	1	1	0.4525	0.04	0.43	0.945	1	1
	C.V.M	1	1	1	0.5875	0.05	0.5875	0.9775	1	1

Table 5.7: Empirical power for a change in the mean (Example 2)

		μ								
		-0.2	-0.15	-0.1	-0.05	0	0.05	0.1	0.15	0.2
$n = 1000$, $l_1 = l_2 = 4$, $k_1 = k_2 = 125$, $B = 500$	K.S	0.9125	0.715	0.4325	0.2	0.09	0.2075	0.3825	0.7175	0.9
	C.V.M	0.925	0.7425	0.4775	0.2225	0.095	0.225	0.4375	0.7775	0.925
$n = 5000$, $l_1 = l_2 = 5$, $k_1 = k_2 = 500$, $B = 500$	K.S	1	0.9975	0.95	0.485	0.0825	0.48	0.9675	1	1
	C.V.M	1	0.9975	0.955	0.5525	0.0925	0.51	0.9775	1	1
$n = 10000$, $l_1 = 5, k_1 = 1000$, $l_2 = 8, k_2 = 625$, $B = 500$	K.S	1	1	1	0.7	0.0625	0.7425	1	1	1
	C.V.M	1	1	1	0.755	0.059	0.78	1	1	1

Table 5.8: Empirical power for a change in the variance (Example 2)

		σ								
		0.6	0.7	0.8	0.9	1	1.1	1.2	1.3	1.4
$n = 1000$, $l_1 = l_2 = 4$, $k_1 = k_2 = 125$, $B = 500$	K.S	0.9975	0.82	0.355	0.157	0.0825	0.14	0.2325	0.51	0.7175
	C.V.M	1	0.8275	0.295	0.1325	0.085	0.135	0.2075	0.44	0.7225
$n = 5000$, $l_1 = l_2 = 5$, $k_1 = k_2 = 500$, $B = 500$	K.S	1	1	0.9925	0.3775	0.775	0.285	0.9025	1	1
	C.V.M	1	1	1	0.3275	0.075	0.2425	0.94	1	1
$n = 10000$, $l_1 = 5, k_1 = 1000$, $l_2 = 8, k_2 = 625$, $B = 500$	K.S	1	1	1	0.6825	0.065	0.5675	1	1	1
	C.V.M	1	1	1	0.665	0.065	0.5425	1	1	1

Example 3

• **Case 1**: Detection of an early change in the coefficients of an AR(1) model.

Step 1a: Generate the test outcomes under H_0

```
test30<-function(n){
X<-data0(n,r1)
s<-seq(0,1,by=0.01);
Vs1<-lapply(1:length(s), FUN=function(j){
ns<-floor(n*s[j])
xs1<-X[1:ns]
xs2<-X[(ns+1):n]
FNs1<-ecdf(xs1)
FNs2<-ecdf(xs2)
ns*(n-ns)*(FNs1(X)-FNs2(X))/(n^(3/2))})
Vs1 <- do.call(rbind,(Vs1))
Vs2<-Vs1^2;
T1<-max(abs(Vs1));T2<-mean(Vs2)
y<-c(X,X[1:(l-1)]);m<-c(1:n);
Tb<-vapply(1:b, FUN.VALUE=numeric(2),function(i){
Xb<-as.vector(sapply(sample(m, k, rep=TRUE),
function(p)y[p:(p+l-1)]))
Vbs1 <- lapply(1:length(s), function(j){
ks<-floor(k*s[j]);
x1sb<-Xb[1:(l*ks)];x2sb<-Xb[(l*ks+1):n];
FNs1b<-ecdf(x1sb);FNs2b<-ecdf(x2sb);
l*ks*(k-ks)*(FNs1b(X)-FNs2b(X))/(k*sqrt(n))})
Vbs1<-do.call(rbind,(Vbs1))
Vbs2<-Vbs1^2;
```

```
c(max(abs(Vbs1)),mean(Vbs2))})
res30<-c(0,0);
if (T1>quantile(Tb[1,],0.95)){res30[1]<-1}
if (T2>quantile(Tb[2,],0.95)){res30[2]<-1}
res30}
```

Step 1b: Generate the test outcomes under H_1

```
test3<-function(n){
X<-data1(n,theta,r1,r2)
s<-seq(0,1,by=0.01);
Vs1<-lapply(1:length(s), FUN=function(j){
ns<-floor(n*s[j]);
xs1<-X[1:ns];xs2<-X[(ns+1):n];
FNs1<-ecdf(xs1);FNs2<-ecdf(xs2);
ns*(n-ns)*(FNs1(X)-FNs2(X))/(n^(3/2))})
Vs1 <- do.call(rbind,(Vs1))
Vs2<-Vs1^2;
T1<-max(abs(Vs1));T2<-mean(Vs2)
y<-c(X,X[1:(l-1)]);m<-c(1:n);
Tb<-vapply(1:b, FUN.VALUE=numeric(2),function(i){
Xb<-as.vector(sapply(sample(m, k, rep=TRUE),
function(p)y[p:(p+l-1)]))
Vbs1<-lapply(1:length(s),function(j){
ks<-floor(k*s[j]);
x1sb<-Xb[1:(l*ks)];x2sb<-Xb[(l*ks+1):n];
FNs1b<-ecdf(x1sb);FNs2b<-ecdf(x2sb);
l*ks*(k-ks)*(FNs1b(X)-FNs2b(X))/(k*sqrt(n))})
Vbs1<- do.call(rbind,(Vbs1))
```

```
Vbs2<–Vbs1∧2;
c(max(abs(Vbs1)),mean(Vbs2))})
res3<–c(0,0);
if (T1>quantile(Tb[1,],0.95)){res3[1]<–1}
if (T2>quantile(Tb[2,],0.95)){res3[2]<–1}
res3}
library(parallel)
pp<–matrix(,nrow=2,ncol=10)
```

Step 2: Compute the empirical size of the test

```
theta<–0.05;r<–0.75;l<–3;k<–4000;b<–500;
cl5<– makeCluster(detectCores())
clusterSetRNGStream(cl5)
clusterExport(cl5,c(”r”,”theta”,”l”,”k”,”b”))
clusterExport(cl5,”data0”)
pp5<–parSapply(cl5,rep(12000,400),test30)
(pp[,5]<–c(mean(pp5[1,]),mean(pp5[2,])))
stopCluster(cl5)
```

Step 3: Compute the empirical power of the test for r2=0.1

```
theta<–0.05;r1<–0.75;r2<–0.1;l<–4;k1<–3000;b<–500;
cl1<– makeCluster(detectCores())
clusterSetRNGStream(cl1)
clusterExport(cl1,c(”r1”,”r2”,”theta”,”l”,”k”,”b”))
clusterExport(cl1,”data1”)
pp1<–parSapply(cl1,rep(12000,400),test3)
(pp[,1]<–c(mean(pp1[1,]),mean(pp1[2,])))
```

stopCluster(cl1)

Step 4: Repeat Step 3 for r1=0.2, 0.3, 0.4, 0.5, 0.6, 0.8, 0.9.

Step 5: Repeat Step 2 and Step 3 for the negative coefficients.

Table 5.9: Empirical power for a change in positive coefficients(Example 3)

		ρ_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.9
$n = 12000$, $l = 3$	K.S	0.4225	0.3725	0.3025	0.275	0.26	0.2775	0.3775	0.08	0.5525	0.955
$k = 4000$, $B = 500$	C.V.M	0.665	0.5775	0.4975	0.39	0.3175	0.3075	0.345	0.078	0.5	0.9275

Table 5.10: Empirical power for a change in negative coefficients(Example 3)

		ρ_2									
		-0.9	-0.8	-0.75	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
$n = 12000$, $l = 3$	K.S	0.6875	0.1225	0.065	0.07	0.095	0.01825	0.2425	0.3025	0.3425	0.3525
$k = 4000$, $B = 500$	C.V.M	0.8025	0.1325	0.0775	0.1	0.245	0.485	0.635	0.7175	0.7725	0.7675

• **Case 2&3**: Detection of a change in the mean or the variance.

It is enough here to replace "data1" by "data2" in the codes of Case 1 and choose the appropriate values for mu and sigma.

Table 5.11: Empirical power for a change in the mean (Example 3)

		μ								
		-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
$n = 10000$, $l = 5$	K.S	0.995	0.795	0.42	0.1775	0.075	0.1575	0.4075	0.8175	0.99
$k = 2000$, $B = 500$	C.V.M	0.995	0.795	0.42	0.1775	0.055	0.1575	0.4075	0.8175	0.99

Table 5.12: Empirical power for a change in the variance (Example 3)

		σ								
		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
$n = 10000$ $l = 5$	K.S	1	0.945	0.225	0.1175	0.065	0.11	0.1775	0.2875	0.52
$k = 2000$ $B = 500$	C.V.M	1	0.99	0.41	0.1075	0.07	0.1	0.1825	0.3925	0.63

Example 4

Repeat the simulations of example 3 with $\theta = 0.5$.

Table 5.13: Empirical power for a change in positive coefficients(Example 4)

		ρ_2									
		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.75	0.8	0.9
$n = 12000$, $l = 3$, $k = 4000$, $B = 500$	K.S	1	1	1	1	1	1	0.6475	0.048	0.8325	1
	C.V.M	1	1	1	1	1	1	0.625	0.0395	0.815	1

Table 5.14: Empirical power for a change in negative coefficients(Example 4)

		ρ_2									
		-0.9	-0.8	-0.75	-0.7	-0.6	-0.5	-0.4	-0.3	-0.2	-0.1
$n = 12000$, $l = 3$, $k = 4000$, $B = 500$	K.S	1	0.8625	0.0675	0.7125	1	1	1	1	1	1
	C.V.M	0.8025	0.1325	0.0775	0.1	0.245	0.485	0.635	0.7175	0.7725	0.7675

Table 5.15: Empirical power for a change in the mean (Example 4)

		μ								
		-0.4	-0.3	-0.2	-0.1	0	0.1	0.2	0.3	0.4
$n = 10000$, $l = 5$, $k = 2000$, $B = 500$	K.S	1	1	1	0.995	0.0625	0.9975	1	1	1
	C.V.M	1	1	1	0.995	0.0675	0.995	1	1	1

Table 5.16: Empirical power for a change in the variance (Example 4)

		σ								
		0.2	0.4	0.6	0.8	1	1.2	1.4	1.6	1.8
$n = 10000$ $l = 5$	K.S	1	1	1	0.99	0.0675	0.9975	1	1	1
$k = 2000$ $B = 500$	C.V.M	1	1	1	0.9	0.065	1	1	1	1

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