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DEAN OF THE FACULTY OF GRADUATE
AND POSTDOCTORAL STUDIES
A STUDY OF HUMAN ERROR IN HEALTH CARE

BY

RAJENDRAN MUTHURAMAN

A Thesis
presented to the University of Ottawa
December 2002
in the partial fulfillment of the
requirements for the degree of

MASTERS OF APPLIED SCIENCES
in
MECHANICAL ENGINEERING

Ottawa-Carleton Institute for
Mechanical and Aerospace Engineering

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Equally important was the constant encouragement, and high expectation from my parents for their moral support, without which this work would not have been possible. Last, but not least I thank my fiancée Melissa for her patience, tolerance and support for the last two years.
Abstract

This study presents an analytical approach to study human error in health care systems. A literature review was conducted on 350 publications on human error in health care system collected from journals, conference proceedings, newspapers, etc. Five mathematical models were developed to analyze human error in health care systems. The Markov method was used to perform analysis of these models. Specific expressions are obtained for human error probabilities, mean time to human death (MTHD), and mean time to health care professional’s error (MTTHPE). A number of useful methods and techniques for performing human error analysis in health care are identified.
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Chapter 1

Human Error in Health Care Systems: A Review

1.1 Introduction

Human error may simply be defined as an inappropriate action, or intention to act, given a goal and the context in which one is trying to reach that goal. In the management and operation of health care systems human error is an increasing focus for public concern. Research on human error over the past 25 years has revealed serious fallacies associated with medical system failures. Institute of Medicine, (IOM) reports roughly 100 000 Americans each year die from preventable errors in hospitals. This makes human error in health care the eighth leading cause of deaths in the US. [238]. The annual financial impact of these errors on the U.S. economy is estimated to be between $17 billion and $29 billion.

Two studies of large samples of hospital admissions, in New York using 1984 data and in Colorado and Utah 1992 data, found that the proportion of hospital admissions experiencing an adverse event, defined as injuries caused by medical management, were 3.7 and 2.9 percent, respectively. The proportion of adverse events attributed to errors (i.e., preventable adverse events) was 58 percents in New York, and 53 percent in Colorado and Utah. Consequently, this led to the increasing number of studies related to human error in health care.

This chapter reviews most of the published literature on human error in medical systems. The literature is collected from journals, conference proceedings, newspapers and general magazines, and books and technical reports as shown in Table 1.1 -1.3. The period covered is from 1977 to 2001.
### Table 1.1 Sources of Reviewed Publications: Journals

<table>
<thead>
<tr>
<th>No</th>
<th>Journal Name</th>
<th>No</th>
<th>Journal Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Accident Analysis and Prevention.</td>
<td>45</td>
<td>Journal of Intravenous Nursing.</td>
</tr>
<tr>
<td>2</td>
<td>American Family Physician.</td>
<td>46</td>
<td>Journal of Nursing Administration</td>
</tr>
<tr>
<td>3</td>
<td>American Industrial Hygiene Association Journal</td>
<td>47</td>
<td>Journal of Professional Nursing</td>
</tr>
<tr>
<td>4</td>
<td>American Journal of Health-System Pharmacy.</td>
<td>48</td>
<td>Journal of Quality Clinical Practice</td>
</tr>
<tr>
<td>5</td>
<td>Anesthesia and Analgesia.</td>
<td>49</td>
<td>Journal of Quality Improvement.</td>
</tr>
<tr>
<td>6</td>
<td>Anesthesia and Intensive Care.</td>
<td>50</td>
<td>Journal of Royal Society of Medicine, (JRSM).</td>
</tr>
<tr>
<td>7</td>
<td>Anesthesia.</td>
<td>51</td>
<td>Journal of the American College of Surgeons.</td>
</tr>
<tr>
<td>9</td>
<td>Annals of Internal Medicine.</td>
<td>53</td>
<td>Journal of Trauma.</td>
</tr>
<tr>
<td>10</td>
<td>Applied Ergonomics.</td>
<td>54</td>
<td>Materials Management in Health Care.</td>
</tr>
<tr>
<td>11</td>
<td>Archives of Pathology and Laboratory Medicine.</td>
<td>55</td>
<td>McKnight’s Long-Term Care News.</td>
</tr>
<tr>
<td>12</td>
<td>Best's Review.</td>
<td>56</td>
<td>Medical Economics.</td>
</tr>
<tr>
<td>13</td>
<td>Biomedical Instrumentation &amp; Technology.</td>
<td>57</td>
<td>Medical Device Diagnosis Industry Magazine.</td>
</tr>
<tr>
<td>14</td>
<td>British Journal of Anesthesia.</td>
<td>58</td>
<td>Medical Care.</td>
</tr>
<tr>
<td>15</td>
<td>British Medical Journal.</td>
<td>59</td>
<td>Medical Marketing &amp; Media.</td>
</tr>
<tr>
<td>16</td>
<td>Business &amp; Health.</td>
<td>60</td>
<td>Modern Healthcare.</td>
</tr>
<tr>
<td>17</td>
<td>Business Insurance.</td>
<td>61</td>
<td>National Institute of health.</td>
</tr>
<tr>
<td>18</td>
<td>Canadian Journal of Anaesthesia.</td>
<td>62</td>
<td>National Underwriter.</td>
</tr>
<tr>
<td>19</td>
<td>Canadian medical Association Journal.</td>
<td>63</td>
<td>Perspectives in Biology and Medicine.</td>
</tr>
<tr>
<td>20</td>
<td>Clinical Biochemistry.</td>
<td>64</td>
<td>Pharmaceutical Executive.</td>
</tr>
<tr>
<td>21</td>
<td>Drug Safety.</td>
<td>65</td>
<td>Philosophical Transactions of the Royal Society of London-Series B: Biological Sciences.</td>
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<tr>
<td>22</td>
<td>Ergonomics.</td>
<td>66</td>
<td>Problems in Anaesthesia.</td>
</tr>
<tr>
<td>23</td>
<td>European Journal of Anaesthesiology.</td>
<td>67</td>
<td>Public Health Reports.</td>
</tr>
<tr>
<td>24</td>
<td>Family Practice News.</td>
<td>68</td>
<td>Rhode Island Medical Journal</td>
</tr>
<tr>
<td>25</td>
<td>FDA Consumer magazine.</td>
<td>69</td>
<td>Radiotherapy &amp; Oncology.</td>
</tr>
<tr>
<td>26</td>
<td>Health Management Technology.</td>
<td>70</td>
<td>Safety in Medicine.</td>
</tr>
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<td>27</td>
<td>Healthcare Executive.</td>
<td>71</td>
<td>Skeptical Inquirer.</td>
</tr>
<tr>
<td>28</td>
<td>Healthcare Financial Management.</td>
<td>72</td>
<td>Social Science Medicine.</td>
</tr>
<tr>
<td>29</td>
<td>Hospitals &amp; Health Network.</td>
<td>73</td>
<td>South African Medical Journal.</td>
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<tr>
<td>30</td>
<td>Hospital Pharmacy.</td>
<td>74</td>
<td>Southern Medical Journal.</td>
</tr>
<tr>
<td>31</td>
<td>Inquiry.</td>
<td>75</td>
<td>Surgery.</td>
</tr>
<tr>
<td>32</td>
<td>Intensive Care Medicine.</td>
<td>76</td>
<td>Surgical Oncology Clinics of North America.</td>
</tr>
<tr>
<td>33</td>
<td>International Anesthesiology Clinics.</td>
<td>77</td>
<td>Technology Review.</td>
</tr>
<tr>
<td>34</td>
<td>International Journal of Radiation Oncology, Biology, Physics.</td>
<td>78</td>
<td>The Lancet.</td>
</tr>
<tr>
<td>36</td>
<td>Joint Commission Journal on Quality Improvement.</td>
<td>80</td>
<td>Topics in Health Information Management.</td>
</tr>
</tbody>
</table>
Table 1.2. Sources of Reviewed Publications: Conference Proceedings, Newspapers, and General Magazines

<table>
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<tr>
<th></th>
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<tbody>
<tr>
<td></td>
<td></td>
<td>12.</td>
<td>Time Magazine.</td>
</tr>
</tbody>
</table>

Table 1.3 Sources of Reviewed Publications: Books & Reports

<table>
<thead>
<tr>
<th>No.</th>
<th>Books/Reports</th>
<th>No.</th>
<th>Books / Reports</th>
</tr>
</thead>
</table>
1.2 Classification of Literature
Adverse outcomes in health care systems with human error are receiving media attention with increased in clinical concern, emphasis on reporting errors, and the pressure to reduce the incidence of such events. An appreciable number of publications in recent years clearly show the fact that attention on human error in medical systems is gaining momentum. In 2000, the British Medical Journal dedicated one full issue on medical error. Fig.1.1 shows the distribution of publications listed at the end of this chapter for the period 1977-2001.

![Graph showing distribution of publications](image)

**Fig 1.1 Trend of Publications on Human Error in Health Care Systems for the Period 1977-2001**

The publications were classified into many categories as presented in Table 1.4: Epidemiological Studies of Human Errors, Methods for Studying Human Errors in Medicine, Nursing and Medication Errors, Medical Equipment Errors, Human Error in Emergency Medicine, Anaesthesia Related Human Error, Human Error in Modern Technology in Medicine, Medical Personnel Effects to Error, Human Error Incident Reporting Systems, Human Factors Related to Medical Error and Miscellaneous (i.e. System Approach to Errors, Mass Media Reports on Human Error in Medicine and Technical Reports on Human Error). Fig 1.2 shows the classifications of references listed at the end of this article.
Table 1.4. Classification of Literature on Human Error in Health Care Systems

<table>
<thead>
<tr>
<th>Serial No</th>
<th>Classifications</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Studies Related to Human Errors in Health Care</td>
<td>21,22,39,46,47,50,52,72,73,95,114,123,133,140,153,177,186,196,208,210,232,259,276,277,287,288,311,312,313,314,316,321,332,340,342,349,350.</td>
</tr>
<tr>
<td>3</td>
<td>Nursing and Medication Errors</td>
<td>10,13,14,29,63,80,81,82,96,98,107,129,132,176,194,202,206,216,244,247,252,264,265,296,307,318,330.</td>
</tr>
<tr>
<td></td>
<td>• System Approach to Errors.</td>
<td>69,152,179,181,207,228,272,278,289.</td>
</tr>
</tbody>
</table>
Fig. 1.2 Pie chart for Table 1.4 breakdowns of literature: a) Studies Related to Human Errors in Health care, b) Methods for Studying Human Errors in Medicine, c) Nursing and Medication Errors, d) Human Error Related to Medical Equipments, e) Human Error in Emergency Medicine, f) Anesthesia Related Human Error, g) Human Error in Modern Technology in Medicine, h) Medical Personnel Effects to Error, i) Human Error Incident Reporting Systems, j) Human Factors Related to Medical Error and k) Miscellaneous. (Numerals denote the number of publications of each breakdown).

Fig. 1.3 Respective histograms for Fig. 1.2 Classifications
Fig. 1.3 Respective histograms for Fig. 1.2 Classifications (continued)

(e)

(i)

(f)

(j)

(g)

(k)

(h)
1.3 Review of Published Literature

This section presents a review of classified literature presented in Table 1.4. Fig. 1.3 presents histograms of publication profiles for each category of Table 1.4.

1.3.1 Studies Related to Human Errors in Health Care

In 1981, Zimmerman, et al. [349] with respect to human error, studied 624 consecutive admissions to a general surgical-medical intensive care unit within a university hospital. In 1983, Zimmerman, et al. [350] conducted a prospective study to examine human error in the functioning of an intensive care unit (ICU). Two years later, in 1985, Keenan and Boyan [186] studied the incidence and causes of cardiac arrest due to anaesthesia over a 15-year period. They concluded the main causes for human error were failure to provide adequate ventilation and overdose of an inhalation agent.

In 1988, Kumar, et al., [196] studied results of mishaps during anaesthesia. They concluded that 80.3% were due to human error. In the following year, Nickel and Shaughnessy [288] conducted a study to examine human error in prescription writing by physicians in a family medicine residency program. In 1991, Brennan, et al., [47] studied the adverse events occurring during hospitalization, and found 48% were due to human oversight [47]. In the same year, they also performed two other similar studies, [46,50]. In 1992, Chopra, et al., [72] examined the reports involving unsafe practices and working conditions during anaesthesia, and concluded that human error was responsible for 75% of the reported incidents.

In 1993, Ludbrook, et al. [208,210] performed analysis of 2000 incident reports, and identified various kinds of human error. In the same year Giraud, et al. [311] conducted a two-year study. They observed 316 iatrogenic complications and sixty-seven percentages of these complications were due to human error. In the same year Williamson, et al. [340] studied the first 2,000 incidents reported to the Australian Incident Monitoring Study (AIMS), and found 83% were due to human error. Information extracted related to system failures from the first 2000 incidents reported ninety-seven percent was due to human error [276]. In the following year Gurwitz, et al. [153] described the adverse and unexpected events resulted due to human error over a 1-year period. Two years later in 1996, Beckmann, et al. [22,21] studied the incidents and accidents caused by human error. Their study objectives were to develop and evaluate an incident reporting system that would reduce these errors.
In 1997, Furness and Lauder [133] conducted a postal questionnaire of 119 pathologists who reported 244 errors, 82 of these errors were due to human error. In 1998, David, et al [95] studied human error in an intensive care unit for a period of one year. In the same year, Reed, et al [259] studied adverse patient occurrences in a University hospital. The purpose of this study was to describe relationships between nursing care and human error. In 1999, Marc de Leval, [114] studied incidence of human error in the United Kingdom health care over an 18 month period. In the same year, Wright [342] studied critical incident reported from 1989 to 1999 and concluded that the majority of incidents were due to human error.

In 1999, Gawande, et al [140] conducted a review of 15,000 randomly selected admissions to Colorado and Utah hospitals during 1992. Their findings provided direction for research identifying the causes of human error. Vincent, et al., [321] in 2000, examined 1014 medical and nursing records and found about half of the events were due to human error, and judged preventable with ordinary standards of care. In the same year, Sexton, et al. [287] surveyed operating theatre and intensive care unit staff about attitudes concerning error, stress, and teamwork. These attitudes were compared with those of airline cockpit crew. In the same year, Morris, [232] reviewed 5600 reports of Australian Patient Safety foundation (APSF), and found 152 reports were due to human error. In 2000, Christakis & Lamont [73] conducted a prospective study that described doctor’s accuracy in treating ill patients.

In the same year three more studied were performed. Espinosa & Nolan [123] conducted a longitudinal study to reduce human errors made in the interpretation of radiographs in an emergency department. Burger, et al. [177] analysed diagnostic errors and classified discrepancies between clinical diagnosis and necropsy findings. Thomas, et al. [316] compared methods and characteristics of Quality in Australian Health Care Study (QAHCS) data and Utah Colorado Study (UTCOS) data. In 2001, Bracco, et al. [39] conducted a 1-year study of 777 critical incidents reports to identify human errors in a multidisciplinary intensive care unit. They found 241 human errors (31%) among these 777 reports.
1.3.2 Methods for Studying Human Errors

In 1986, Tarcinale, et al [310] explained a practical teaching plan, which could reduce human errors. In the same year, Widman & Tong [337] used a system known as Einthoven system to deal with human errors in health care systems. Three years later, Brennan, et al [49] evaluated a process for identifying adverse events due to human error by reviewing 360 medical records from two teaching hospitals. In the same year, Vincent [320] proposed various methods to study medical accidents due to hospital personnel. In 1994, Cohen, et al [78] showed how failure mode and effect analysis (FMEA) approach could be adapted to a hospital environment by using a continuous quality improvement practice to reduce human error. In the following year, Jones and Hunter [183] used Delphi Method to analyse human error in health care systems. Two years later, Richard and Heights, [263] discussed some of the possible ways to reduce surgical errors. In the following year, Andrews, et al. [5] proposed an alternative strategy for studying adverse events in medical care.

In 1998, Hollnagel [165] presented Cognitive Reliability and Error Analysis concerning complex industrial systems. In the same year, Kirwan, [189, 190] outlined thirty-eight approaches of error identification and proposed a framework or tool-kit approach for human error identification. In the same year, Stanton, et al. [306] suggested that human error identification techniques in general may be acquired with relative ease and provide reasonable error predictions.

In 1999, Naylor, [236] illustrated methods for reporting medical mistakes and misconduct by medical personnel. In the same year, McConnell [220] explained how engineering, behavioural and organizational changes could help nurse managers reduce or eliminate needle stick injuries in their units. In 1999, Narumi, et al. [235] analyzed incidental and accidental events in nursing care and concluded that unnatural working hours, such as evening duty, night duty falling next to a holiday, two consecutive night-duty shifts, and two consecutive evening-duty shifts were major factors in the occurrence of human errors. In 2000, Handler, et al. [155] described the findings of a consensus committee created to address the definition, measurement and identification of human error in emergency medicine. In the same year, Wenz, et al. [336] formulated practical methods to improve patient safety by using novel patient identification systems and reviewed the literature of human error measurement in medicine.

In 2000, McIsaac & Butler, [223] recommended a new clinical approach to help physicians deal with the uncertainty associated with the decision to prescribe an antibiotic. In the same year,
Meyhoorn, et al. [227] developed a new signal processing using Bayesian logic. Their evaluation showed that the Bayesian Confidence Propagation Neural Network (BCPNN) approach had a high and promising predictive value in identifying early signals of new adverse drug reactions caused by human error. Again in the same year, Reason, [258] viewed and explained human error in two different ways: the person approach and the system approach. In another publication in 2000, Vincent, et al. [322] explained how to investigate and analyse clinical incidents resulting from human error. In the same year, John, [245] compared the current ways concerning human errors and poor quality in health care to the approach taken by other industries. In 2001, Thomas, et al. [315] introduced a new approach useful to improve human error rates by incorporating use of skills, rules, and knowledge for effective management.

1.3.3 Nursing and Medication Errors


Charles [216], in the same year stated that medication errors commonly arise because two drugs have similar names and packaging. In the following year, Bates, et al. [13] explained the additional resource associated with an adverse drug event (ADE) to identify human error. In 1998 Phillips, et al [247] examined all US death certificates between 1983 and 1993; these certificates indicate the cause of death, race, sex, and inpatient/outpatient status. They found more than 50 percent of the deaths were due to adverse drug events. In the same year, Leape, et al. [206] evaluated the efficacy of his two interventions for preventing serious medication errors.
In the following year, Coleman [80] analyzed various methods for dealing with medication errors to protect the patients, the caregiver, and the institution from further harm. Again in the same year, O’Shea [244] discussed various factors contributing to medication errors. His review examined what constitutes a medication error and the contributing factors in medication errors. A total of eight publications appeared in 2000. Ferner [129] focused on errors in administering anaesthetics and in prescribing and giving medicines and Varricchio [318] reported an incident of human error where a pharmacy technician used an entire volume of diluents prescribed for a patient. Wechsler [330] concluded that the increased focus on medication safety puts pressure on manufacturers to carefully examine the actions which industry must take to reduce the error rate. Bates [15] used information technology to reduce rates of medication errors in hospitals and Caldwell [63] discussed a software that reduces medication errors. Richards, et al. [265] proposed the use of the simpler bolus thrombolytic agents which may reduce emergency department medication errors and thus improve overall clinical outcome. Conlan [82] discussed the ongoing research on medication errors and Leape, et al. [202] summarized lessons from a breakthrough series to reduce adverse drug events. In 2001, Preboth, [252] discussed paediatric patients medication errors.

1.3.4 Human Errors Related to Medical Equipment

In 1977, Rendell-Baker [261] reviewed most of the hazards resulting from human error in anaesthetic gas machines and discussed various proposals for their elimination. In 1984, Roelofse & Shipton [270] demonstrated that preventable mishaps resulting from human error contribute to anaesthetic risk and concluded that the incidence of anaesthetic-associated deaths has fallen steadily since 1935. In 1990, Arearese [8] explained Food and Drug Administration’s (FDA) role in medical device user education. In the following year, Belkin [24] discussed human and mechanical failures in health care. Again in 1991, Nobel [239] described the adverse effects of medical device failures due to humans. Two years later, Webb, et al. [327] found that 9% of human failures were due to ‘pure’ equipment failure according to pre-defined criteria from their analysis of 2000 incident reports at Royal Adelaide Hospital. In the same year, Bogner, [37] emphasised that the human failure mode analysis is a necessary tool in designing medical devices.
In 1994, Bogner [32] described human performance with respect to medical devices. In the same year, Hyman [172] explained the occurrence of errors in medical equipment use. Again in the same year, Senders [286] suggested the FDA to inspect and approve all the medical devices, whether they meet the criteria of being safe or not. In 1994, Wikland [338] in his book pointed out the importance of considering human factors in medical equipment design. In the following year, Wallace and Richard [323] analysed the medical device software related failures. Again in the same year, Souhrada [300] emphasised the importance of buying right medical device, for desired operation could reduce human error and Rachin [254] discussed various human factors issues with medical devices. In 1996, Burlington [58] stated the need for the FDA to take a closer look at how new medical devices are designed to ensure proper attention being given to human error prevention.

In 1997, Caplan et al. [64] studied the adverse anaesthetic outcomes arising from gas delivery equipment due to human error. In the same year, Maddox, [212] explained how proper designing of medical devices will minimize human error. Again in the same year, FDA made problem reporting with medical devices mandatory [174] and established a task Force to evaluate the system for managing the risks of FDA-approved medical products [214]. Also, in 1997 Laura, et al. [198] studied Abbott PCA Infuser, a commonly used medical device, and found that human error was responsible for 68% of fatalities and serious injuries. In 1999, Weinger [333] reported human error occurrence in anaesthesia pumps and Bogner [36] explained how human factors testing are yielding important data regarding safe and effective medical device and alarm designs that take into account the users’ cognitive limitations. In 2000, Dhillon [163] presented a list of medical devices having a high incidence of human error. In the same year, Sawyer [280] encouraged manufacturers to improve the safety of medical devices and equipment by reducing the likelihood of human error.

1.3.5 Human Error in Emergency Medicine

In 1986, Knaus, et al. [192] stratified hospital's patients by individual risk of death using diagnosis, indication for treatment, and Acute Physiology and Chronic Health Evaluation (APACHE) II scores. Their findings support the hypothesis that the degree of coordination of intensive care personnel significantly influences individual’s risk of death. In 1988, Robert [268] pointed out the preventable deaths in emergency medicine. In 1991, Davis, et al. [99] studied the
significance of critical care errors (CCEs) on preventable mortality and morbidity in a regionalized system of trauma care. In 1994, Gaba [137] discussed the behaviours of operational personnel, issues of organizational structure and managerial decisions of an intensive care unit, which has a direct effect on human error. In the following year, Ferner [128] described the confusion that can be caused when different drugs are labelled in an identical manner in emergency departments. In 1999, Leape, et al. [200] measured the effect of pharmacist participation in the intensive care unit on the rate of preventable adverse drug events (ADEs) caused by human errors. In the same year, Leape with the help of Wears, L.R., [326] formulated a new approach called ‘New Look’ to reduce human error in emergency departments.

In 2000, a total of fifteen publications appeared. Davidson [104] explained how the Data Elements for Emergency Department system (DEEDS) could reduce human error in emergency department. Dorevitch & Frost [109] discussed the occupational hazards of the emergency physicians. Adams and Bohan [2] discussed how engineering human factors and operational procedures, promoting team coordination, and standardizing care processes can decrease error. They concluded that these efforts should be coupled to systematic analysis of errors that occur. Brady, et al. [41] determined the rate of error in emergency physician (EP) interpretation of the cause of electrocardiographic (ECG) ST-segment elevation (STE) in adult chest pain patients and Brasel, et al. [42] introduced a new public model for more effective way to address medical injury. Clarke, et al. [75] described how a computer-based system is used to objectively critique the actual care given to patients with respect to process errors in reasoning.

Famularo, et al. [124] initiated a systematic analysis of errors made during the diagnostic workup in emergency departments and Glick, et al. [144] discussed misdiagnosis of serious neurologic conditions that represent a significant challenge in the field of human errors in medicine. Pope, et al. [160] determined the incidence of failure of hospitalize patients with acute cardiac ischemia and Hobgood, et al. [164] evaluated emergency medicine residency detectors (EMRDs), which identifies and reports clinical human errors made by emergency residents. Kyriacou, and Coben [197] discussed the objectives for researching emergency medicine (EM) errors.

Sacchetti, et al. [279] introduced an emergency information form (EIF) for containing patient-specific information on essential diagnostic and therapeutic interventions with respect to errors. Wears, et al. [324)& [325] discussed the board reaction generated by the Institute of Medicine’s (IOM) report, “To Err Is Human: Building a Safer Health System”. The authors described several pitfalls that must be avoided, in case the opportunity to reduce medical errors is

1.3.6 Anaesthesia Related Human Error

In 1978, Epstein [122] discussed the morbidity and mortality from anaesthesia. In the same year, Cooper, et al. [89] used a modified critical incident reporting technique in a retrospective examination of the characteristics of human error and equipment failure in anaesthetic practice. They found that most of the preventable incidents involved human error (i.e., 82 percent). In 1980, Taylor [309] described the essentials of non-invasive monitoring in anaesthesia to reduce human error in emergency departments. In the same year, Cooper, et al. [237] suggested that many human errors are due to or are encouraged by poor design of devices, connectors, equipment and, the whole system. Also, they discussed various preventive measures to avoid human errors. Also, in 1989, Abramson, et al. [1] discussed the adverse occurrences in intensive care units caused by human error.

In 1981, Craig and Wilson [91] conducted a survey of anaesthetic misadventures and concluded that human error was more frequently responsible for such misadventures than equipment failures. In 1984, Davies and Strunin [97] discussed various anaesthetic human errors and the risk of death directly attributable to anaesthesia. In the same year, Cooper, et al. [88] analysed major errors and equipment failures in anaesthesia management and observed that human error is the dominant issue in anaesthesia mishaps. Also, in 1984, McDonald [222] gathered data on the frequency of human errors associated with the practice of anaesthesiology and investigated the human factor aspect of anaesthesia accidents in the United States.

In 1985, Cooper [85] explained how errors and equipment failures in anaesthesia were impossible to prevent. In the same year, Emergency Care Research Institute (ECRI) [103] suggested that much more needs to be done, especially in "human-factors" areas such as improved training, consistent use of pre-anaesthesia checklists, and anaesthetists' willingness to enhance their vigilance by using appropriate monitoring equipment.

Two years later, Gaba, et al. [135] discussed anaesthetic mishaps as the breaking chain of accident evolution. In 1989, Gaba [136] outlined the components of a dynamic decision-making process that successfully protects patients in almost all cases of human error. In the same year, Cooper and Gaba [87] presented a strategy for preventing anaesthesia accidents due to human
error. Also, in 1989, Cullen [93] suggested a different approach to risk management to evaluate anaesthetic care and Gopher, et al. [145] investigated the nature and causes of human errors in the intensive care unit (ICU). In 1990, DeAnda and Gaba [102] indicated that although most incidents are simple and do not progress into more serious incidents, but human error remains ubiquitous, and that formal training and education should include recognition of events and the responses to them, in addition to prevention. In 1992, Runciman, et al. [275] provided a classification of human error in health care systems.


Also, in 1999, Busse and Johnson [60] investigated incidents in an adult intensive care unit (ICU). Their human error analysis approach stresses the importance of taking cognitive factors into account. In 2000, Gaba [134] reported that human error in anaesthesiology is increasing because of the use of non-traditional investigative techniques. In the same year, Nakata, et al. [234] suggested that physician’s risk attitudes couldn’t be predicted by their specialties or gender. Also, in 2000, Gravenstein, [149] concluded that most mishaps have a multifactorial cause in which human error plays a significant part and only good design of anaesthesia machines, ventilators, and monitors can prevent some, but not all, human errors.

1.3.7. Human Error and Modern Technology in Medicine

year, Flickinger, et al [131] investigated human error frequency in setting stereotactic coordinates for gamma knife radio surgery to determine what quality assurance safeguards is necessary. In the following year, Sheridan and Thompson [292] explained how it is important for doctors, nurses and other healthcare workers to be trained in the technology of computers. Also, in 1994, Bates, et al. [15] evaluated the potential ability of computerized information systems to identify and prevent adverse events in medical patients.

Two years later, Bradley, et al. [40] examined various issues in human error and how researchers are designing systems to minimize it. In 1997, Stahlhut, et al. [305] developed a curriculum in medical informatics that focuses on human error in clinical medicine. In the following year, Busse and Johnson [61, 62] argued that the analysis of human error in medicine would greatly benefit from representing a cognition-based error model within a cognitive architecture. In the same year, Bogner [33] chaired a conference on “Technology and human error in Medicine”. This conference discussed the problem of human error as a system problem along with various human error identification techniques. In 1997, Beith [23] discussed how human factors engineering can play a positive role in the development of telemedicine.

Two years later, Bates, et al. [51] discussed their experiences with measuring and improving quality in healthcare using information systems. In 2000, Angelo [6] argued how Internet helps to reduce medical errors and Internet based systems help to reduce liability for caregivers as they detect drug interaction risks, allergies, contraindications, etc. In the same year, Curtin and Simpson [94] discussed various simple principles to alleviate human errors and they concluded that the most effective approach to reducing error is to use modern technology to force change. Also, in the same year, Peters, et al. [250] implemented and assessed a rule based computerized prescribing system with the main aim of improving the safety of prescriptions and the administration of drugs.

Also, in 2000, Webster [329] discussed the Henry Ford System that was introduced in 1995. This computerized drug-ordering system reduced deaths by 55 percent and Carnall [65] stated that by using computers, doctors could reduce most of the errors in medicine.

1.3.8. Medical Personnel Reaction to Error

In 1990, Beiser [31] emphasised the importance of reporting physician’s mistakes so that they could learn from these mistakes. In the following year, Wu, et al. [344, 345] suggested various
methods to help house keepers learn from their mistakes and to institute constructive changes in practice. In 1992, Christensen, et al. [74] described how physicians think and feel about their perceived mistakes. They also examined how physician’s prior beliefs and manners of coping with mistakes may influence their emotional responses. In 1994, Perper, [246] proposed various strategies to improve adverse events reporting. He explained how it is important for the hospitals and health care providers to report such events. In the same year, Richard, et al. [262] studied various factors affecting physicians in healthcare systems. Also, in 1994, Bogner [161] discussed teamwork among physicians and house officers to reduce errors.

In the following year, Bell [25] explained the importance of medical personnel’s knowledge in their respective areas of specialization, and how this can reduce errors in healthcare. In the same year, Gorman [146] reported a disturbing case that killed a patient because of physician error. In 1996, Kennedy, et al. [266] illustrated the fact that the medical training health care workers should encourage a spirit of candour, humility, and partnership with patients. In 1997, Apgar [7] demonstrated that patients want full disclosure if a mistake occurs and he recommended that physicians should acknowledge the error in some way. In the same year, Crane [92] discussed how good doctors can avoid bad errors. Also, in the same year, Wu, et al. [343] discussed the ethical and practical issues in disclosing medical mistakes to patients by the healthcare providers.

In 1999, Horton [166] stated that how medical errors contribute to learning in medicine, but fear of disclosure prevents many physicians from benefiting from their errors. In the same year, Kramer and Hamm [195] stressed the point of being honest with the mistakes the physicians make and reporting them to avoid such kinds of errors in future.

In the same year, Casarett and Helms [67] concluded with the discussion of strategies to achieve the desired balance between the use of a systems approach and a personal-responsibility approach to managing errors in academic medical centres. In 2000, a total of nine publications appeared. Cooper [86] reported that some of the incidents where physicians who were responsible for series of human error in hospitals and industrial accidents and similar sentiments were echoed by Spencer [302]. Dutton [113] concluded that until health professionals can admit that they are not omniscient without fear of being sued, there seems little hope for a voluntary system of reporting errors. Hatlie [158] stated that merely reporting the incident, without an analysis of the cascade of events leading to it would not serve any preventive purpose and Pietro, et al. [248] illustrated the concerns, fears and practical problems that physicians face in conducting an evaluation of misinterpreted prostate biopsies.
Eisenberg [118, 119] discussed in the case of physicians how it is important to defend the problem before offending it and O’Leary [242] and Rao [225] believed to improve the safety of patients care, those directly providing the care must engage in the improvement process and feel safe in doing so. In 2001, Ivy & Cohn [178] concluded that additional training for physicians in management, leadership, and communication techniques might improve patient safety and quality of care.

1.3.9. Human Error Incident Reporting Systems

In 1992, Short, et al. [295] discussed an critical incident reporting system of quality assurance in an anaesthetic department and concluded that human error was the factor in 80% of the incidents. In the following year, Hart, et al. [157] suggested that by implementing a prospective, confidential, non-punitive incident reporting system, it is possible to reduce most of the human errors in intensive care units. In 1995, Bates, et al. [12] determined the frequency with which adverse drug events result in an incident report (IR) in hospitalized patients. In the following year, Bradley, et al. [17, 18, 19] designed, developed, and implemented different kinds of medical event-reporting systems for use in transfusion medicine to improve transfusion safety. In 1997, Altshuler et al. [4] in an effort to reduce the incidence of human errors, developed three automated donor-recipient identification systems. In the same year, Staender, et al. [304] set up a Critical Incident Reporting System (CIRS) to collect anonymous critical incidents in anaesthesia using a reporting form on the Internet. In 1998, Battles, et al. [16] implemented a medical event reporting system and Billings, et al. [28] discussed the NASA Aviation Safety Reporting System respect to medical event reporting system. In the same year, Chassin [71] discussed Six Sigma levels of quality for application in health care to reduce human errors.

A total of five publications appeared in 2000. Stump [308] developed a one-page medication-use variance report for collecting key data elements including root causes of errors, patient outcomes, and possible ways to prevent similar incidents. Ibojie and Urbaniak [173] presented a retrospective review of transfusion errors in a large teaching hospital using an internal reporting system. Barach and Small [11] discussed reporting and preventing medical mishaps using non-medical near miss reporting systems, which were institutionalized in aviation, nuclear power technology, petrochemical processing, steel production, military operations and air transportation. Saul, et al. [331] developed a confidential reporting system for detecting adverse events in a teaching hospital. Busse and Wright [59] explained incident analysis theory and
methodology from non-medical areas and applied them to an incident reporting scheme in an Intensive Care Unit. Scott [282] suggested that the reporting systems in all hospitals must be mandatory to avoid future errors. In 2001, Rascona, et al [257] evaluated adverse event reporting using a computer-based medical incident reporting system (MIRS).

1.3.10 Human Factors Related to Medical Errors

In 1981, O’Rourke, [243] studied various human factors contributing to medical errors. In the same year, Pickett and Triggs [274] discussed how human factor specialists can embark on human performance analysis and studies within the health care. In 1989, Keller [187] discussed human factors issues in surgical devices. In the following year, Rasmussen [256] described the concept of human error and analysed its intimate relation with human adaptation and learning with respect to health care. In 1991, Gaba [139] discussed the human performance issues in anaesthesia patient’s safety. In 1994, Cook & Woods [84] discussed human performance in large medical systems and how the failure of these systems is closely linked and Botney and Gaba [38] discussed the human factors issues in critical care monitoring. In 1997, Welch [334] emphasized that the health care industry is becoming aware of the cost of human error and is turning to human factor engineering (HFE) for answers. In the same year, Marc [100] pointed out that the influence of human factors on surgical outcomes must be taken into account and appropriate corrective action must be taken to prevent human errors.

In 1998, Joice, et al. [184] reviewed a human reliability analysis (HRA) approach, for application in assessing human error in endoscopic surgical performance. In the same year Bogner [35] described various human factors, which affects the regular medical care systems. Also, in 1998, Welch [328, 335] stressed the importance of using human factor engineering (HFE) within health care facilities—hospitals, clinics, nursing homes, etc. In the same year, Green [150] discussed the issues relating to the psychology of human error in healthcare systems. The importance of human factors Engineering in error and medical education was clearly pointed out by Gosbee [147]. In 2000, Dhillon [106] explained the human factors and human error occurrence issues in medical care and Kaye and Crowley [185] described how hazards related to medical device use should be addressed using human factors engineering during device development. In the same year, Egger [116] quoted that ProMutual, the largest provider of physician liability insurance in U.S. believes that stressed-out doctors result in more medical

1.3.11 Miscellaneous

a) System Approaches to Errors

In 1989, King [188] reported that medical accidents claimed 96,000 lives that year. In 1990, Souhrada [298] described a device that automatically delivers the right medication at the proper time; thus reducing nursing time and the margin of error. In 1991, Gaba and Howard [138] discussed the reports of the conference on human error in anaesthesia. In 1993, Horan [169] examined arguments ‘Should human errors be blamed for accidents or can they be attributed to poor design?’ and evaluated an ergonomic approach for use in the planning stage that might reduce accidents. In the same year, Leape [205] discussed general aspects on human error in medicine. In 1994, Bogner [34] edited a book completely devoted to human error in medicine. Cott [90] presented various causes of human error in medicine and mechanisms to prevent them. In 1997, Feldman and Roblin [126] discussed the framework of current methods of hospital quality appraisal focussed on clinician error. In 1998, Fisher [130] concluded that eliminating medical error is a sound risk management and Mcmanus [224] discussed the Hong Kong medical association’s guidelines to prevent medical blunders. In the same year, Scheffler and Zipperer [281] discussed the issues of enhancing patient safety and reducing errors in health care and Leape, et al. [115, 201] concluded that a proper system analysis could identify potential medical errors before they happen. In 1999, Moyer [233] reported that medical mistakes are a leading cause of death and disability in hospitals. In the same year, similar sentiments were expressed in Refs. [301, 76, 203, 346].

A total of twelve publications appeared in 2000. Ferman [127] discussed the safety measures concerned with the patients admitted to a critical care unit and George and Gerlin [142, 143] presented medical error and its current trend. Chassin, et al. [70] identified issues related to the quality of healthcare in the United States and according to James and Hammond [180] the control of variations and feedback to physicians will improve medical practice. Moeller, et al. [229] discussed the performance of the European Foundation for Quality Management (EFQM) excellence model within German health care organisations and Ref [260] proposed the creation of a log of errors in health care setting. Newhall [238] reported that 100,000 deaths are due to medical error each year and Hallon [154] reported that as per a recent survey patients fear
medical error and want better protection from such mistakes. Helmreich [162] surveyed cockpits and operating theatres and concluded that pilots and doctors have common interpersonal problem areas and similarities in professional culture. Hughes, et al. [170] explained various causes of death due to medical error and Eisenberg [120] discussed the latest research facilities on patient’s safety. In 2001, Billings & Woods [27] presented various examples of medical error due to the system failure and Dunn [112] discussed the common error in medicine. In the same year, critical issues of health care systems were discussed in Ref [175].

b) Mass Media Reports on Medical Error

In 1999, Ref [207] stated that for the success of all members of the health care enterprises must participate in a coordinated effort to identify and establish effective practices that may reduce human error in medicine. In 2000, the President of USA called for a nationwide system of reporting medical errors in the wake of IOM report [69, 181]. In the same year, Pretzer [228] reported that the United States Congress passed Patient Safety and Errors Reduction Act, Medical Error Reduction Act, and Voluntary Error Reduction and Improvement in Patient Safety Act to reduce human error in medicine and Gunn [152] reviewed all the IOM reports published in 1999. In 2001, Jackson [179] discussed how media exaggerates the medical error news and cited some examples to prove it.

c) Organisation’s Reports on Medical Error.

In 1995, Eisele and Watkins [117] evaluated a survey of workplace violence in the Department of Energy facilities with respect to medical errors. In 1999, a IOM publication [193] reviewed studies quantifying medical errors in health care and recommendations for eliminating such errors. In the same year, Mc Cray [221] studied the medical systems in U.S. to identify errors. In 2000, Brennan [48] pointed out a key problem overlooked by the IOM report on medical error (i.e., reporting of errors by the physicians) and concluded to address the problem the system of malpractice litigation must be reformed. In the same year, Gebhart [141] evaluated a program launched to cut errors in hospitals and Cook & Woods [83] reported a wide-ranging discussion about safety, accidents, and research in health care in a workshop. Also, in 2000, a state agency for Health Care Research and Quality (AHRQ) [168] developed a workshop to help State health care policymakers to understand the causes of medical errors.
1.4 Study Objective

The main objective of this study is to show the growing concern for human error in health care systems by reviewing 350 publications along with identifying human error analysis methods and developing appropriate mathematical models to conduct analytical studies in the said area.

The need for development of mathematical models is to predict human error probability in health care systems and to develop expressions for mean time to human death (MTHD) and mean time to health care professional’s error (MTTHPE).

1.5 Organization of Study

This study is presented in five chapters.

Chapter 1 provides a brief introduction of human error in health care systems and a detailed review of published literature.

Chapter 2 identifies useful methods used to analyze human error in health care systems extracted from the reviewed literature.

Chapter 3 presents human reliability analysis of two health care professional - patient models considering a human having health problems and his/her diagnosis/remedial measures are subject to error.
Chapter 4 presents three mathematical models concerned with reliability evaluation of health care professionals subject to multiple modes of error.

Chapter 5 presents the conclusions and recommendations for future studies.
Chapter 2

Human Error Analysis Methods in Health Care Systems

2.1 Introduction

Human errors have become a very important issue in health care, owing to the vast number of associated deaths each year. Human error analysis involves the systematic identification and evaluation of the possible errors that may be made by health care professionals and other personnel in the health care system. In the past, many techniques were developed to perform human error analysis in various disciplines. Some of the commonly used human error analysis techniques as follows:

- Failure Mode Effect Analysis (FMEA)
- Probability Tree Method
- Technique for Human Error Rate Prediction (THERP)
- Fault Tree Analysis
- Cognitive Analysis of Human Error
- Throughput Ratio Method
- Task Analysis

A recent report by the Institute of Medicine in the United States and subsequent government action on its recommendations has heightened awareness regarding human error in health care as a public health risk [238]. Over the years, many techniques and methods have been developed to perform human error analysis for the purpose of reducing the occurrence of human error in health care. This chapter presents seven human error analysis methods extracted from published literature on human error in health care systems.

2.2 Method 1: Cognitive Analysis of Human Error

One of the most important academic developments of the past few decades has been the birth of an exciting new interdisciplinary field called cognitive science. The word cognitive refers to perceiving and knowing. Thus cognitive science is the science of mind. Cognitive scientists view the human mind as a complex system that receives, stores, retrieves, transforms, and
transmit information. There are five major topic areas in cognitive science: knowledge representation, language, learning, thinking, and perception. The analysis of human error in incidents and accidents in health care has received considerable attention. Several taxonomies exist, ranging from behavioural classifications to classification of error according to underlying cognitive mechanisms. Current cognitive user models enable interface designers to describe, analyze and predict various aspects of human cognition.

Some of the publications directly or indirectly dealing with cognitive analysis of human error in health care are as follows:

- Busse [59,60,61,62] used Interacting Cognitive Subsystems (ICS) to illustrate the modelling of human error within a cognitive architecture in an intensive care unit.
- Harrison, et al. [156] used a cognitive analysis technique to study the adverse events in health care systems.
- Spencer [302] stressed the importance of using cognitive science to study human error in health care systems.

2.3 Method II: Failure Mode and Effect Analysis (FMEA)
This technique was developed in the aerospace industry and is concerned with identifying mistakes that will happen before they happen in addition to determining whether the consequences of those mistakes would be tolerable or intolerable. More specifically FMEA is a method used for the identification of potential error types in order to define its effect on the examined object and to classify the error types with regard to criticality or persistency.

The FMEA process consists of the following steps:

- Identify FMEA goals and levels
- Define procedures, basic rules, and criteria for the FMEA realization
- Determine system with regard to functions, interfaces, operating phases, operating modes, and environment
- Analyze functional and reliability block diagrams or error tree diagrams to illustrate the processes
- Determine interconnections, and dependencies
- Identify potential error types
- Evaluate error types and their effects
• Identify measures to prevent the occurrence of errors
• Evaluate the effects suggested measures
• Document the results

Cohen [78] used this method in 1994 to analyze human error in health care. This method was also applied in Ref [212]. Kieffer [188] deformed human intervention related risk by using Failure Mode Effect Analysis (FMEA) and Passey [246] pointed out that FMEA is an excellent technique for making accurate risk assessments for products and processes in health care system.

2.4 Method III: Bayesian Analysis of Error
Bayesian analysis is named after Thomas Bayes, an English clergyman and a mathematician, Bayesian logic is a branch of logic applied to decision making and inferential statistics that deals with probability inference: using the knowledge of prior events to predict future events. Bayes first proposed his theorem in 1763 and it basically states that the only way to quantify a situation with an uncertain outcome is through determining its probability.
Meyboom et al [227] developed a new signalling process using Bayesian logic to identify adverse drug reaction. Maddox [212] discussed this method as one of the human error analysis techniques in health care system.

2.5 Method IV: Root Cause Analysis (RCA)

Root cause analysis widely used to investigate major industrial accidents. Root cause analysis (RCA) has its foundations in industrial psychology and human factors engineering. Many experts have used this method for the investigation of human error occurrences in medicine. In 1997, the Joint Commission on the Accreditation of Healthcare Organizations (JCAHO) mandated the use of RCA in the investigation of adverse events in accredited hospitals [205].

Root cause analyses systematically search out latent or system failures that underlie adverse events or near misses. There is insufficient evidence in the medical literature to support RCA as a proven patient safety practice, however it may represent an important qualitative tool that is complementary to other techniques employed in error reduction. When applied appropriately, RCA may illuminate targets for change, and, in certain healthcare contexts, may generate testable hypotheses. The use of RCA merits more consideration, as it lends a formal structure to efforts to learn from past mistakes.
Based on the concepts of active and latent errors root cause analysis is generally broken down into the following two steps [213]:

1. Data collection: This is concerned with collecting data through structured interviews, document review, and/or field observation. These data are used to generate a sequence or timeline of events preceding and following the event.

2. Data analysis: This is an iterative process with the goals of determining the following:
   - Establish how the event happened by identifying active failures in the sequence.
   - Establish why the event happened by identifying latent failures in the sequence.

Some of the publications directly or indirectly dealing with this method in health care systems is as follows:
   - Leape [205]
   - Reason [258]
   - Shea [290]
   - Shinn [293]
   - Spath [301]

2.6 Method V: Reason's Analysis of Error.

Reason's [101] theory distinguishes between active failures and latent conditions. Active failures are errors and violations that are committed by people at the service delivery end of the system (e.g., pilots, control room operators, financial traders, a ship's crew, and, in cardiac surgery, the operating room team). Active failures by these people have an immediate impact on safety. Latent conditions result from poor decisions made by the higher management in an organization, (e.g., by regulators, governments, designers, and manufacturers. Latent conditions lead to weaknesses in the organization's defences, thus increasing the likelihood of active failures.

Figure 2.1 shows the organizational accidents model tailored to a health care system. The accident trajectory is represented by the penetration of the levels of defence by an arrow. The holes represent latent and active failures that have breached successive levels of defence. When the arrow penetrates all the levels of defence, an adverse event (a death or a near miss) occurs.
Fig 2.3 Generic organizational accident model applied to health care systems [Reason, 101]

Reason [101,258], Gaba [134], Leape [205] and Bogner [23] recommended this method to analyze human error in health care system.

2.7 Method VI: Critical Incident Reporting Technique

The methods discussed above are most often used in analytical settings. That is, researchers convene a group of people and perform an analysis on paper. However, when a product is already sold and being used, researchers can use the critical incident technique. This technique can play a vital role in analysing human error in healthcare systems. In this method, users typically are asked whether they have observed or been involved in near-accidents or injuries related to the product. Critical incidents are valuable indicators that a component or usage scenario might be hazardous and thus needs further examination.

Since 1970s, critical incident studies have been performed to investigate anaesthetic mishaps. Some examples of the application of the application of the Critical Incident reporting Technique are as follows:
• Investigation of Intensive Care Unit Critical Incidents at the Williamson Hospital [342].
• Development of a critical incident reporting system in an Australian hospital [22].
• Use of critical incident reporting systems in anaesthesia environment [294, 295, 304]
Chapter 3

HEALTH CARE PROFESSIONAL - PATIENT MODELS

3.1 Introduction
In the proceeding chapters, a detail review of published literature was conducted to determine the status of human error in health care systems along with extracting useful methods for performing human error analysis. Researchers agreed on the fact that human errors in health care systems are caused by bad systems, not bad people [302]. Just like in the case of a power plant, a hospital or an operating room within the hospital can be considered a system. This chapter presents two mathematical models for performing human reliability analysis in health care system. The Markov method was used to develop equations for the models. Expressions for state probability and mean time to human error are developed. Some plots of these expressions are presented.

3.2 Model I
This model represents a human having health problems and his/her diagnosis/remedial measures are subject to error. More specifically, a human with health problems visits a health care professional (i.e., doctor, nurse, etc.). In turn, due to the action of a health care professional, the human can either become normal, still retain health problems or his/her health becomes worse due to human error; or dies due to or without a human error. There are two possible outcomes of a health care professional’s action from the state when the human still retains the health problem or his/her health become worse due to human error: the human’s health becomes normal or human dies. The human from normal state can also die due to natural cause (e.g. non curable diseases). The human state space diagram is shown in Fig.3.1.

The following assumptions are associated with this model:
- All occurrences (i.e., health problems, human errors, deaths and remedial measures) are independent.
- Health problem, human error, death, and curable rates are constant.

The numerals in the boxes of Fig.3.1 denote system states. The following symbols are associated with this model:

\[ i \]

is the \( i \)th system state: \( i = 0 \) (human in normal health), \( i = 1 \) (human with health problems visits a health care professional), \( i = 2 \) (human still having problems
due to human error by the health care professional), $i = 3$ (human dies due to natural causes), $i = 4$ (human dies due to human error by health care professional).

$P_i(t)$ is the probability that the human is in state $i$ at time $t$; for $i = 0, 1, 2, 3, 4$.

$\theta$ is the constant human health problem rate.

$\mu_1$ is the constant human curable rate from state 1.

$\mu_2$ is the constant human curable rate from state 2.

$\lambda_1$ is the constant health care professional's error rate from state 1 to state 2.

$\lambda_2$ is the constant health care professional's error rate from state 2 to state 4.

$\lambda_3$ is the constant death rate of the human due to natural causes when in normal health.

$\lambda_4$ is the constant health care professional's error rate that results in human death.

$\lambda_5$ is the constant death rate of the human due to natural causes from state 1.

$s, t$ is the Laplace transform variable, time in months.

**Figure 3.1** State Space diagram of Model I

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Model Analysis

With the aid of the Markov method, the system of differential equations associated with Fig 3.1 is as follows:

\[
\frac{dP_0(t)}{dt} = \mu_2 P_2(t) + \mu_1 P_1(t) - (\theta + \lambda_3) P_0(t) \tag{3.1}
\]

\[
\frac{dP_1(t)}{dt} = \theta P_1(t) - (\mu_1 + \lambda_1 + \lambda_4 + \lambda_2) P_1(t) \tag{3.2}
\]

\[
\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\mu_2 + \lambda_2) P_2(t) \tag{3.3}
\]

\[
\frac{dP_3(t)}{dt} = \lambda_3 P_0(t) + \lambda_5 P_1(t) \tag{3.4}
\]

\[
\frac{dP_4(t)}{dt} = \lambda_2 P_2(t) + \lambda_4 P_1(t) \tag{3.5}
\]

At t=0, P_0(0)=1, and all other initial condition probabilities are equal to zero.

Laplace transforms of the solutions to Eqs. (3.1)-(3.5) are as follows:

\[
P_0(s) = \frac{(s + b)(s + c)}{(s + a)(s + b)(s + c) - (s + c)\theta \mu_1 + \lambda_1 \mu_2 \theta} \tag{3.6}
\]

\[
P_1(s) = \frac{\theta(s + c)}{(s + a)(s + b)(s + c) - (s + c)\theta \mu_1 - \lambda_1 \mu_2 \theta} \tag{3.7}
\]

* Detailed analysis is provided in Appendix A.
\[ P_2(s) = \frac{\lambda_1 \theta}{(s + a)(s + b)(s + c) - (s + c) \theta \mu_1 - \lambda_1 \mu_2 \theta} \quad (3.8) \]

\[ P_3(s) = \left( \frac{\lambda_3 (s + b)(s + c)}{(s + a)(s + b)(s + c) - (s + c) \theta \mu_1 - \lambda_1 \mu_2 \theta} \right) + \frac{\lambda_3 \theta (s + c)}{(s + a)(s + b)(s + c) - (s + c) \theta \mu_1 - \lambda_1 \mu_2 \theta} \right) / s \quad (3.9) \]

\[ P_4(s) = \left( \frac{\lambda_4 \theta (s + c)}{(s + a)(s + b)(s + c) - (s + c) \theta \mu_1 - \lambda_1 \mu_2 \theta} \right) + \frac{\lambda_3 \lambda_4 \theta}{(s + a)(s + b)(s + c) - (s + c) \theta \mu_1 - \lambda_1 \mu_2 \theta} \right) / s \quad (3.10) \]

where,
\[ a = \theta + \lambda_3 \]
\[ b = \mu_1 + \lambda_1 + \lambda_3 + \lambda_4 \]
\[ c = \lambda_2 + \mu_2 \]

Taking inverse Laplace transforms of Eqs. (3.6)- (3.10) we get:

\[ P_0(t) = \frac{(z + c)(b + z)e^{(zt)}}{(y - z)(-z + x)} + \frac{(x + c)(b + x)e^{(xt)}}{(-y + x)(-z + x)} - \frac{(c + y)(b + y)e^{(yt)}}{(-y + x)(y - z)} \quad (3.11) \]

\[ P_1(t) = \theta \left( \frac{(-c - y)e^{(yt)}}{(y - z)(-y + x)} + \frac{(x + c)e^{(xt)}}{(-y + x)(-z + x)} + \frac{(z + c)e^{(zt)}}{(y - z)(-z + x)} \right) \quad (3.12) \]

\[ P_2(t) = \theta \lambda_1 \left( \frac{e^{(yt)}}{(y - z)(-y + x)} + \frac{e^{(xt)}}{(-y + x)(-z + x)} + \frac{e^{(zt)}}{(y - z)(-z + x)} \right) \quad (3.13) \]
\[ P_3(t) = -\frac{c(\lambda_3 b + \lambda_3 b)}{z x y} + \frac{(x + c)(x \lambda_3 + \lambda_3 b)}{x (z + x) (-y + x)} \]
\[ -\frac{(c + y)(y \lambda_3 + \lambda_3 b + \lambda_3 \theta)}{(-y + x) (y - z) y} + \frac{(z + c)(\lambda_3 b + z \lambda_3 + \lambda_3 \theta)}{(-z + x) (y - z) z} \]  

(3.14)

\[ P_4(t) = \theta \left( \frac{(-x_1 - \lambda_4 y - \lambda_4 c)}{y (y-z) (-y + x)} + \frac{\lambda_4 c + x_1 + \lambda_4 x}{(-y + x) x (-z + x)} + \frac{\lambda_4 c + x_1 + z \lambda_4}{(y-z) (-z + x) z} \right) \]
\[ + \frac{-\lambda_4 c - x_1}{z x y} \]  

(3.15)

where,

\[ x_1 = \lambda_1 \lambda_2 \]

\[ x = -\frac{k_1}{3} - \frac{2^{1/3} (N)}{3 (M)^{1/3}} + \frac{(M)^{1/3}}{3 \ 2^{1/3}} \]

\[ y = -\frac{k_1}{3} + \frac{(1 + i \sqrt{3}) (N)}{3 \ 2^{2/3} (M)^{1/3}} - \frac{(1 - i \sqrt{3}) (M)^{1/3}}{6 \ 2^{1/3}} \]

\[ z = -\frac{k_1}{3} + \frac{(1 - i \sqrt{3}) (N)}{3 \ 2^{2/3} (M)^{1/3}} - \frac{(1 + i \sqrt{3}) (M)^{1/3}}{6 \ 2^{1/3}} \]

\[ k_1 = a + b + c \]

\[ k_2 = (ab + (a + b) c - \theta \mu_1) \]

\[ k_3 = abc - \lambda_1 \mu_2 \theta - \theta \mu_1 c \]

\[ M = -2 \ k_3^3 + 9 \ k_1 \ k_2 + \sqrt{4 \ (-k_1^2 + 3 k_2)^3 + (-2 \ k_1^3 + 9 \ k_1 \ k_2 - 27 \ k_3)^2} - 27 \ k_3 \]

\[ N = -k_1^2 + 3 \ k_2 \]

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For the specified values of model parameters, the plots of Equations (3.11)-(3.15) are shown in Fig. 3.2.

\[
\lambda_1=0.003, \lambda_2=0.002, \lambda_3=0.0009 \\
\lambda_4=0.001, \lambda_5=0.0005, \mu_1=0.01, \mu_2=0.005 \\
\theta=0.05
\]

![Graph](image)

**Figure 3.2.** State probability plots for Model 1

The probability of the human being alive is

\[
R(t) = P_0(t) + P_1(t) + P_2(t)
\]  
(3.16)
For the values of model parameters, the plots of Equation (3.16) are shown in Fig. 3.3.

\[ \lambda_3=0.003, \lambda_4=0.0009 \\
\lambda_2=0.001, \lambda_1=0.0005, \mu_1=0.01, \mu_2=0.005 \\
\theta=0.05 \]

**Figure 3.3.** The probability of human being alive plot

Mean time to human death is given by

\[
\text{MTHD} = \int_0^\infty (P_0(t) + P_1(t) + P_2(t)) \, dt
\]

\[
= \frac{b \, c + \theta \, c + \lambda_1 \, \theta}{a \, b \, c - \theta \, \mu_1 \, c - \lambda_1 \, \mu_2 \, \theta}
\]

(3.17)

where

MTHD is the mean time to human death.
For the values of model parameters, the plots of Equation (3.17) are shown in Fig. 3.4.

\[ \lambda_3 = 0.0009, \lambda_2 = 0.002, \lambda_5 = 0.0005, \mu_1 = 0.01, \mu_2 = 0.005, \theta = 0.05 \]

\[ \lambda_4 = 0.001, \lambda_4 = 0.002, \lambda_4 = 0.003 \]

Figure 3.4 Mean time to human death plots

Plot Discussion

The plots of Eqs. (3.11) – (3.15) are shown in Fig. 3.2. These plots show that for given value of \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \), the value of the \( P_i(t) \) decreases or increases as \( t \) increases for \( i = 0, 1, 2, 3 \) and 4. Another set of plots of Eq. (3.16) is shown in Fig. 3.3. These plots show that for the value of \( \lambda_1, \lambda_2, \lambda_3, \lambda_4 \) and \( \lambda_5 \), the value of \( R(t) \) decreases as \( t \) increases, and \( R(t) \) decreases as \( \lambda_2 \) increases. The plots of the Eq. (3.17) are shown in Fig. 3.4. These plots show that for the value of \( \lambda_1 \) and \( \lambda_4 \), the value of MTHD decreases as \( \lambda_4 \) increases.
3.3 Model II

This model represents a further development to Model I. More specifically, in this case, the health care professional is identified as doctor, a nurse or both. Again a human with health problems visits a doctor, a nurse, or both. In turn, due to the action of a doctor, a nurse, or both the human can either become normal, still retain health problems or his/her health becomes worse due to doctor/nurse’s error, or dies due to or without a human error.

There are two possible outcomes of doctor’s/nurse’s actions from the state when the human still retains the health problem or his/her health become worse due to human error: the human’s health becomes normal or the human dies. The human from normal state can also die due to natural causes. The human state space diagram is shown in Fig.3.5. The following assumptions are associated with this model:

- All occurrences (i.e., health problems, doctor error, nurse error, deaths, and remedial measures) are independent.
- Health problem, doctor error, nurse error, death and curable rates are constant.

The numerals in the boxes of Fig.3.5 denote system states. The following symbols are associated with this model:

\[ i \]

is the \( i \)th system state: \( i = 0 \) (human in normal health), \( i = 1 \) (human with health problems visits a health care professional), \( i = 2 \) (human still having problems due to human error by the Nurse), \( i = 3 \) (human still having problems due to human error by the Doctor), \( i = 4 \) (human dies due to natural causes), \( i = 5 \) (human dies due to human error by the Nurse), \( i = 6 \) (human dies due to human error by the Doctor).

\[ P_i(t) \]

is the probability that the human is in state \( i \) at time \( t \); for \( i = 0,1,2,3,4,5,6. \)

\[ \theta \]

is the constant human health problem rate.

\[ \mu_1 \]

is the constant human curable rate from state 1.

\[ \mu_2 \]

is the constant human curable rate from state 2.

\[ \mu_3 \]

is the constant human curable rate from state 3.

\[ \lambda_1 \]

is the constant nurse’s error rate from state 1 to state 2.

\[ \lambda_2 \]

is the constant nurse’s error rate from state 2 to state 5.

\[ \lambda_3 \]

is the constant death rate of the human due to natural causes when in normal health.
\( \lambda_4 \) is the constant death rate of the human due to natural causes from state 1.

\( \lambda_5 \) is the constant Doctor’s error rate from state 1 to state 3.

\( \lambda_6 \) is the constant Doctor’s error rate from state 3 to state 6.

\( \lambda_7 \) is the constant nurse’s error rate that results in human death.

\( \lambda_8 \) is the constant Doctor’s error rate that results in human death.

\( s \) is the Laplace transform variable

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**Figure 3.5** State –Space diagram of Model II
Model Analysis

With the aid of Markov method, the system of differential equations associated with Fig 3.5 is as follows:

\[
\frac{dP_0(t)}{dt} = \mu_2 P_2(t) + \mu_1 P_1(t) + \mu P_3(t) - (\theta + \lambda_4) P_0(t) \tag{3.19}
\]

\[
\frac{dP_1(t)}{dt} = \theta P_0(t) - (\mu_1 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8) P_1(t) \tag{3.20}
\]

\[
\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\mu_2 + \lambda_2) P_2(t) \tag{3.21}
\]

\[
\frac{dP_3(t)}{dt} = \lambda_4 P_1(t) - (\lambda_6 + \mu_3) P_3(t) \tag{3.22}
\]

\[
\frac{dP_4(t)}{dt} = \lambda_5 P_1(t) + \lambda_4 P_1(t) \tag{3.23}
\]

\[
\frac{dP_5(t)}{dt} = \lambda_5 P_2(t) + \lambda_5 P_1(t) \tag{3.24}
\]

\[
\frac{dP_6(t)}{dt} = \lambda_6 P_1(t) + \lambda_6 P_3(t) \tag{3.25}
\]

At t=0 , \(P_0(0)=1\), and all other initial condition probabilities are equal to zero.

Laplace transforms of the solutions to Eqs. (3.19)-(3.25) are as follows:

\[
P_0(s) = \frac{(s + X) (s + Y) (s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4} \tag{3.26}
\]

where,

\[X = \mu_1 + \lambda_4 + \lambda_5 + \lambda_8 + \lambda_1 + \lambda_7\]

\[Y = \mu_2 + \lambda_2\]

- Detailed analysis is given in Appendix B.
\[ k_1 = U + X + Y + Z \]
\[ Z = \lambda_6 + \mu_3 \]
\[ U = \theta + \lambda_3 \]
\[ k_2 = -\mu_1 \theta + U X + (U + X) Y + (U + X + Y) Z \]
\[ k_3 = -\mu_2 \lambda_1 \theta - \mu_3 \lambda_5 \theta - \mu_1 \theta Y - \mu_1 \theta Z + U X Y + (U X + (U + X) Y) Z \]
\[ k_4 = U X Y Z - \mu_2 \lambda_1 \theta Z - \mu_3 \lambda_5 \theta Y - \mu_1 \theta Y Z \]

\[ P_1(s) = \frac{\theta P_0(s)}{s + X} \quad (3.27) \]

\[ P_2(s) = \frac{\lambda_1 \theta P_0(s)}{(s + X)(s + Y)} \quad (3.28) \]

\[ P_3(s) = \frac{\lambda_5 P_1(s)}{s + Z} \quad (3.29) \]

\[ P_4(s) = \frac{\lambda_2 P_0(s) + \lambda_4 P_1(s)}{s} \quad (3.30) \]

\[ P_5(s) = \frac{\lambda_2 P_2(s) + \lambda_4 P_1(s)}{s} \quad (3.31) \]

\[ P_6(s) = \frac{\lambda_6 P_3(s) + \lambda_8 P_1(s)}{s} \quad (3.32) \]
Taking inverse Laplace transform of Eqs. (3.26) – (3.32) we get:

\[
P_0(t) = \frac{(x+Z)(x+Y)(x+X)e^{(xt)}}{(x-w)(x-z)(x-y)} - \frac{(Z+y)(y+Y)(y+X)e^{(yt)}}{(x-y)(y-z)(-w+y)}
+ \frac{(z+Z)(Y+z)(z+X)e^{(zt)}}{(z-w)(y-z)(x-z)} - \frac{(w+Z)(Y+w)(X+w)e^{(wt)}}{(z-w)(-w+y)(x-w)}
\]

\[
P_1(t) = \theta \left( -\frac{(w+Z)(Y+w)e^{(wt)}}{(z-w)(-w+y)(x-w)} + \frac{(z+Z)(Y+z)e^{(zt)}}{(x-z)(y-z)(z-w)}
- \frac{(Z+y)(y+Y)e^{(yt)}}{(-w+y)(y-z)(x-y)} + \frac{(x+Z)(x+y)e^{(xt)}}{(x-w)(x-z)(x-y)} \right)
\]

\[
P_2(t) = \theta \lambda_1 \left( \frac{(-w-Z)e^{(wt)}}{(z-w)(-w+y)(x-w)} + \frac{(z+Z)e^{(zt)}}{(y-z)(x-z)(z-w)}
+ \frac{(-Z-y)e^{(yt)}}{(-w+y)(y-z)(x-y)} + \frac{(x+Z)e^{(xt)}}{(x-w)(x-z)(x-y)} \right)
\]

\[
P_3(t) = \theta \lambda_5 \left( \frac{(-Y-w)e^{(wt)}}{(z-w)(-w+y)(x-w)} + \frac{(Y+z)e^{(zt)}}{(z-w)(y-z)(x-z)}
+ \frac{(-y-Y)e^{(yt)}}{(x-y)(y-z)(-w+y)} + \frac{(x+y)e^{(xt)}}{(x-w)(x-z)(x-y)} \right)
\]

\[
P_4(t) = \frac{ZY(\lambda_3 X + \lambda_4 \Theta)}{wyzx} - \frac{(w+Z)(Y+w)(\lambda_3 w + \lambda_3 X + \lambda_4 \Theta)e^{(wt)}}{(z-w)(-w+y)(x-w)}
+ \frac{(z+Z)(Y+z)(z \lambda_3 + \lambda_4 \Theta + \lambda_3 X)e^{(zt)}}{(z-w)(y-z)(x-z)}
- \frac{(Z+y)(Y+Y)(\lambda_3 X + \lambda_4 \Theta + y \lambda_3) e^{(yt)}}{y(x-y)(y-z)(-w+y)}
+ \frac{(x+Z)(x+Y)(x \lambda_3 + \lambda_3 X + \lambda_4 \Theta) e^{(xt)}}{x(x-y)(x-z)(x-w)}
\]
\[ P_s(t) = \theta \left( \frac{(w + Z)(\lambda_1 Y + \lambda_2 + \lambda_1 + w \lambda_7)}{(x - w)(z - w)(w + y)} e^{(w)t} + \frac{(z + Z)(\lambda_1 \lambda_1 + \lambda_7 Y + z \lambda_2)}{(y - z)(z - w)(x - z)} e^{(z)t} \right. \\
\left. - \frac{(Z + y)(\lambda_1 Y + y \lambda_2 + \lambda_1 \lambda_1)}{(w + y)(y - z)(x - y)} e^{(y)t} + \frac{(x + Z)(\lambda_2 \lambda_1 + \lambda_7 Y + \lambda_7 x)}{(x - w)(x - z)(x - y)} e^{(x)t} \right) \\
+ \frac{Z(\lambda_7 Y + \lambda_2 \lambda_1)}{w y z x} \right) \\
\]
(3.38)

\[ P_s(t) = \theta \left( \frac{(\lambda_6 + \lambda_8 Z + \lambda_8 w)(Y + w)}{(x - w)(z - w)(w + y)} e^{(w)t} + \frac{(\lambda_6 \lambda_5 + \lambda_8 Z + z \lambda_8)(Y + z)}{(x - z)(y - z)(z - w)} e^{(z)t} \right. \\
\left. - \frac{(\lambda_8 Z + \lambda_6 \lambda_5 + y \lambda_8)(y + Y)}{y(w + y)(y - z)(x - y)} e^{(y)t} + \frac{(\lambda_6 \lambda_5 + x \lambda_8 + \lambda_8 Z)(x + Y)}{x(x - w)(x - z)(x - y)} e^{(x)t} \right) \\
+ \frac{Y(\lambda_6 \lambda_5 + \lambda_8 Z)}{x y z w} \right) \\
\]
(3.39)

where,

\( x, y, z, \text{ and } w \) are the roots for Laplace transform variable \( s \)

\[
x = \frac{k_1}{4} - \frac{1}{2} \sqrt{Q - \frac{2k_2}{3}} - \frac{1}{2} \sqrt{-Q - \frac{4k_2}{3} - \frac{-k_1^2 + 4k_1k_2 - 8k_3}{4 \sqrt{Q - \frac{2k_2}{3}}}}
\]

\[
y = \frac{k_1}{4} - \frac{1}{2} \sqrt{Q - \frac{2k_2}{3}} + \frac{1}{2} \sqrt{-Q - \frac{4k_2}{3} - \frac{-k_1^2 + 4k_1k_2 - 8k_3}{4 \sqrt{Q - \frac{2k_2}{3}}}}
\]

\[
z = \frac{k_1}{4} + \frac{1}{2} \sqrt{Q - \frac{2k_2}{3}} + \frac{1}{2} \sqrt{-Q - \frac{4k_2}{3} - \frac{-k_1^2 + 4k_1k_2 - 8k_3}{4 \sqrt{Q - \frac{2k_2}{3}}}}
\]

\[
w = \frac{k_1}{4} + \frac{1}{2} \sqrt{Q - \frac{2k_2}{3}} - \frac{1}{2} \sqrt{-Q - \frac{4k_2}{3} - \frac{-k_1^2 + 4k_1k_2 - 8k_3}{4 \sqrt{Q - \frac{2k_2}{3}}}}
\]

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where,
\[ Q = \frac{2^{1/3} N}{O^{1/3}} + \frac{O^{1/3}}{32^{1/3}} + \frac{k_1^2}{4} \]
\[ O = M + \sqrt{M^2 - 4 N^3} \]
\[ M = 2k_2^3 - 9k_1k_2k_3 + 27k_2^2 + 27k_1^2k_4 - 72k_2k_4 \]
\[ N = k_2^2 - 3k_1k_3 + 12k_4 \]

For the specified values of the model parameters, the plots of Equations (3.33) – (3.39) are shown in Fig. 3.6.

\[ \lambda_1 = 0.005, \lambda_2 = 0.005, \lambda_3 = 0.001, \lambda_4 = 0.002, \mu_3 = 0.004, \]
\[ \lambda_5 = 0.006, \lambda_6 = 0.004, \lambda_7 = 0.002, \lambda_8 = 0.001, \]
\[ \mu_1 = 0.008, \mu_2 = 0.005, \theta = 0.05, \]

**Figure 3.6. State Probability plots for Model II**
The probability of the human being alive is given by,

$$R(t) = P_0(t) + P_1(t) + P_2(t) + P_3(t)$$

(3.40)

Mean time to human death is given by,

$$MTDH = \int_0^\infty [R(t) + P_1(t) + P_2(t) + P_3(t)]dt$$

$$= \frac{\lambda_1 \theta Z + \lambda_5 \theta Y + Z \theta X + \theta Z Y}{UXYZ - \mu_2 \lambda_1 \theta Z - \mu_3 \lambda_5 \theta Y - \mu_1 \theta YZ}$$

(3.41)

For the specified values of model parameters, the plots of the Equation (3.41) are shown in Fig. 3.7.

![Figure 3.7 Mean time to human death plot for Model II](image-url)

\(\lambda_1 = 0.002\quad \lambda_2 = 0.005, \lambda_4 = 0.002, \mu_3 = 0.004, \lambda_3 = 0.001,\)
\(\lambda_6 = 0.004, \lambda_7 = 0.002, \lambda_8 = 0.001,\)
\(\mu_1 = 0.008, \mu_2 = 0.005, \theta = 0.05,\)
Plot Discussion

The plots of Eqs. (3.33)-(3.39) are shown in Fig. 3.6. These plots show that for given values of $\lambda_1$, $\lambda_2$, $\lambda_3$, $\lambda_4$, $\lambda_5$, $\lambda_6$, $\lambda_7$ and $\lambda_8$, the value of $P_1(t)$ decreases or increases as $t$ increases. Fig. 3.7 shows plots for Eq. (3.41). These plots show the effect of varying $\lambda_1$ and $\lambda_5$ on the MTHD of the model. Furthermore, it is observed that the MTHD decreases with increasing values of nurse’s error rate ($\lambda_1$).
Chapter 4

MULTIPLE ERROR MODE MODELS

4.1 Introduction
This chapter represents three mathematical models concerned with reliability evaluation of health care professionals subject to multiple error models. The Markov method was used to develop equations for the models. Expressions for state probability and mean time to health care professional’s error are developed. Some plots of these expressions are presented.

4.2 Model III
This model represents a health care professional performing diagnosis/remedial measures associated with human having health problems. The professional can commit two types of errors: retractable (repairable) or non-retractable (non-repairable). The state space diagram is shown in Fig. 4.1. The following assumptions are associated with this model:

- All error and other occurrences are independent.
- Error and error retractable rates are constant.

The numerals in the boxes of Fig.4.1 denote system states. The following symbols are associated with this model:

\( i \) is the \( ith \) system state: \( i = 0 \) (health care professional performing his/her task normally), \( i = 1 \) (health care professional committed a non-repairable error), \( i = 2 \) (health care professional committed a repairable error).

\( P_i(t) \) is the probability that the health care professional is in state \( i \) at time \( t \); for \( i = 0,1,2 \).

\( \lambda_{nr} \) is the constant health care professional’s non-repairable error rate.

\( \lambda_r \) is the constant health care professional’s repairable error rate.

\( \mu \) is the constant health care professional’s error retractable rate.

\( s \) is the Laplace transform variable.

\( t \) is the time in months.
Figure 4.1. State Space diagram of Model III

Model Analysis

With the aid of the Markov method we write down the following system of differential equations for Fig. 4.1:

\[
\frac{dP_0(t)}{dt} = (\mu)P_2(t) - (\lambda_{nr} + \lambda_r)P_0(t) \quad (4.1)
\]

\[
\frac{dP_1(t)}{dt} = \lambda_{nr}P_0(t) \quad (4.2)
\]

\[
\frac{dP_2(t)}{dt} = -(\mu)P_2(t) + \lambda_rP_0(t) \quad (4.3)
\]

* Detailed analysis is provided in Appendix B
At time $t = 0$, $P_0(0) = 1$, $P_1(0) = 0$, $P_2(0) = 0$.

Taking Laplace transforms of Equations (4.1) – (4.3) we get,

$$P_0(s) = \frac{s + \mu}{s^2 + s \mu + \lambda_{nr} s + \lambda_{nr} \mu + \lambda_r s}$$  \hspace{1cm} (4.4)

$$P_1(s) = \frac{\lambda_{nr} (s + \mu)}{(s^2 + s \mu + \lambda_{nr} s + \lambda_{nr} \mu + \lambda_r s) s}$$ \hspace{1cm} (4.5)

$$P_2(s) = \frac{\lambda_r}{s^2 + s \mu + \lambda_{nr} s + \lambda_{nr} \mu + \lambda_r s}$$ \hspace{1cm} (4.6)

Taking inverse Laplace transform of Eqs. (4.4) – (4.6) we get

$$P_0(t) = \frac{(-y - \mu) e^{(y)t} + (\mu + x) e^{(x)t}}{x - y}$$ \hspace{1cm} (4.7)

$$P_1(t) = \lambda_{nr} \left[ \frac{(-y - \mu) e^{(y)t}}{(x - y) y} + \frac{(\mu + x) e^{(x)t}}{x (x - y)} + \mu \right]$$ \hspace{1cm} (4.8)

$$P_2(t) = \frac{\lambda_r (-e^{(y)t} + e^{(x)t})}{x - y}$$ \hspace{1cm} (4.9)
where
\( x, y \) are the roots of the Laplace transform variable \( s \).

\[
x = -\frac{1}{2} a_1 + \frac{1}{2} \sqrt{a_1^2 - 4 a_2}
\]

\[
y = -\frac{1}{2} a_1 - \frac{1}{2} \sqrt{a_1^2 - 4 a_2}
\]

\[
a_1 = \mu + \lambda_r + \lambda_{nr}
\]

\[
a_2 = \mu \lambda_{nr}
\]

For the specified values of model parameters, the plots of Equations (4.7) –(4.9) are shown in Fig. 4.2. These plots show that for specified value of \( \lambda_r, \lambda_{nr}, \) and \( \mu \), the value of the \( P_i(t) \) increases or decreases as \( t \) increases for \( i = 0,1, \) and \( 2 \).

![Figure 4.2 State probability plots for Model III](image-url)

\( \lambda_r = 0.003, \lambda_{nr} = 0.008, \mu = 0.003 \)
For $\mu = 0$, the reliability of the health care professional is given by

$$R(t) = P_{0}(t)$$  \hspace{1cm} (4.10)$$

Mean time to health professional error (MTTHPE) is given by

$$MTTHPE = \int_{0}^{\infty} R(t) \, dt = \frac{1}{\lambda_{r} + \lambda_{nr}}$$  \hspace{1cm} (4.11)$$

For the given values of model parameters the plots of the Equation (4.11) are shown in Fig. 4.3. These plots show the effect of varying $\lambda_{r}$ and $\lambda_{nr}$ on the MTTHPE of the model. Further more, it is observed that the MTTHPE increases with the increasing values of $\lambda_{r}$.

**Figure 4.3.** Mean time to health care professional error plots for Model III
4.3 Model IV

This is an extension to model III. More specifically, the model assumes that a health care professional performing diagnosis/remedial oriented tasks associated with a human having health problems is subject to normal and stressful environments instead of the normal environment only. The other factors associated with the model remain the same. The model state space diagram is shown in Fig. 4.4. The following assumptions are associated with this model:

- All error and other occurrences are independent.
- The rate of changing health care professional work environment from normal abnormal (stressful) and vice versa is constant.
- Error and error retractable rates are constant.

The numerals in the boxes of Fig.4.1 denote system states. The following symbols are associated with this model:

\[ i \] is the \( i \)th system state: \( i = 0 \) (health care professional performing in normal environment), \( i = 1 \) (health care professional committed a non-repairable (non-retractable) error in normal environment), \( i = 2 \) (health care professional committed a repairable (retractable) error in normal environment), \( i = 3 \) (health care professional performing in abnormal (stressful) environment), \( i = 4 \) (health care professional committed a non-repairable error in abnormal environment), \( i = 5 \) (health care professional committed a repairable error in abnormal environment).

\[ P_i(t) \] is the probability that the health care professional is in state \( i \) at time \( t \); for \( i = 0,1,2,3,4,5 \).

\[ \theta_a \] is the transition rate from normal to abnormal (stressful) state.

\[ \theta_n \] is the transition rate from abnormal (stressful) to normal state.

\[ \lambda_{nr1} \] is the constant health care professional’s non-repairable (non-retractable) error rate from state 0 to state 1.

\[ \lambda_{r1} \] is the constant health care professional’s repairable (retractable) error rate from state 0 to state 2.

\[ \mu_{r1} \] is the constant health care professional’s repairable (retractable) rate from state 2.
\( \lambda_{mr2} \) is the constant health care professional’s non-repairable (non-retractable) error rate from state 3 to state 4.

\( \lambda_{r2} \) is the constant health care professional’s repairable (retractable) error rate from state 3 to state 5.

\( \mu_2 \) is the constant health care professional’s repairable (retractable) rate from state 5.

\( s \) is the Laplace transform variable.

**Figure 4.4.** State space diagram of Model IV
Model Analysis

Using the Markov method we write down the following differential equations shown for Fig. 4.4: *

\[
\frac{dP_0(t)}{dt} = (\theta_0)P_0(t) + (\mu_1)P_2(t) - (\lambda_{m1} + \lambda_{r1} + \theta_a)P_0(t) \\
\frac{dP_1(t)}{dt} = \lambda_{r1}P_0(t) \\
\frac{dP_2(t)}{dt} = -(\mu_1)P_2(t) + \lambda_2P_0(t) \\
\frac{dP_3(t)}{dt} = (\theta_0)P_0(t) + (\mu_2)P_3(t) - (\lambda_{m2} + \lambda_{r2} + \theta_a)P_3(t) \\
\frac{dP_4(t)}{dt} = \lambda_{r2}P_3(t) \\
\frac{dP_5(t)}{dt} = -(\mu_2)P_4(t) + \lambda_{r2}P_4(t)
\]

At time \( t = 0 \),

\[ P_0 (0) = 1, \ P_1 (0) = 0, \ P_2 (0) = 0, \ P_3 (0) = 0, \ P_4 (0) = 0, \ P_5 (0) = 0. \]

* Detailed analysis is provided in Appendix B.
Using Eqs. (4.12) - (4.17), we get the following Laplace transforms of the state probabilities:

\[ P_0(s) = \frac{s^3 + s^2 a_1 + s a_2 + a_3}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4} \]  
(4.18)

\[ P_1(s) = \frac{\lambda_{nr2} P_0(s)}{s} \]  
(4.19)

\[ P_2(s) = \frac{\lambda_{rl} P_0(s)}{s + \mu_1} \]  
(4.20)

\[ P_3(s) = \frac{\theta_a P_0(s) (s + \mu_2)}{(s + B) (s + \mu_2) - \mu_2 \lambda_{r2}} \]  
(4.21)

\[ P_4(s) = \frac{\lambda_{nr2} P_3(s)}{s} \]  
(4.22)

\[ P_5(s) = \frac{\lambda_{r2} P_3(s)}{s + \mu_2} \]  
(4.23)

where

\[ a_1 = B + \mu_2 + \mu_1 \]
\[ a_2 = B \mu_2 - \mu_2 \lambda_{r2} + (B + \mu_2) \mu_1 \]
\[ a_3 = (B \mu_2 - \mu_2 \lambda_{r2}) \mu_1 \]
\[ k_1 = A + B + \mu_2 + \mu_1 \]
\[ k_2 = -\theta_n \theta_a + A (B + \mu_2) + B \mu_2 - \mu_2 \lambda_{r2} + (A + B + \mu_2) \mu_1 - \mu_1 \lambda_{rl} \]
\[ k_3 = -\theta_n \theta_a \mu_1 - \theta_n \theta_a \mu_2 + A (B \mu_2 - \mu_2 \lambda_{r2}) + (A (B + \mu_2) + B \mu_2 - \mu_2 \lambda_{r2}) \mu_1 - \mu_1 \lambda_{rl} (B + \mu_2) \]
\[ k_4 = A \frac{(B \mu_2 - \mu_1 \lambda_{r_2}) \mu_1 - \theta_n \theta_a \mu_1 \mu_2 - \mu_1 \lambda_{r_1} (B \mu_2 - \mu_2 \lambda_{r_2})}{\lambda_{r_1} + \lambda_{nr_1} + \theta_a} \]

\[ A = \lambda_{r_1} + \lambda_{nr_1} + \theta_a \]

\[ B = \lambda_{r_2} + \lambda_{nr_2} + \theta_n \]

Taking inverse Laplace transforms of Eqs. (4.18) – (4.23) we get the following time dependent probabilities:

\[ P_0(t) = \frac{(z^3 + z^2 a_1 + a_2 z + a_3) e^{(zi)}}{(-w + z) (y - z) (z + x)} + \frac{(-a_3 - a_2 y - y^3 - a_1 y^2) e^{(yi)}}{(y - w) (y - z) (x - y)} \]

\[ + \frac{(-w^2 a_1 - a_3 - w^3 - w a_2) e^{(wi)}}{(-w + z) (y - w) (x - w)} + \frac{(x^3 + a_1 x^2 + a_3 + a_2 x) e^{(xi)}}{(x - y) (-z + x) (x - w)} \]

(4.24)

\[ P_1(t) = \lambda_{nr_2} \frac{(z^3 + z^2 a_1 + a_2 z + a_3) e^{(zi)}}{z (-w + z) (y - z) (-z + x)} + \frac{(-a_3 - a_2 y - y^3 - a_1 y^2) e^{(yi)}}{(y - z) (y - w) y (x - y)} \]

\[ + \frac{(-w^2 a_1 - a_3 - w^3 - w a_2) e^{(wi)}}{w (y - w) (x - w) (-w + z)} + \frac{(x^3 + a_1 x^2 + a_3 + a_2 x) e^{(xi)}}{(-z + x) (x - y) (x - w) x} + \frac{a_3}{x y z w} \]

(4.25)

\[ P_2(t) = \lambda_{r_1} \left( \frac{(-\mu_1 + a_3 - \mu_1 a_2 + \mu_1^2 a_1) e^{(-\mu_1 t)}}{a_4} + \frac{(z^3 + z^2 a_1 + a_2 z + a_3) e^{(zi)}}{(z + \mu_1) (-w + z) (y - z) (-z + x)} \right) \]

\[ + \frac{(-a_3 - a_2 y - y^3 - a_1 y^2) e^{(yi)}}{(y - z) (y + \mu_1) (x - y) (y - w)} + \frac{(-w^2 a_1 - a_3 - w^3 - w a_2) e^{(wi)}}{(y - w) (x - w) (\mu_1 + w) (-w + z)} \]

\[ + \frac{(x^3 + a_1 x^2 + a_3 + a_2 x) e^{(xi)}}{(-z + x) (x - w) (\mu_1 + x) (x - y)} \]

(4.26)

\[ P_3(t) = \theta_a \left( \frac{(\mu_2 + z) (z + \mu_1) e^{(zi)}}{(-z + x) (-w + z) (y - z)} - \frac{(y + \mu_2) (y + \mu_1) e^{(yi)}}{(y - w) (y - z) (x - y)} \right) \]

\[ - \frac{(w + \mu_2) (\mu_1 + w) e^{(wi)}}{(x - w) (-w + z) (y - w)} + \frac{(\mu_2 + x) (\mu_1 + x) e^{(xi)}}{(x - w) (-z + x) (x - y)} \]

(4.27)
\[ P_4(t) = \theta_a \frac{\lambda_m}{\tau} \left( \frac{(\mu_2 + z)(z + \mu_1)e^{zt}}{(y-z)z(-w+z)(-z+x)} - \frac{(y + \mu_2)(y + \mu_1)e^{yt}}{(y-w)(y-z)(x-y)y} \right. \]
\[ \left. - \frac{(w + \mu_2)(\mu_1 + w)e^{wt}}{w(x-w)(-w+z)(y-w)} + \frac{(\mu_2 + x)(\mu_1 + x)e^{xt}}{(x-w)x(x-y)(-z+x)} + \frac{\mu_1 \mu_2}{zxyw} \right) \] (4.28)

\[ P_5(t) = \theta_a \frac{\lambda_m}{\tau} \left( \frac{(z + \mu_1)e^{zt}}{(-w + z)(y-z)(-z+x)} + \frac{(-y - \mu_1)e^{yt}}{(y-z)(x-y)(y-w)} \right. \]
\[ \left. + \frac{(-\mu_1 - w)e^{wt}}{(-w + z)(y-w)(x-w)} + \frac{(\mu_1 + x)e^{xt}}{(x-w)(x-y)(-z+x)} \right) \] (4.29)

where

\[ x, y, z, w \] are the roots of the Laplace transform variable \( s \)

\[ x = \frac{k_1 - 1}{4} + \frac{1}{2} \sqrt{\frac{2k_2}{Q - \frac{2k_2}{3}}} + \frac{1}{2} \sqrt{-\frac{4k_2}{Q - \frac{2k_2}{3}} - \frac{k_1^3}{3} + 4k_1k_2 - 8k_3} \]

\[ y = \frac{k_1 - 1}{4} + \frac{1}{2} \sqrt{\frac{2k_2}{Q - \frac{2k_2}{3}}} + \frac{1}{2} \sqrt{-\frac{4k_2}{Q - \frac{2k_2}{3}} - \frac{k_1^3}{3} + 4k_1k_2 - 8k_3} \]

\[ z = \frac{k_1 + 1}{4} + \frac{1}{2} \sqrt{\frac{2k_2}{Q - \frac{2k_2}{3}}} + \frac{1}{2} \sqrt{-\frac{4k_2}{Q - \frac{2k_2}{3}} - \frac{k_1^3}{3} + 4k_1k_2 - 8k_3} \]

\[ w = \frac{k_1 + 1}{4} + \frac{1}{2} \sqrt{\frac{2k_2}{Q - \frac{2k_2}{3}}} - \frac{1}{2} \sqrt{-\frac{4k_2}{Q - \frac{2k_2}{3}} - \frac{k_1^3}{3} + 4k_1k_2 - 8k_3} \]
where

\[
Q = \frac{2^{1/3}}{3} \frac{N}{O^{1/3}} + \frac{O^{1/3}}{3} + \frac{k_1^2}{2^{1/3} + 4}
\]

\[
O = M + \sqrt{M^2 - 4 N^3}
\]

\[
M = 2 k_2^2 - 9 k_1 k_2 k_3 + 27 k_3^2 + 27 k_1^2 k_4 - 72 k_2 k_4
\]

\[
N = k_2^2 - 3 k_1 k_3 + 12 k_4
\]

For the specified values of the model parameters the plots of the Equations (4.24) – (4.29) are shown in Fig. 4.5. These plots show that for specified value of \(\lambda_{r1}, \lambda_{mr1}, \lambda_{r2}, \lambda_{mr2}, \mu_1\) and \(\mu_2\), the value of the \(P_i(t)\) increases or decreases as \(t\) increases for \(i = 0, 1, 2, 3, 4\) and 5.

\[\lambda_{r1} = 0.003, \lambda_{r2} = 0.005\]
\[\lambda_{mr1} = 0.005, \lambda_{mr2} = 0.008, \mu_1 = 0.005, \mu_2 = 0.008, \]
\[\theta_a = 0.05, \theta_n = 0.05\]

**Figure 4.5** State Probability plots for Model IV
The reliability of the health care professional performing both in normal and abnormal conditions is given by

\[ R(t) = P_0(t) + P_2(t) \]  \hspace{1cm} (4.30)

For the specified values of the model parameters, the plots of equation (4.30) are shown in Fig 4.6

\[ \lambda_{r_2} = 0.005, \lambda_{r_1} = 0.005, \]
\[ \lambda_{n_2} = 0.008, \mu_1 = 0.005, \mu_2 = 0.008, \]
\[ \theta_a = 0.05, \theta_n = 0.05 \]

![Figure 4.6 Reliability plots for Model IV](image)

The mean time to health care professional error (MTTHPE) is given by

\[ MTTHPE = \int_0^\infty R(t) \, dt \]  \hspace{1cm} (4.31)

\[ MTTHPE = \frac{B - \lambda_{r_2} + \theta_a}{AB - A \lambda_{n_2} - \lambda_{r_1} B + \lambda_{r_1} \lambda_{r_2} - \theta_a \theta_n} \]

MTTHPE is the mean time to health care professional error.
4.4 Model V

This model represents a health care professional performing diagnosis/remedial oriented tasks associated with a human having health problems. The professional can make three types of errors: the human dies, the human’s health gets deteriorated or the human is unaffected. The error is either retracted from the deteriorated state of the human or the human dies. The state space diagram is shown in Fig. 4. 7. The Markov method was used to develop equations for the model. Expression for state probability and mean time to health care professional error are developed and some plots of these expressions are presented.

The following assumptions are associated with this model:

- All error and other occurrences are independent.
- Error and other rates are constant.

The numerals in the boxes of Fig.4.7 denote system states. The following symbols are associated with this model:

\[ i \] is the ith system state: \( i = 0 \) (healthcare professional performing normally ), \( i = 1 \) (health care professional made the error and the patient dies), \( i = 2 \) health care professional made an error and patient’s health deteriorated, \( i = 3 \) (patient’s health unaffected by the action of health care professional).

\[ P_i (t) \] is the probability that the health care professional is in state \( i \) at time \( t \); for \( i = 0,1,2,3 \).

\[ \theta \] is the constant transition rate from state 0 to state 3.

\[ \lambda_1 \] is the constant human death rate from state 0 to state 1.

\[ \lambda_2 \] is the constant human error rate from state 0 to state 2.

\[ \lambda_3 \] is the constant human error rate from state 2 to state 1.

\[ \mu \] is the error retractable rate from state 2.

\( s \) is the Laplace transform variable
Figure 4.7 state space diagrams for Model V

Model Analysis

With the aid of Markov method, the system of differential equations associated with Fig. 4.7 is as follows: *

\[
\begin{align*}
\frac{dP_0(t)}{dt} &= \mu P_2(t) - (\lambda_1 + \theta + \lambda_2) P_0(t) \\
\frac{dP_1(t)}{dt} &= \lambda_3 P_2(t) + \lambda_1 P_0(t) \\
\frac{dP_2(t)}{dt} &= (\mu + \lambda_3) P_2(t) + \lambda_2 P_0(t) \\
\frac{dP_3(t)}{dt} &= \theta P_0(t)
\end{align*}
\]

(4.32) (4.33) (4.34) (4.35)
* Detailed analysis is provided in Appendix B

At time \( t = 0 \),

\[
P_0(0) = 1, \quad P_1(0) = 0, \quad P_2(0) = 0, \quad P_3(0) = 0.
\]

Solving the Equations (4.32) – (4.35) we get the following Laplace transforms of the state probabilities:

\[
P_0(s) = \frac{s + B}{(s + A)(s + B) - \mu \lambda_2} \tag{4.36}
\]

\[
P_1(s) = \frac{\lambda_2}{(s + A)(s + B) - \mu \lambda_2} \tag{4.37}
\]

\[
P_2(s) = \frac{\left(\frac{\lambda_3 \lambda_2}{s + B} + \lambda_1\right) P_0(s)}{s} \tag{4.38}
\]

\[
P_3(s) = \frac{\theta P_0(s)}{s} \tag{4.39}
\]

where

\[
A = \lambda_1 + \theta + \lambda_2
\]

\[
B = \mu + \lambda_3
\]
Taking the inverse Laplace transforms of Eqs. (4.36) – (4.39) we get the following time dependent probabilities:

\[ P_0(t) = \frac{(-\lambda_3 - y - \mu) \, e^{yt} + (\lambda_3 + \mu + x) \, e^{xt}}{x - y} \]  \hspace{1cm} (4.40)

\[ P_1(t) = \frac{(-y \lambda_1 - \lambda_3 \mu - \lambda_1 \lambda_2 - \lambda_3 \lambda_2) \, e^{yt} + (\lambda_1 x + \lambda_1 \mu + \lambda_1 \lambda_3 + \lambda_3 \lambda_2) \, e^{xt}}{(x - y) \, y} + \frac{\lambda_3 \lambda_2 + \lambda_1 \mu + \lambda_1 \lambda_3}{x \, y} \]  \hspace{1cm} (4.41)

\[ P_2(t) = \frac{\lambda_2 \left( -e^{yt} + e^{xt} \right)}{x - y} \]  \hspace{1cm} (4.42)

\[ P_3(t) = \theta \left( \frac{(-\lambda_3 - y - \mu) \, e^{yt}}{(x - y) \, y} + \frac{(\lambda_3 + \mu + x) \, e^{xt}}{x \, (x - y)} + \frac{\mu + \lambda_3}{x \, y} \right) \]  \hspace{1cm} (4.43)

where

\[ x, \ y \] are the roots of the Laplace transform variable \( s \)

\[ x = -\frac{1}{2} B - \frac{1}{2} A + \frac{1}{2} \sqrt{B^2 - 2AB + A^2 + 4\mu \lambda_2} \]

\[ y = -\frac{1}{2} B - \frac{1}{2} A - \frac{1}{2} \sqrt{B^2 - 2AB + A^2 + 4\mu \lambda_2} \]
The plots for state 1 probability Equation (4.41) are shown in Fig. 4.8. These plots show the effect of $\lambda_1$ on the state probability of the system. Further more, it is observed that the probability increases with the increasing values of $\lambda_1$.

\[
\lambda_3 = 0.02, \lambda_2 = 0.03, \mu = 0.003, \\
\theta = 0.05
\]

\[
\lambda_1 = 0.06
\]

\[
\lambda_1 = 0.03
\]

\[
\lambda_1 = 0.003
\]

**Figure 4.8** State 1 Probabilities plots for Model V

For given values of the model parameters, the plots of Eqs (4.40) – (4.43) are shown in Fig 4.9.
These plots show that for specified value of $\lambda_1$, $\lambda_2$, $\lambda_3$, and $\mu$, the value of the $P_i(t)$ increases or decreases as $t$ increases for $i = 0, 1, 2, \text{and } 3$.

\[
\begin{align*}
\lambda_1 &= 0.005, \\
\lambda_2 &= 0.03, \\
\lambda_3 &= 0.02, \\
\mu &= 0.003, \\
\theta &= 0.05
\end{align*}
\]

![Graph showing state probability plots for Model V](image)

Fig. 4.9 State probability plots for Model V

The reliability of the health care professional is given by

\[ R(t) = P_0(t) + P_3(t) \]  
(4.44)

The mean time to health care professional error is given by

\[ MTTHPE = \int_0^\infty (P_0(t) + P_3(t)) dt \]  
(4.45)

\[
= \frac{\lambda_2}{\lambda_1 \mu + \lambda_1 \lambda_3 + \theta \mu + \theta \lambda_3 + \lambda_2 \lambda_3}
\]


Chapter 5
Discussion, Conclusions and Recommendations

5.1 Discussion

The overall thesis can be divided basically into two sections, the literature review and the mathematical modeling of human error in health care systems.

The literature on human error in health care system is growing rapidly, however is scattered across a wide variety of fields and is not well represented in the traditional medical literature. This study presents a detail guide to some of the cited works of human error in health care systems. A comprehensive literature review was conducted from 350 published literatures collected from journals, newspaper, conference proceedings, health care organization reports and books [351]. The literatures are classified into 11 different categories. Human error analysis methods in health care systems are identified.

This study has presented a mathematical reliability and human error analysis of health care system models with human errors and natural causes. The systems incorporate elements of health care professionals and human having health problems. The analysis considers system with constant human error rates, transition rates and constant remedial (curable) error rates. Generalized expressions for state probability, human error probability, system reliability, mean time to human death and mean time to health care professional’s error are presented. Some plots of these expressions are shown.
5.2 Conclusions

The main results obtained from this study can be summarized as follows:

1. The literature on human error in health care system is growing rapidly but is scattered across a wide variety of medical fields.

2. Institute of Medicine (IOM), reports roughly 100 000 Americans each year die from preventable errors in health care [238]. Surprisingly, from all the literature collected and reviewed only few studies have been done to shed light on this area of health care. In the 350 collected literatures only 37 were the human error studies conducted in health care systems.

3. A significant number of dangerous human errors occur in health care systems. Many of these errors are attributed to problems of communication between the physicians and nurses. Applying human factor engineering concepts to the study of the weak points will help reduce such errors.

4. Researchers in health care are using various methods such as Failure Mode Effect Analysis (FMEA) and Root Cause Analysis (RCA), which are used in technical systems to study human error in health care. More proactive methods used in technical systems should be used to study human error in health care.

5. Physicians and nurses were the major contributors to the error. Proper training how to operate the medical equipment will reduce large amount of human error.

6. Most of the methods used to analyze human error in health care were done after the error occurs. There is not enough published literature on predicting human error probability and reliability of the health care system.

7. The analytical section of this report intends to satisfy the need for predicting human error probability subject to different health professional’s error rates. In this section, the mean time to human death and mean time to human error expressions were developed. These expressions and their plots indicate that the occurrence of human error significantly impacts the reliability of healthcare professionals. All in all, mathematical models such as these can help to improve human reliability in health care.
5.3 Recommendations for Further Study

1. Constant human error rates and curable rates are assumed in this thesis. The models could be studied further with the actual data collected from the hospital incident reporting system to predict health care professional's error probability and the obtain mean time to human death.

2. The models developed in this study can be used in hospitals to evaluate their departments with specific modifications according to their needs.

3. Models in this thesis are specific models; these models can be expanded with more failure modes other than health care professional's error, for example prescription error, medical technician error in anaesthesia department, human documentation error.

4. The models discussed in this thesis assumed constant error and curable rates, which can be studied further with time dependent error and curable rates.
References.


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72. Chopra, V., Bovill, J.G., Koornneef, J.F., Reported Significant Observations During


84. Cook, R.I., Woods, D.D., Operating at the Sharp End: The Complexity of Human Error,


110. Doyal, L., The Mortality of Medical Mistakes, Doctors-Obligations and Patients-Expectation,


164. Hobgood, C. D., John, O., Swart, L., Emergency Medicine Resident Errors: Identification and
174.Improving Patient Care by Reporting Problems with Medical Devices, A MedWatch Education Article, Food and Drug Administration, Rockville, Maryland, September 1997.


221. McCray, S., Medical Errors in the US Medical System, Institute of Medicine, National Academy of Medicine, National Academy Press, Washington, DC, USA, 1999.


226. Mehra, R.H., Eagle, K. A., Missed Diagnosis of Acute Coronary Syndromes in the


251. Posner, K.L., Freund, P.R., Trends in Quality of Anaesthesia Care Associated with Changing


263. Richard, J., Identifying Ways to Reduce Surgical Errors, *Journal of American Medical*


Emergency Medicine, Vol. 36, No.2, 2000, pp. 142-144.


Grove, IL, 1995.


Appendix A

Model I

Model Analysis

With the aid of the Markov method, the system of differential equations associated with Fig 3.1 is as follows:

\[
\frac{dP_0(t)}{dt} = \mu_1 P_1(t) + \mu_2 - \mu_3 P_0(t) + P_1(t) + \lambda_3
\]  
(1)

\[
\frac{dP_1(t)}{dt} = \theta P_0(t) - (\mu_1 + \lambda_1 + \lambda_5) P_1(t)
\]  
(2)

\[
\frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\lambda_2 + \mu_2) P_2(t)
\]  
(3)

\[
\frac{dP_3(t)}{dt} = \lambda_3 P_0(t) + \lambda_5 P_1(t)
\]  
(4)

\[
\frac{dP_4(t)}{dt} = \lambda_4 P_1(t) + \lambda_2 P_2(t)
\]  
(5)

Laplace Transforms

Taking Laplace transforms of the solutions to Eqs. (1) - (5)

Applying the initial conditions \( t=0, P_0(0)=1 \), and all other initial condition probabilities are equal to zero.

\[(s + \theta + \lambda_3) P_0(s) - \mu_1 P_1(s) - \mu_2 P_2(s) = 1 \]  
(6)

\[(s + b) P_1(s) - \theta P_0(s) = 0 \]  
(7)
\begin{align*}
(s + c)P_2(s) - \lambda P_1(s) &= 0 
\tag{8} \\
(s - \lambda_3)P_3(s) - \lambda_3 P_0(s) &= 0 
\tag{9} \\
(s - \lambda_2)P_4(s) - \lambda_4 P_1(s) &= 0 
\tag{10} \\
\end{align*}

Solving for $P_0(s)$

Substituting the values of $P_1(s)$ and $P_2(s)$ from the equations (7) – (8) in equation (6) we get:

\[ P_0(s) = \frac{(s + b)(s + c)}{(s + a)(s + b)(s + c) - (s + c)\Theta \mu_1 + \lambda_1 \mu_2 \Theta} \]

\( \tag{11} \)

Let

\[ a = \Theta + \lambda_3 \]

\[ b = \mu_1 + \lambda_1 + \lambda_5 + \lambda_4 \]

\[ c = \lambda_2 + \mu_2 \]
expanding the denominator of the above equation (11) and separating the coefficients of \( s \) we get the following equation

\[
P_0(s) = \frac{(s + b)(s + c)}{s^3 + (a + b + c)s^2 + (a b + (a + b)c - \theta \mu_1)s + a b c - \lambda_1 \mu_2 \theta - c \theta \mu_1}
\]

Let

\[
\begin{align*}
k_1 &= a + b + c \\
k_2 &= (ab + (a + b)c - \Theta \mu_1) \\
k_3 &= abc - \lambda_1 \mu_2 \theta - \Theta \mu_1 c
\end{align*}
\]

Now we can write the above equation in the simplified form as:

\[
P_0(s) = \frac{(s + b)(s + c)}{s^3 + k_1 s^2 + k_2 s + k_3}
\]

Applying Heaviside’s rule for partial fraction to solve the above equation we get

\[
P_0(s) = \frac{x b + b c + x c + x^2}{(y + x)(-z + x)(-s + x)} + \frac{b c + b y + y^2 + c y}{(y - z)(-y + x)(s - y)} + \frac{z b + b c + z c + z^2}{(-y z + x y + z^2 - x z)(s - z)}
\]

(12)
where

\(x, y \text{ and } z\) are the roots of the equation \([s^3 + k_1 s^2 + k_2 s + k_3]\)

Computer software Maple V is used to obtain the roots of the above equations and the inverse laplace transform of the Equations (12), (14), (16), (18) and (20).

\[
x = - \frac{k_1}{3} - \frac{2^{1/3}}{3} \left( \frac{N}{M} \right)^{1/3} + \frac{(M)^{1/3}}{3^{21/3}}
\]

\[
y = - \frac{k_1}{3} + \frac{(1 + i \sqrt{3})}{3^{22/3}} \left( \frac{N}{M} \right)^{1/3} - \frac{(1 - i \sqrt{3})}{6^{21/3}} (M)^{1/3}
\]

\[
z = - \frac{k_1}{3} + \frac{(1 - i \sqrt{3})}{3^{22/3}} \left( \frac{N}{M} \right)^{1/3} - \frac{(1 + i \sqrt{3})}{6^{21/3}} (M)^{1/3}
\]

\[
M = -2k_1^2 + 9k_1k_2 + \sqrt{4 \left( -k_1^2 + 3k_2 \right)^3 + \left( -2k_1^2 + 9k_1k_2 - 27k_3 \right)^2 - 27k_3}
\]

\[
N = -k_1^2 + 3k_2
\]

**Inverse Laplace**

Taking the inverse Laplace of the above equation we get the following equation of the state probability \(P_0(t)\):

\[
P_0(t) = \frac{(z + c)(b + z)e^{(zt)}}{(y - z)(-z + x)} + \frac{(x + c)(b + x)e^{(xt)}}{(-y + x)(-z + x)} - \frac{(c + y)(b + y)e^{(yt)}}{(-y + x)(y - z)}
\]

**Solving for \(P_1(s)\)**

Re-writing the equation (7) we get:
\[ P_1(s) = \frac{\theta}{s + b} P_0(s) \]

Substituting the value of \( P_0(s) \) from the equation (12) into the above equation and applying the Heaviside’s rule for partial fraction we get the following equation:

\[
P_1(s) = \frac{\theta (z + c)}{(-y z + x y + z^2 - x z) (s - z)} - \frac{(c + y) \theta}{(y - z) (-y + x) (s - y)}
- \frac{\theta (x + c)}{(-y + x) (-z + x) (-s + x)}
\]

(14)

**Inverse Laplace**

Taking the inverse Laplace of the equation (14) we get the following equation of the state probability \( P_1(t) \):

\[
P_1(t) = \theta \left( \frac{(-c - y) e^{yt}}{(y - z) (-y + x)} + \frac{(x + c) e^{xt}}{(-y + x) (-z + x)} + \frac{(z + c) e^{zt}}{(y - z) (-z + x)} \right)
\]

(15)

**Solving for \( P_2(s) \)**

Re-writing the equation (8) we get:

\[
P_2(s) = \frac{\lambda_1 P_1(s)}{s + c}
\]
Substituting the value of $P_1(s)$ from the equation (14) into the above equation and applying the Heaviside’s rule for partial fraction we get the following equation:

$$
P_2(s) = \frac{\theta \lambda_1}{(-y z + x y + z^2 - x z)(s-z)} - \frac{\theta \lambda_1}{(y-z)(-y+x)(s-y)} - \frac{\theta \lambda_1}{(-y+x)(-z+x)(-s+x)}
$$

(16)

**Inverse Laplace**

Taking the inverse Laplace of the equation (16) we get the following equation of the state probability $P_2(t)$:

$$
P_2(t) = \theta \lambda_1 \left( -\frac{e^{(yt)}}{(y-z)(-y+x)} + \frac{e^{(xt)}}{(-y+x)(-z+x)} + \frac{e^{(zt)}}{(y-z)(-z+x)} \right)
$$

(17)

**Solving for $P_3(s)$**

Re-writing the equation (9) we get:

$$
P_3(s) = \frac{\lambda_3 P_0(s) + \lambda_5 P_1(s)}{s}
$$
Substituting the value of \( P_I(s) \) from the equation (14) and \( P_0(s) \) from equation (12) into the above equation and applying the Heaviside's rule for partial fraction we get the following equation:

\[
P_3(s) = -\frac{c (\lambda_5 \theta + \lambda_3 b)}{z \times y \times s} + \frac{\lambda_3 b \times c + \lambda_5 \theta \times c + \lambda_3 b \times c + \lambda_3 c \times \lambda_3 \times \lambda_3 \times \lambda_2 \times \lambda_3 \times \lambda_2^2}{(-z + x) \times (y - z) \times z \times (s - z)} \]

\[\]

\[\frac{y \times \lambda_3 b + y \times \lambda_3 c + y \times \lambda_5 \theta + \lambda_5 \theta \times c + \lambda_3 b \times c + \lambda_3 y \times \lambda_3 \times \lambda_3 \times \lambda_2 \times \lambda_3 \times \lambda_2^2}{(-y + x) \times (y - z) \times y \times (s - y)} \]

\[\frac{x \times \lambda_3 b \times c + x \times \lambda_3 c + x \times \lambda_5 \theta + \lambda_5 \theta \times c + \lambda_3 b \times c}{x \times (\lambda_3 b \times c + \lambda_3 c \times \lambda_3 \times \lambda_2 \times \lambda_3 \times \lambda_2^2)} \times (-x \times z \times x \times y \times y \times z) \times (s - x) \]

\[\text{(18)}\]

**Inverse Laplace**

Taking the inverse Laplace of the equation (18) we get the following equation of the state probability \( P_3(t) \):

\[
P_3(t) = -\frac{c (\lambda_5 \theta + \lambda_3 b)}{z \times y} + \frac{(x + c) \times (x \times \lambda_3 \times \lambda_5 \theta + \lambda_5 \theta \times c)}{x \times (-z + x) \times (-y + x)} \]

\[\]

\[\]

\[\frac{-(c + y) \times (y \times \lambda_3 \times \lambda_5 \times \lambda_3 \times \lambda_5 \theta \times e^{yt})}{(-y + x) \times (y - z) \times y} \]  + \[\frac{(z + c) \times (\lambda_5 \times b + \lambda_5 \times \lambda_3 \times \lambda_3 \times \lambda_5 \theta \times e^{zt})}{(-z + x) \times (y - z) \times z} \]

\[\text{(19)}\]

**Solving for \( P_4(s) \)**

Re-writing the equation (10) we get:

\[
P_4(s) = \frac{\lambda_4 P_1(s) + \lambda_2 P_2(s)}{s} \]

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Substituting the value of $P_f(s)$ from the equation (14) and $P_2(s)$ from equation (16) into the above equation and applying the Heaviside’s rule for partial fraction we get the following equation:

$$
P_4(s) = -rac{(\lambda_4 c + x_1) \theta}{z x y s} - \frac{(\lambda_4 c + x_1 + \lambda_4 x) \theta}{(-z + x) x (-y + x) (-s + x)} + \frac{\theta (x_1 + \lambda_4 y + \lambda_4 c)}{(-y + x) (y - z) y (-s + y)} + \frac{\theta (\lambda_4 c + x_1 + z \lambda_4)}{z (-y z + x y + z^2 - x z) (s - z)}$$

(20)

**Inverse Laplace**

Taking the inverse Laplace of the equation (20) we get the following equation of the state probability $P_4(t)$:

$$
P_4(t) = \theta \left( \frac{(-x_1 - \lambda_4 y - \lambda_4 c)}{y (y - z) (-y + x)} e^{(y t)} + \frac{(\lambda_4 c + x_1 + \lambda_4 x)}{(-y + x) x (-z + x)} e^{(x t)} + \frac{(\lambda_4 c + x_1 + z \lambda_4)}{(y - z) (-z + x) z} e^{(z t)} + \frac{-\lambda_4 c - x_1}{z x y} \right)
$$

(21)

The probability of the human being alive is

$$
R(t) = P_0(t) + P_1(t) + P_2(t)
$$

(22)
Mean time to human death (MTHD) is given by

\[ \text{MTHD} = \int_0^\infty (P_0(t) - P_1(t) - P_2(t)) \, dt \]

or

\[ \text{MTHD} = \lim_{s \to 0} R(s) = \frac{b \, c + \theta \, c + \lambda_1 \, \theta}{a \, b \, c - \theta \, \mu_1 \, c - \lambda_1 \, \mu_2 \, \theta} \]  

where

\[ R(s) = P_0(s) + P_1(s) + P_2(s) \]

MTHD is the mean time to human death.
Plots of the Equations (13), (15), (17), (19), (21) are shown in Fig (A.1)-(A.5)

\[ \lambda_1=0.003, \lambda_2=0.0009 \]
\[ \lambda_2=0.001, \lambda_3=0.0005, \mu_1=0.01, \mu_2=0.005 \]
\[ \theta=0.05 \]

Fig A.1 State 0 Probability plot for Model 1
Fig A.2  State 1 Probability plots for Model 1

\[ \lambda_2 = 0.002, \lambda_3 = 0.0009, \lambda_4 = 0.001, \lambda_5 = 0.0005, \]
\[ \mu_1 = 0.01, \mu_2 = 0.005, \theta = 0.05 \]
$\lambda_1=0.003, \lambda_2=0.0009$
$\lambda_4=0.001, \lambda_3=0.0005, \mu_1=0.01, \mu_2=0.005$
$\theta=0.05$

Fig A.3 State 2 Probability plot for Model 1
Fig A.4 State 3 Probability plots for Model 1

\[ \lambda_1 = 0.003, \lambda_2 = 0.002, \lambda_4 = 0.001, \lambda_5 = 0.0005, \]
\[ \mu_1 = 0.01, \mu_2 = 0.005, \theta = 0.05 \]
Fig A.5 State 4 Probability plots for Model 1

\[ \lambda_1 = 0.003, \lambda_2 = 0.002, \lambda_3 = 0.0009, \lambda_4 = 0.0005, \mu_1 = 0.01, \mu_2 = 0.005, \theta = 0.05 \]
Model II

Model Analysis

With the aid of Markov method, the system of differential equations associated with Fig 3.5 is as follows:

\[ \frac{dP_0(t)}{dt} = \mu_2 P_2(t) + \mu_1 P_1(t) + \mu P_3(t) - (\theta + \lambda_3)P_0(t) \]  \hspace{1cm} (1)

\[ \frac{dP_1(t)}{dt} = \theta P_0(t) - (\mu_1 + \lambda_4 + \lambda_2 + \lambda_7 + \lambda_8)P_1(t) \]  \hspace{1cm} (2)

\[ \frac{dP_2(t)}{dt} = \lambda_1 P_1(t) - (\mu_2 + \lambda_2)P_2(t) \]  \hspace{1cm} (3)

\[ \frac{dP_3(t)}{dt} = \lambda_5 P_1(t) - (\lambda_6 + \mu_3)P_3(t) \]  \hspace{1cm} (4)

\[ \frac{dP_4(t)}{dt} = \lambda_3 P_0(t) + \lambda_4 P_1(t) \]  \hspace{1cm} (5)

\[ \frac{dP_5(t)}{dt} = \lambda_2 P_2(t) + \lambda_7 P_1(t) \]  \hspace{1cm} (6)

\[ \frac{dP_6(t)}{dt} = \lambda_8 P_1(t) + \lambda_8 P_3(t) \]  \hspace{1cm} (7)
Laplace Transforms

Applying Laplace transforms to Eqs. (1)-(7) is as follows:

At $t=0$, $P_0(0)=1$, and all other initial condition probabilities are equal to zero.

\[ s \ P_0(s) - 1 = \mu_2 \ P_2(s) + \mu_1 \ P_1(s) + \mu_3 \ P_3(s) - (\theta + \lambda_3) \ P_0(s) \]  \hspace{1cm} (8)

\[ s \ P_1(s) = \theta \ P_0(s) - (\mu_1 + \lambda_1 + \lambda_4 + \lambda_5 + \lambda_7 + \lambda_8) \ P_1(s) \]  \hspace{1cm} (9)

\[ s \ P_2(s) = \lambda_1 \ P_1(s) - (\mu_2 + \lambda_2) \ P_2(s) \]  \hspace{1cm} (10)

\[ s \ P_3(s) = \lambda_2 \ P_1(s) - (\mu_3 + \lambda_6) \ P_3(s) \]  \hspace{1cm} (11)

\[ s \ P_4(s) = \lambda_3 \ P_0(s) - \lambda_4 \ P_1(s) \]  \hspace{1cm} (12)

\[ s \ P_5(s) = \lambda_2 \ P_2(s) - \lambda_7 \ P_1(s) \]  \hspace{1cm} (13)

\[ s \ P_6(s) = \lambda_8 \ P_1(s) - \lambda_6 \ P_3(s) \]  \hspace{1cm} (14)
Solving for $P_0(s)$

Substituting the values of $P_1(s)$, $P_2(s)$ and $P_3(s)$ from the equations (9) - (10) in equation (8) we get:

$$P_0(s) = \frac{(s + X)(s + Y)(s + Z)((s + W)(s + X)(s + Y)(s + Z) - \mu_2 \lambda_1 \theta (s + Z) - \mu_3 \lambda_5 \theta (s + Y) - \mu_1 \theta (s + Y)(s + Z))}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4}$$

Expanding the denominator of the above equation and separating the coefficients of $s$ we get the following equation:

$$P_0(s) = \frac{(s + X)(s + Y)(s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4}$$  \hspace{1cm} (15)

where,

$X = \mu_1 + \lambda_4 + \lambda_5 + \lambda_8 + \lambda_1 + \lambda_7$

$Y = \mu_2 + \lambda_2$

$Z = \lambda_6 + \mu_3$

$U = \theta + \lambda_3$

$k_1 = U + X + Y + Z$

$k_2 = -\mu_2 \lambda_1 \theta + U X + (U + X) Y + (U + X + Y) Z$

$k_3 = -\mu_2 \lambda_1 \theta - \mu_3 \lambda_5 \theta - \mu_1 \theta Y - \mu_1 \theta Z + U X Y + (U X + (U + X) Y) Z$

$k_4 = U X Y Z - \mu_2 \lambda_1 \theta Z - \mu_3 \lambda_2 \theta Y - \mu_1 \theta Y Z$
Applying Heaviside's rule for partial fraction to solve the above equation (15) we get,

\[
\begin{align*}
P_0(s) &= \frac{Yx^2 + xZy + x^3 + xXY + xZX + x^2Z + XYZ + XX^2}{(x-w)(x-z)(x-y)(s-x)} \\
&\quad - \frac{ZYy + ZXY + XYZ + XyY + Yy^2 + y^3 + Zy^2 + Xy^2}{(-w+y)(y-z)(x-y)(s-y)} \\
&\quad + \frac{zZX + z^2Y + zZY + zXY + z^3 + z^2Z + XYZ + z^2X}{(z-w)(x+y+z+xz)(s-z)} \\
&\quad - \frac{wXY + w^2Y + XYZ + w^3 + w^2X + w^2Z + wZX + wZY}{(x+y+z+w-w^3+yw^2+w^2z+zw^2-xz+w-xyw)(s-w)}
\end{align*}
\]

(16)

where,

\[x, \ y, \ z, \ w\] are the roots of the equation \[s^4 + k_1s^3 + k_2s^2 + k_3s + k_4\]

Computer software Maple V is used to obtain the roots of the above equation and the inverse laplace transform of the Equations (16), (19), (23), (26), (30) and (32).

\[
x = \frac{k_1}{4} - \frac{1}{2} \sqrt{\frac{Q - \frac{2k_2}{3}}{3}} - \frac{1}{2} \sqrt{-\frac{Q - \frac{4k_2}{3}}{3} - \frac{-\frac{k_1^2 + 4k_1k_2 - 8k_3}{4}}{\sqrt{\frac{Q - \frac{2k_2}{3}}{3}}}}
\]

\[
y = -\frac{k_1}{4} - \frac{1}{2} \sqrt{\frac{Q - \frac{2k_2}{3}}{3}} + \frac{1}{2} \sqrt{-\frac{Q - \frac{4k_2}{3}}{3} - \frac{-\frac{k_1^2 + 4k_1k_2 - 8k_3}{4}}{\sqrt{\frac{Q - \frac{2k_2}{3}}{3}}}}
\]

\[
z = \frac{k_1}{4} + \frac{1}{2} \sqrt{\frac{Q - \frac{2k_2}{3}}{3}} + \frac{1}{2} \sqrt{-\frac{Q - \frac{4k_2}{3}}{3} - \frac{-\frac{k_1^2 + 4k_1k_2 - 8k_3}{4}}{\sqrt{\frac{Q - \frac{2k_2}{3}}{3}}}}
\]

\[
w = -\frac{k_1}{4} + \frac{1}{2} \sqrt{\frac{Q - \frac{2k_2}{3}}{3}} - \frac{1}{2} \sqrt{-\frac{Q - \frac{4k_2}{3}}{3} - \frac{-\frac{k_1^2 + 4k_1k_2 - 8k_3}{4}}{\sqrt{\frac{Q - \frac{2k_2}{3}}{3}}}}
\]

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where,

\[
Q = \frac{2^{1/3} N}{3} + \frac{O^{1/3}}{32^{1/3}} + \frac{k^2_4}{4}
\]

\[
O = M + \sqrt{M^2 - 4 N^3}
\]

\[
M = 2k^2_2 - 9k_1 k_2 k_3 + 27 k^2_3 + 27 k^2_1 k_4 - 72 k_2 k_4
\]

\[
N = k^2_2 - 3 k_1 k_3 + 12 k_4
\]

**Inverse Laplace**

Taking the inverse Laplace of the above equation (16) we get the following equation of the state probability \(P_0(t)\):

\[
P_0(t) = \frac{(x + Z)(x + Y)(x + X) e^{x t}}{(x - w)(x - z)(x - y)} - \frac{(Z + y)(y + Y)(y + X) e^{y t}}{(x - y)(y - z)(-w + y)}
\]

\[
+ \frac{(z + Z)(Y + z)(z + X) e^{z t}}{(z - w)(y - z)(x - z)} - \frac{(w + Z)(Y + w)(X + w) e^{w t}}{(z - w)(-w + y)(x - w)}
\]

(17)

**Solving for \(P_1(s)\)**

Substituting the values of \(P_0(s)\) from the equations (16) in equation (9)

we get:

\[
P_1(s) = \frac{\theta P_0(s)}{s + X}
\]

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\[ P_1(s) = \frac{\theta (s + Y) (s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4} \]  

(18)

Applying Heaviside's rule for partial fraction to solve the above equation (18) we get,

\[ P_1(s) = -\frac{(w Y + Z Y + w^2 + w Z) \theta}{(x - w) (-w + y) (z - w) (s - w)} + \frac{(z Z + z Y + Z Y + z^2) \theta}{(x - z) (y - z) (z - w) (s - z)} \]

\[ -\frac{(Z y + Z Y + Y y + y^2) \theta}{(x - y) (-y z + z w - y w + y^2) (s - y)} \]

\[ + \frac{(y z - y z w - x^2 y + x y w - x^2 z + x^3 + x z w - x^2 w) \theta}{(xy z - y z w - x^2 y + x y w - x^2 z + x^3 + x z w - x^2 w) (s - x)} \]  

(19)

**Inverse Laplace**

Taking the inverse Laplace of the above equation (18) we get the following equation of the state probability \( P_1(t) \):

\[ P_1(t) = \theta \left( -\frac{(w + Z) (Y + w) e^{(w) t}}{(z - w) (-w + y) (x - w)} + \frac{(z + Z) (Y + z) e^{(z) t}}{(x - z) (y - z) (z - w)} \right) \]

\[ -\frac{(Z + y) (y + Y) e^{(y) t}}{(-w + y) (y - z) (x - y)} + \frac{(x + Z) (x + Y) e^{(x) t}}{(x - w) (x - z) (x - y)} \]  

(20)

**Solving for \( P_2(s) \)**

Re-writing the equation (10) we get

\[ P_2(s) = \frac{\lambda_1 \theta P_0(s)}{(s + X) (s + Y)} \]  

(21)
Substituting the values of $P_0(s)$ from the equations (16) in equation (21)

\[ P_2(s) = \frac{\lambda_1 \theta (s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 S + k_4} \]  

(22)

Applying Heaviside's rule for partial fraction to solve the above equation (22) we get,

\[
P_2(s) = -\frac{(w + Z) \theta \lambda_1}{(x - w)(-w + y)(z - w)(s - w)} + \frac{(z + Z) \theta \lambda_1}{(y - z)(x - z)(z - w)(s - z)} \\
- \frac{\theta \lambda_1 (Z + y)}{(-xyz + xzw - xyw + y^2x + y^2z + y^2w - yzw - y^3)(s - y)} \\
+ \frac{\theta \lambda_1}{(x - y)(-xz + z w + x^2 - x w)(s - x)}
\]  

(23)

**Inverse Laplace**

Taking the inverse Laplace of the above equation (23) we get the following equation of the state probability $P_2(t)$:

\[
P_2(t) = \theta \lambda_1 \left( \frac{(-w - Z) e^{(zt)}}{(z - w)(-w + y)(x - w)} + \frac{(z + Z) e^{(zt)}}{(y - z)(x - z)(z - w)} \\
+ \frac{(-Z - y) e^{(yt)}}{(-w + y)(y - z)(x - y)} + \frac{(x + Z) e^{(xt)}}{(x - w)(x - z)(x - y)} \right)
\]  

(24)
Solving for $P_3(s)$

Re-writing the equation (11) we get

$$P_3(s) = \frac{\lambda_5 P_1(s)}{s + Z}$$

Substituting the values of $P_1(s)$ from the equations (18) in the above equation we get:

$$P_3(s) = \frac{\lambda_5 \theta (s + Y)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4} \quad (25)$$

Applying Heaviside's rule for partial fraction to solve the above equation (25) we get,

$$P_3(s) := \frac{(x + Y) \theta \lambda_5}{(x - w)(x - z)(x - y)(s - x)} - \frac{(y + Y) \theta \lambda_5}{(-w + y)(y - z)(x - y)(s - y)}$$

$$+ \frac{(Y + z) \theta \lambda_5}{(z - w)(x y - y z + z^2 - x z)(s - z)}$$

$$- \frac{\theta \lambda_5 (Y + w)}{(x y z - y z w - w^3 + y w^2 + w^2 x + z w^2 - x z w - x y w)(s - w)} \quad (26)$$
Inverse Laplace

Taking the inverse Laplace of the above equation (26) we get the following equation of the state probability $P_3(t)$:

$$
P_3(t) = \theta \lambda_3 \left( \frac{(-Y - w)}{(z - w)(-w + y)(x - w)} + \frac{(Y + z)}{(z - w)(y - z)(x - z)} \right) + \frac{(-y - Y)}{(x - y)(y - z)(-w + y)} + \frac{(x + Y)}{(x - w)(x - z)(x - y)} \right)$$

(27)

Solving for $P_4(s)$

Rewriting the equation (12) we get,

$$
P_4(s) = \frac{\lambda_3 P_3(s) + \lambda_4 P_1(s)}{s}
$$

(28)

Substituting the values of $P_1(s)$ from equations (18) and $P_0(s)$ from equation (15) in the above equation, we get

$$
P_4(s) = \frac{\frac{\lambda_3 (s + X)(s + Y)(s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4} + \frac{\lambda_4 (s + Y)(s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4}}{s}
$$

(29)
Applying Heaviside's rule for partial fraction to solve the equation (29) and taking inverse Laplace we get,

\[
P_4(t) = \frac{Z \ Y (\lambda_3 \ X + \lambda_4 \ \Theta)}{w \ z \ y \ x} \ - \ \frac{(w + Z) \ (Y + w) \ (\lambda_3 \ w + \lambda_4 \ X + \lambda_4 \ \Theta) \ e^{(w t)}}{(z - w) \ (-w + y) \ (x - w) \ w} \\
+ \ \frac{(z + Z) \ (Y + z) \ (-\lambda_3 + \lambda_4 \ \Theta + \lambda_3 \ X) \ e^{(z t)}}{(z - w) \ (y - z) \ (x - z) \ z} \\
+ \ \frac{(Z + y) \ (y + Y) \ (\lambda_3 \ X + \lambda_4 \ \Theta + y \ \lambda_3) \ e^{(y t)}}{y \ (x - y) \ (y - z) \ (-w + y)} \\
+ \ \frac{(x + Z) \ (x + Y) \ (x \lambda_3 + \lambda_3 \ X + \lambda_4 \ \Theta) \ e^{(x t)}}{x \ (x - y) \ (x - z) \ (x - w)}
\]  

(30)

**Solving for \( P_5(s) \)**

Re-writing the equation (13) we get,

\[
P_5(s) = \frac{\lambda_2 \ P_2(s) + \lambda_4 \ P_1(s)}{s}
\]  

(31)

Substituting the values of \( P_1(s) \) from equations (18) and \( P_2(s) \) from equation (22) in the above equation, we get

\[
P_5(s) := \frac{\lambda_2 \ \lambda_1 \ \theta \ (s + Z) + \lambda_4 \ \theta \ (s + Y) \ (s + Z)}{s^4 + k_1 \ s^3 + k_2 \ s^2 + k_3 \ s + k_4} + \frac{\lambda_2 \ \lambda_1 \ \theta \ (s + Z)}{s^4 + k_1 \ s^3 + k_2 \ s^2 + k_3 \ s + k_4}
\]  

(32)
Applying Heaviside's rule for partial fraction to solve the equation (32) and taking inverse Laplace we get

\[
P_3(t) = \Theta \left( -\frac{(w + Z) (\lambda_7 Y + \lambda_2 \lambda_1 + w \lambda_7) e^{(w t)}}{(x - w) w (z - w) (-w + y)} + \frac{(z + Z) (\lambda_2 \lambda_1 + \lambda_7 Y + z \lambda_7) e^{(z t)}}{(y - z) z (z - w) (x - z)} \right.
\]

\[
- \frac{(Z + y) (\lambda_7 Y + y \lambda_7 + \lambda_2 \lambda_1) e^{(y t)}}{(-w + y) (y - z) (x - y) y} + \frac{(x + Z) (\lambda_2 \lambda_1 + \lambda_7 Y + \lambda_7 x) e^{(x t)}}{(x - w) x (x - z) (x - y)}
\]

\[
+ \frac{Z (\lambda_7 Y + \lambda_2 \lambda_1)}{w z y x}
\]

(33)

**Solving for \( P_6(s) \)**

Re-writing the equation (14) we get,

\[
P_6(s) = \frac{\lambda_6 P_3(s) + \lambda_8 P_1(s)}{s}
\]

(34)

Substituting the values of \( P_1(s) \) from equations (18) and \( P_3(s) \) from equation (25) in the above equation, we get

\[
P_6(s) := \frac{\lambda_6 \lambda_2 \Theta (s + Y)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4} + \frac{\lambda_8 \Theta (s + Y) (s + Z)}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4}
\]

(35)
Applying Heaviside's rule for partial fraction to solve the equation (35) and taking inverse Laplace we get

\[
P_6(t) = \theta \left( -\frac{(\lambda_6 \lambda_5 + \lambda_8 Z + \lambda_8 w) (Y + w) e^{(w t)}}{(x - w) w (z - w) (-w + y)} + \frac{(\lambda_6 \lambda_5 + \lambda_8 Z + z \lambda_8) (Y + z) e^{(z t)}}{(x - z) (y - z) z (z - w)} \right. \\
- \frac{(\lambda_8 Z + \lambda_6 \lambda_5 + y \lambda_8) (y + Y) e^{(y t)}}{y (-w + y) (y - z) (x - y)} + \frac{(\lambda_6 \lambda_5 + x \lambda_8 + \lambda_8 Z) (x + Y) e^{(x t)}}{x (x - w) (x - z) (x - y)} \\
\left. + \frac{Y (\lambda_6 \lambda_5 + \lambda_8 Z)}{x y z w} \right)
\]

(36)
Model III

Model Analysis

With the aid of the Markov method we write down the following system of differential equations for Fig. 4.1:

\[ \frac{dP_0(t)}{dt} = (\mu) P_0(t) - (\lambda_{mr} + \lambda_r) P_0(t) \]  \hspace{1cm} (1)

\[ \frac{dP_1(t)}{dt} = \lambda_{mr} P_0(t) \]  \hspace{1cm} (2)

\[ \frac{dP_2(t)}{dt} = -(\mu) P_2(t) + \lambda_r P_0(t) \]  \hspace{1cm} (3)

Laplace Transforms

Taking Laplace transforms of Equations (1) – (3)

Applying the initial conditions at time t = 0, \( P_0(0) = 1 \), \( P_1(0) = 0 \), \( P_2(0) = 0 \).

We get,

\[ s P_0(s) - 1 = \mu P_2(s) - (\lambda_{mr} + \lambda_r) P_0(s) \]  \hspace{1cm} (4)

\[ s P_1(s) = \lambda_{mr} P_0(s) \]  \hspace{1cm} (5)
\[ s \ P_1(s) = \lambda_{nr} \ P_0(s) \]  

(5)

\[ s \ P_2(s) = \lambda_r \ P_0(s) + \mu \ P_2(s) \]  

(6)

**Solving for** \( P_0(s) \)

Substituting the values of \( P_2(s) \) from equation (6) into equation (4) we get:

\[ P_0(s) = \frac{s + \mu}{(s + \lambda_{nr} + \lambda_r)(s + \mu) - \mu \lambda_r} \]  

(7)

\[ P_0(s) = \frac{s + \mu}{s^2 + s \mu + \lambda_{nr} s + \lambda_{nr} \mu + \lambda_r s} \]

expanding the denominator of the equation (7) and separating the coefficients of \( s \), we get the following equation,

\[ P_0(s) = \frac{s + \mu}{s^2 + s a_1 + a_2} \]  

(8)

where

\[ a_1 = \mu + \lambda_r + \lambda_{nr} \]

\[ a_2 = \mu \lambda_{nr} \]
Applying Heaviside's rule for partial fraction to solve equation (8), we get

\[ P_0(s) = -\frac{\mu + x}{(-y + x)(-s + x)} + \frac{y + \mu}{(-y + x)(-s + y)} \] (9)

where

\[ x, y \] are the roots of the Laplace transform variable \( s \).

Computer software Maple V is used to obtain the roots of the above equations and the inverse Laplace transform of Equations (9), (12) and (15).

\[ x = -\frac{1}{2} a_1 + \frac{1}{2} \sqrt{a_1^2 - 4 a_2} \]

\[ y = -\frac{1}{2} a_1 - \frac{1}{2} \sqrt{a_1^2 - 4 a_2} \]

**Inverse Laplace**

Taking the inverse Laplace of the above equation (9) we get the equation for state probability \( P_0(t) \)

\[ P_0(t) = \frac{(-y - \mu) e^{(y t)} + (\mu + x) e^{(x t)}}{x - y} \] (10)
Solving for $P_1(s)$

Substituting the values of $P_0(s)$ from equation (7) in to equation (5) we get:

$$P_1(s) := \frac{\lambda_{nr} (s + \mu)}{(s^2 + s \mu + \lambda_{nr} s + \lambda_{nr} \mu + \lambda_r s) s} \tag{11}$$

Applying Heaviside’s rule for partial fraction to solve equation (11), we get

$$P_1(s) = \frac{\mu \lambda_{nr}}{x y s} - \frac{\lambda_{nr} (y + \mu)}{y (x - y) (s - y)} + \frac{\lambda_{nr} (\mu + x)}{x (x - y) (s - x)} \tag{12}$$

Inverse Laplace

Taking the inverse laplace of the above equation (12) we get the equation for state probability $P_1(s)$

$$P_1(t) = \lambda_{nr} \left( \frac{(-y - \mu) e^{(y t)}}{(x - y) y} + \frac{(\mu + x) e^{(x t)}}{x (x - y)} + \frac{\mu}{y x} \right) \tag{13}$$
Solving for $P_2(s)$

Substituting the values of $P_0(s)$ from equation (7) in to equation (6) we get:

$$P_2(s) := \frac{\lambda_r}{s^2 + s \mu + \lambda_{nr} s + \lambda_{nr} \mu + \lambda_r s} \quad (14)$$

Applying Heaviside’s rule for partial fraction to solve equation (14), we get

$$P_2(s) := -\frac{\lambda_r}{(x-y)(s-y)} + \frac{\lambda_r}{(x-y)(s-x)} \quad (15)$$

Inverse Laplace

Taking the inverse laplace of the above equation (15) we get the equation for state probability $P_2(s)$

$$P_2(t) = \frac{\lambda_r (e^{(y)t} + e^{(x)t})}{x-y} \quad (16)$$
Plots of the Equation (10), (13) and (16) are shown in Fig (B.1) – Fig (B.3)

\[ \lambda_m = 0.003, \quad \mu = 0.005 \]

**Fig B.1** State 0 plot for Model III
Fig B.2 State 1 plot for Model III

\( \lambda = 0.005, \mu = 0.005 \)

\( \lambda_m = 0.009 \)

\( \lambda_m = 0.006 \)

\( \lambda_m = 0.003 \)
$\lambda_{tr} = 0.003, \ \mu = 0.005$

Fig B.3 State 2 plot for Model III
Model IV

Model Analysis

Using the Markov method we write down the following differential equations shown for Fig. 4.4:

\[
\frac{dP_0(t)}{dt} = \theta_0 P_3(t) + \mu_1 P_1(t) - (\lambda_{nr1} + \lambda_{r1} + \theta_0) P_0(t)
\]  

(1)

\[
\frac{dP_1(t)}{dt} = \lambda_{r1} P_0(t)
\]  

(2)

\[
\frac{dP_2(t)}{dt} = \lambda_{r1} P_0(t) - \mu_0 P_2(t)
\]  

(3)

\[
\frac{dP_3(t)}{dt} = \theta_0 P_0(t) + \mu_2 P_5(t) - (\lambda_{nr2} + \lambda_{r2} + \theta_0) P_3(t)
\]  

(4)

\[
\frac{dP_4(t)}{dt} = \lambda_{r2} P_3(t)
\]  

(5)

\[
\frac{dP_5(t)}{dt} = \lambda_{r2} P_3(t) - \mu_2 P_5(t)
\]  

(6)
Taking Laplace transforms of Equations (1) – (6)

Applying the initial conditions at time \( t = 0 \), \( P_0(0) = 1, P_1(0) = 0, P_2(0) = 0, P_3(0) = 0, P_4(0) = 0, P_5(0) = 0 \).

\[
\begin{align*}
  sP_0(s) - 1 &= \theta_n P_3(s) + \mu_1 P_2(s) - (\lambda_{nr1} + \lambda_{r1} + \theta_a) P_0(s) \\
  sP_1(s) &= \lambda_{r1} P_0(s) \\
  sP_2(s) &= \lambda_{r1} P_0(s) + \mu_1 P_2(s) \\
  sP_3(s) &= \theta_a P_0(s) + \mu_2 P_3(s) - (\lambda_{nr2} + \lambda_{r2} + \theta_n) P_3(s) \\
  sP_4(s) &= \lambda_{r2} P_3(s) \\
  sP_5(s) &= \lambda_{r2} P_3(s) - \mu_2 P_5(s)
\end{align*}
\]
Solving for $P_0(s)$

Re-arranging Equation (7), (10) and (12) we get,

$$P_0(s) = \frac{s^3 + s^2 a_1 + s a_2 + a_3}{s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4}$$  \hspace{1cm} (13)

Where

\[ a_1 = B + \mu_2 + \mu_1 \]
\[ a_2 = B \mu_2 - \mu_2 \lambda_{r_2} + (B + \mu_2) \mu_1 \]
\[ a_3 = (B \mu_2 - \mu_2 \lambda_{r_2}) \mu_1 \]
\[ k_1 = A + B + \mu_2 + \mu_1 \]
\[ k_2 = -\theta_n \phi_a + A (B + \mu_2) + B \mu_2 - \mu_2 \lambda_{r_2} + (A + B + \mu_2) \mu_1 - \mu_1 \lambda_{r_{fl}} \]
\[ k_3 = -\theta_n \phi_a \mu_1 - \theta_n \phi_a \mu_2 + A (B \mu_2 - \mu_2 \lambda_{r_2}) + (A (B + \mu_2) + B \mu_2 - \mu_2 \lambda_{r_2}) \mu_1 - \mu_1 \lambda_{r_{fl}} (B + \mu_2) \]
\[ k_4 = A (B \mu_2 - \mu_2 \lambda_{r_2}) \mu_1 - \theta_n \phi_a \mu_1 \mu_2 - \mu_1 \lambda_{r_{fl}} (B \mu_2 - \mu_2 \lambda_{r_2}) \]
\[ A = \lambda_{r_{fl}} + \lambda_{mr_{fl}} + \theta_n \phi_a \]
\[ B = \lambda_{r_2} + \lambda_{mr_2} + \theta_n \]

Applying Heaviside's rule for partial fraction to solve equation (13), we get

$$P_0(s) := -\frac{a_2 x + a_3 + a_1 x^2 + x^3}{(-y z w + x y z + x y w - x^2 y - x^2 z - x^2 w + x^3 + x z w) (-s + x)} + \frac{a_2 y + a_1 y^2 + a_3 + y^3}{(x - y) (-y z + w z - w y + y^2) (-s + y)} + \frac{z^3 + z^2 a_1 + a_2 z + a_3}{(-z + x) (-z + y) (z - w) (s - z)} + \frac{a_3 + w^2 a_1 + w a_2 + w^3}{(z - w) (-w + y) (-w + x) (s - w)}$$  \hspace{1cm} (14)
where

\[ x, y, z, w \] are the roots of the Laplace transform variable \( s \) of the Equation

\[ s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4 \]

Computer software Maple V is used to obtain the roots of the above equations and the inverse laplace transform of Equations (18), (21), (24), (27) and (30).

\[ x = \frac{k_1}{4} - \frac{1}{2} \sqrt{Q - \frac{2 k_2}{3} + \frac{1}{2}} \sqrt{-Q - \frac{4 k_2}{3} - \frac{-k_1^3 + 4 k_1 k_2 - 8 k_3}{4 \sqrt{Q - \frac{2 k_2}{3}}}} \]

\[ y = \frac{k_1}{4} - \frac{1}{2} \sqrt{Q - \frac{2 k_2}{3} + \frac{1}{2}} \sqrt{-Q - \frac{4 k_2}{3} - \frac{-k_1^3 + 4 k_1 k_2 - 8 k_3}{4 \sqrt{Q - \frac{2 k_2}{3}}}} \]

\[ z = \frac{k_1}{4} + \frac{1}{2} \sqrt{Q - \frac{2 k_2}{3} + \frac{1}{2}} \sqrt{-Q - \frac{4 k_2}{3} - \frac{-k_1^3 + 4 k_1 k_2 - 8 k_3}{4 \sqrt{Q - \frac{2 k_2}{3}}}} \]

\[ w = -\frac{k_1}{4} + \frac{1}{2} \sqrt{Q - \frac{2 k_2}{3} - \frac{1}{2}} \sqrt{-Q - \frac{4 k_2}{3} - \frac{-k_1^3 + 4 k_1 k_2 - 8 k_3}{4 \sqrt{Q - \frac{2 k_2}{3}}}} \]

where

\[ Q = \frac{2^{1/3}}{3} \left( \frac{O^{1/3} + k_1^2}{O^{1/3} 2^{1/3} + \frac{4}{3}} \right) \]

\[ O = M + \sqrt{M^2 - 4 N^3} \]
\[ M = 2k_2^3 - 9k_1k_2k_3 + 27k_3^2 + 27k_1^2k_4 - 72k_2k_4 \]

\[ N = k_2^2 - 3k_1k_3 + 12k_4 \]

**Inverse Laplace**

Taking the inverse laplace of the above equation (14) we get the equation for state probability \( P_0(s) \), we get

\[
P_0(t) = \frac{(z^3 + z^2a_1 + a_2z + a_3)e^{zt}}{(-w + z)(y - z)(-z + x)} + \frac{(-a_3 - a_2y - y^3 - a_1y^2)e^{yt}}{(y - w)(y - z)(x - y)}
\]

\[+ \frac{(-w^2a_3 - a_3 - w^3 - wa_2)e^{wt}}{(-w + z)(y - w)(x - w)} + \frac{(x^3 + a_1x^2 + a_3 + a_2x)e^{xt}}{(x - y)(-z + x)(x - w)} \]  \hspace{1cm} (15)

**Solving for \( P_1(s) \)**

Re-arranging equation (8), we get

\[ P_1(s) = \frac{\lambda_{nr}P_0(s)}{s} \]

Substituting the value of \( P_0(s) \) from equation (13) in the above equation we get

\[ P_1(s) = \frac{\lambda_{nr}(s^3 + s^2a_1 + s^2a_2 + a_3)}{(s^4 + k_1s^3 + k_2s^2 + k_3s + k_4)s} \]  \hspace{1cm} (16)
Applying Heaviside’s rule for partial fraction to solve equation (16), we get

\[ P_1(s) := \frac{\lambda_{mr} a_3}{x y z w s} + \frac{\lambda_{mr} (x^2 + a_1 x^2 + a_3 + a_2 x)}{x (x z w - x^2 z - x^2 w + x^3 - y z w + x y z - x^2 y + x y w) (s - x)} \]

\[ - \frac{(a_1 y^2 + a_2 y + a_3) \lambda_{mr}}{(y - z) (y - w) y (x - y) (s - y)} + \frac{\left( z^3 + z^2 a_1 + a_2 z + a_3 \right) \lambda_{mr}}{(-z + x) (-y w + y z + z w - z^2) z (s - z)} \]

\[ - \frac{(a_3 + w^3 + w a_2 + w^2 a_1) \lambda_{mr}}{(x - w) (-w + z) w (y - w) (s - w)} \]  

(17)

**Inverse Laplace**

Taking the inverse laplace of the above equation (17) we get the equation for state probability \( P_1(s) \), we get

\[ P_1(t) = \lambda_{mr} \left[ \frac{(z^3 + z^2 a_1 + a_2 z + a_3) e^{zt}}{z (-w + z) (y - z) (-z + x)} + \frac{(-a_3 - a_2 y - y^3 - a_1 y^2) e^{yt}}{(y - z) (y - w) y (x - y)} \right. \]

\[ + \frac{(-w^2 a_1 - a_3 - w^3 - w a_2) e^{wt}}{w (y - w) (x - w) (-w + z)} + \left. \frac{(x^3 + a_1 x^2 + a_3 + a_2 x)}{(-z + x) (x - y) (x - w) x} + \frac{a_3}{x y z w} \right] \]  

(18)
Solving for $P_2(s)$

Re-arranging equation (9), we get

$$P_2(s) = \frac{\lambda_{rl} P_0(s)}{s + \mu_1}$$

Substituting the value of $P_0(s)$ from equation (13) in the above equation we get

$$P_2(s) = \frac{\lambda_{rl} (s^3 + s^2 a_4 + s a_2 + a_3)}{(s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4) (s + \mu_1)}$$  \hspace{1cm} (19)

Applying Heaviside’s rule for partial fraction to solve equation (19), we get

$$P_2(s) := -\lambda_{rl} \left( \mu_1^3 - a_2 + \mu_1 a_2 - \mu_1^2 a_1 \right) \left( z \mu_1^3 + z \mu_1^2 y + \mu_1 x y z + z \mu_1^2 x + w \mu_1^2 x + w \mu_1^3 + \mu_1^2 x y + \mu_1^4 + x \mu_1^3 + \mu_1^3 y + \mu_1 x y w + w \mu_1^2 y + x y z w + \mu_1 x z w + w z \mu_1^2 + \mu_1 y z w \right) \left( s + \mu_1 \right))$$

$$+ \lambda_{rl} \left( x^3 + a_1 x^2 + a_2 x \right)$$

$$\frac{\lambda_{rl}}{(s + x) (s + w) (s + x) (s + y) (s + x)}$$

$$- \frac{(a_1 y^2 + a_2 y + y^3 + a_3) \lambda_{rl}}{(s + y) (s + y) (s + y) (s + y)}$$

$$+ \frac{(z + z^2 a_1 + a_2 z + a_3) \lambda_{rl}}{(z + \mu_1) (x y z - y z^2 - x z^2 + z^3 - x y w + y z w - z^2 w + x z w) (s + z)}$$

$$- \frac{(a_3 + w^3 + w a_2 + w^2 a_1) \lambda_{rl}}{(-w + z) (s + w) (s + w) (s + w)}$$  \hspace{1cm} (20)
Inverse Laplace

Taking the inverse laplace of the above equation (20) we get the equation for state probability \( P_2(s) \), we get

\[
P_2(t) = \lambda_{rl} \left( \frac{(-\mu_1^3 + \mu_1 a_2 + \mu_1^2 a_1) e^{(-\mu_1 t)}}{a_4} + \frac{(z^3 + z^2 a_1 + a_2 z + a_3) e^{(z t)}}{(z + \mu_1) (-w + z) (y - z) (-z + x)} \right.

\[+ \frac{(-a_3 - a_2 y - y^3 - a_1 y^2) e^{(y t)}}{(y - z) (y + \mu_1) (x - y) (y - w)} + \frac{(-w^2 a_1 - a_3 - w^3 - w a_2) e^{(w t)}}{(y - w) (x - w) (\mu_1 + w) (-w + z)} \]

\[
\left. + \frac{(x^3 + a_1 x^2 + a_3 + a_2 x) e^{(x t)}}{(-z + x) (x - w) (\mu_1 + x) (x - y)} \right) \tag{21}\]

Solving for \( P_3(s) \)

Re-arranging equation (10), we get

\[
P_3(s) = \frac{\theta_a P_0(s) (s + \mu_2)}{(s + B) (s + \mu_2) - \mu_2 \lambda_{r2}} \]

Substituting the value of \( P_0(s) \) from Equation (13) in the above equation we get

\[
P_3(s) = \frac{\theta_a (s + \mu_2) (s^3 + s^2 a_1 + s a_2 + a_3)}{(s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4) ((s + B) (s + \mu_2) - \mu_2 \lambda_{r2})} \tag{22}\]
Applying Heaviside's rule for partial fraction to solve equation (22), we get

\[ P_3(s) := \frac{\Theta_a \left( x^2 + \mu_1 \mu_2 + x \mu_2 + x \mu_1 \right)}{(x z w - x^2 z - x^2 w + x^3 - y z w + x y z - x^2 y + x y w) (s - x) - (y \mu_1 + y^2 + \mu_1 \mu_2 + y \mu_2) \Theta_a} + \frac{(z^2 + \mu_1 \mu_2 + z \mu_1 + z \mu_2) \Theta_a}{(y - w) (y - z) (x - y) (s - y) + (-z + x) (-w + z) (y - z) (s - z)} \]

\[ - \frac{(w \mu_1 + w \mu_2 + w^2 + \mu_1 \mu_2) \Theta_a}{(x - w) (-y w + w^2 + y z - z w) (s - w)} \]  

(23)

**Inverse Laplace**

Taking the inverse laplace of the above equation (23) we get the equation for state probability \( P_3(s) \), we get

\[ P_3(t) = \Theta_a \left( \frac{(\mu_2 + z) (z + \mu_1) e^{(zt)}}{(-z + x) (-w + z) (y - z) (s - y)} \right) - \frac{(y + \mu_2) (y + \mu_1) e^{(yt)}}{(y - w) (y - z) (x - y)} \]

\[ - \frac{(w + \mu_2) (\mu_1 + w) e^{(wr)}}{(x - w) (-w + z) (y - w)} + \frac{(\mu_2 + x) (\mu_1 + x) e^{(xr)}}{(x - w) (-z + x) (x - y)} \]  

(24)
Solving for $P_4(s)$

Re-arranging equation (11), we get

$$P_4(s) = \frac{\lambda_{nr_2} P_3(s)}{s}$$

Substituting the value of $P_3(s)$ from Equation (22) in the above equation we get

$$P_4(s) = \frac{\lambda_{nr_2} \theta_a (s + \mu_2) (s^3 + s^2 a_1 + sa_2 + a_3)}{(s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4) ((s + B) (s + \mu_2) - \mu_2 \lambda_{nr_2}) s}$$

(25)

Applying Heaviside’s rule for partial fraction to solve equation (25), we get

$$P_4(s) = \frac{\theta_a \lambda_{nr_2} \mu_1 \mu_2}{zxyw} \frac{(w \mu_1 + w \mu_2 + w^2 + w \mu_2) \theta_a \lambda_{nr_2}}{(y - w) (-w + z) (x - w) w (s - w)} + \frac{(z^2 + \mu_1 \mu_2 + z \mu_1 + z \mu_2) \theta_a \lambda_{nr_2}}{(y - z) z (z w + x z - z^2 - w x) (s - z)} - \frac{\theta_a \lambda_{nr_2} (y \mu_1 + y^2 + \mu_1 \mu_2 + y \mu_2)}{y (x z w - y z w - x y z + y^2 z + y^2 w - x y w + y^2 x - y^3) (s - y)} + \frac{(x^2 + \mu_1 \mu_2 + x \mu_2 + x \mu_1) \theta_a \lambda_{nr_2}}{(x - w) x (x - y) (-z + x) (s - x)}$$

(26)
Inverse Laplace

Taking the inverse laplace of the above equation (26) we get the equation for state probability \( P_4(s) \), we get

\[
P_4(t) = \theta_a \lambda_{mr2} \left( \frac{(\mu_2 + z)(z + \mu_1)e^{zt}}{(y-z)(-w+z)(-z+x)} - \frac{(y + \mu_2)(y + \mu_1)e^{yt}}{(y-w)(y-z)(x-y)y} 
\right.

\left. - \frac{(w + \mu_2)(\mu_1 + w)e^{wt}}{w(x-w)(-w+z)(y-w)} + \frac{(\mu_2 + x)(\mu_1 + x)e^{xt}}{(x-w)x(x-y)(-z+x)} + \frac{\mu_1 \mu_2}{z x y w} \right)
\]  \hspace{1cm} (27)

Solving for \( P_4(s) \)

Re-arranging equation (12), we get

\[
P_5(s) = \frac{\lambda_{r2} P_3(s)}{s + \mu_2}
\]

Substituting the value of \( P_3(s) \) from \( \text{Equation (22)} \) in the above equation we get

\[
P_5(s) = \frac{\lambda_{r2} \theta_a (s^3 + s^2 a_1 + s a_2 + a_3)}{(s^4 + k_1 s^3 + k_2 s^2 + k_3 s + k_4)((s + B)(s + \mu_2) - \mu_2 \lambda_{r2})}
\]  \hspace{1cm} (28)
Applying Heaviside’s rule for partial fraction to solve equation (28), we get

\[
P_5(s) := - \frac{\theta_a \lambda r_2 (\mu_1 + w)}{(y w^2 - x y w - w^3 + w^2 x + x y z - y z w + z w^2 - x z w)} \frac{(z + \mu_1) \theta_a \lambda r_2}{(y + \mu_1) \theta_a \lambda r_2} \]
\[
+ \frac{(-w + z)(-y z + x y - x z + z^2)}{(y - w)(x - y)(y - z)(s - y)} \frac{(\mu_1 + x) \theta_a \lambda r_2}{(x - w)(x - y)(-z + x)(s - x)}
\]

(29)

**Inverse Laplace**

Taking the inverse laplace of the above equation (29) we get the equation for state probability \( P_5(s) \), we get

\[
P_5(t) = \theta_a \lambda r_2 \left( \frac{(z + \mu_1) e^{(zt)}}{(-w + z)(y - z)(-z + x)} + \frac{(-y - \mu_1) e^{(yt)}}{(y - z)(x - y)(y - w)} \right)
\]
\[
+ \frac{(-\mu_1 - w) e^{(wt)}}{(-w + z)(y - w)(x - w)} + \frac{(\mu_1 + x) e^{(xt)}}{(x - w)(x - y)(-z + x)}
\]

(30)

Plots of Equation (18), (210, (24), 27) and (30) are shown in Fig (B.4) – (B.9)

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Fig B.4 State 0 plots for Model IV

\[\lambda_{r2} = 0.005, \lambda_{r1} = 0.003,\]
\[\lambda_{n2} = 0.008, \mu_1 = 0.005, \mu_2 = 0.008,\]
\[\theta_a = 0.05, \theta_n = 0.05\]

P\(_0(t)\) vs. t

\[\lambda_{r1} = 0.006\]

\[\lambda_{r1} = 0.003\]

\[\lambda_{r1} = 0.009\]

Fig B.5 State 1 Plots for Model IV

\[\lambda_{n1} = 0.01\]

\[\lambda_{n1} = 0.008\]

\[\lambda_{n1} = 0.005\]

P\(_1(t)\) vs. t

\[\lambda_{r2} = 0.005, \lambda_{r1} = 0.003,\]
\[\lambda_{n2} = 0.008, \mu_1 = 0.005, \mu_2 = 0.008,\]
\[\theta_a = 0.05, \theta_n = 0.05\]
**Fig B.6** State 2 Plots for Model IV

\[
\begin{align*}
\lambda_{r_2} &= 0.005, \lambda_{r_1} = 0.005, \\
\lambda_{m_2} &= 0.008, \mu_1 = 0.005, \mu_2 = 0.008, \\
\theta_a &= 0.05, \quad \theta_n = 0.05
\end{align*}
\]

\[
\lambda_{r_1} = 0.009
\]

\[
\lambda_{r_1} = 0.006
\]

\[
\lambda_{r_1} = 0.003
\]

**Fig B.7** State 3 Plots for Model IV

\[
\begin{align*}
\lambda_{r_1} &= 0.003, \lambda_{m_1} = 0.005, \\
\lambda_{m_2} &= 0.008, \mu_1 = 0.005, \mu_2 = 0.008, \\
\theta_a &= 0.05, \quad \theta_n = 0.05
\end{align*}
\]

\[
\begin{align*}
\lambda_{r_2} &= 0.005 \\
\lambda_{r_2} &= 0.009 \\
\lambda_{r_2} &= 0.015
\end{align*}
\]
\[
\lambda_{r1} = 0.003, \lambda_{r2} = 0.005, \lambda_{mr1} = 0.005, \\
\mu_1 = 0.005, \mu_2 = 0.008, \theta_n = 0.05, \\
\theta_n = 0.05
\]

\[
\lambda_{mt2} = 0.016 \\
\lambda_{mt2} = 0.012 \\
\lambda_{mt2} = 0.008
\]

*Fig B.8 State 4 Plots for Model IV*

\[
\lambda_{r1} = 0.003, \lambda_{mr1} = 0.005, \\
\lambda_{mt2} = 0.008, \mu_1 = 0.005, \mu_2 = 0.008, \\
\theta_n = 0.05, \theta_n = 0.05
\]

\[
\lambda_{r2} = 0.01 \\
\lambda_{r2} = 0.008 \\
\lambda_{r2} = 0.005
\]

*Fig B.9 State 5 Plots for Model IV*
Model V

Model Analysis

With the aid of Markov method, the system of differential equations associated with Fig. 4.7 is as follows:

\[
\frac{dP_0(t)}{dt} = \mu P_2(t) - (\lambda_1 + \theta + \lambda_2) P_0(t) \tag{1}
\]

\[
\frac{dP_1(t)}{dt} = \lambda_3 P_2(t) + \lambda_1 P_0(t) \tag{2}
\]

\[
\frac{dP_2(t)}{dt} = (\mu + \lambda_3) P_2(t) + \lambda_2 P_0(t) \tag{3}
\]

\[
\frac{dP_3(t)}{dt} = \theta P_0(t) \tag{4}
\]

Laplace Transforms

At time \( t = 0 \),

\[
P_0(0) = 1, P_1(0) = 0, P_2(0) = 0, P_3(0) = 0.
\]

\[
s P_0(s) - 1 = \mu P_2(s) - (\lambda_1 + \theta + \lambda_2) P_0(s) \tag{5}
\]

\[
s P_1(s) = \lambda_3 P_2(s) + \lambda_1 P_0(s) \tag{6}
\]

\[
s P_2(s) = (\mu + \lambda_3) P_2(s) + \lambda_2 P_0(s) \tag{7}
\]
\[ s \, P_3(s) = \theta \, P_0(s) \] 

**Solving for \( P_0(s) \)**

Re-arranging Equation (5), and substituting the value of \( P_2(s) \) from equation (7) we get

\[
P_0(s) = \frac{s + B}{(s + A)(s + B) - \mu \lambda_2} \tag{9}
\]

where

\[ A = \lambda_1 + \theta + \lambda_2 \]

\[ B = \mu + \lambda_3 \]

expanding the denominator of the equation (9) and separating the coefficients of \( s \), we get the following equation

\[
P_0(s) = \frac{s + B}{s^2 + s \, B + A \, s + A \, B - \mu \lambda_2} \tag{10}
\]

Applying Heaviside’s rule for partial fraction to solve equation (10), we get

\[
P_0(s) = -\frac{B + x}{(x - y)(-s + x)} + \frac{y + B}{(x - y)(-s + y)} \tag{11}
\]
where

\[ x, y \] are the roots of the Laplace transform variable \( s \).

Computer software Maple V is used to obtain the roots of the above equations and the inverse laplace transform of Equations (12), (12), () and (15).

\[
x = -\frac{1}{2} B - \frac{1}{2} A + \frac{1}{2} \sqrt{B^2 - 2AB + A^2 + 4\mu \lambda_2}
\]

\[
y = -\frac{1}{2} B - \frac{1}{2} A - \frac{1}{2} \sqrt{B^2 - 2AB + A^2 + 4\mu \lambda_2}
\]

**Inverse Laplace**

Taking the inverse laplace of the above equation (11) we get the equation for state probability \( P_0 (s) \)

\[
P_0(t) = \frac{(-\lambda_3 - y - \mu) e^{(yt)} + (\lambda_3 + \mu + x) e^{(xt)}}{x - y}
\]  \hspace{1cm} (12)

**Solving for \( P_1 (s) \)**

Re-arranging Equation (6), and substituting the value of \( P_2 (s) \) from equation (7) and \( P_0 (s) \) from equation (10),we get

\[
P_1 (s) = \frac{\lambda_2}{(s + A) (s + B) - \mu \lambda_2}
\]  \hspace{1cm} (13)
Applying Heaviside's rule for partial fraction to solve equation (13), we get

\[ P_1(s) := \frac{\lambda_3 \lambda_2 + \lambda_1 B}{x y s} - \frac{\lambda_1 x + \lambda_1 B + \lambda_3 \lambda_2}{x (x - y) (-s + x)} + \frac{\lambda_1 B + \lambda_3 \lambda_2 + y \lambda_1}{y (x - y) (-s + y)} \]  \hspace{1cm} (14)

**Inverse Laplace**

Taking the inverse laplace of the above equation (14) we get the equation for state probability \( P_1(s) \)

\[ P_1(t) = \frac{(-y \lambda_1 - \lambda_1 \mu - \lambda_1 \lambda_3 - \lambda_3 \lambda_2)}{(x - y) y} e^{(x t)} + \frac{(\lambda_1 x + \lambda_1 \mu + \lambda_1 \lambda_3 + \lambda_3 \lambda_2)}{x (x - y)} e^{(x t)} + \frac{\lambda_3 \lambda_2 + \lambda_1 \mu + \lambda_1 \lambda_3}{x y} \]  \hspace{1cm} (15)

**Solving for \( P_2(s) \)**

Re-arranging Equation (7), and substituting the value of \( P_0(s) \) from equation (10), we get

\[ P_2(s) = \frac{\lambda_2}{s^2 + s B + A s + A B - \mu \lambda_2} \]  \hspace{1cm} (16)
Applying Heaviside’s rule for partial fraction to solve equation (16), we get

\[ P_2(s) := -\frac{\lambda_2}{(x-y)(-s+x)} + \frac{\lambda_2}{(x-y)(-s+y)} \]  

(17)

**Inverse Laplace**

Taking the inverse laplace of the above equation (17) we get the equation for state probability \( P_2(s) \)

\[ P_2(t) = \frac{\lambda_2 \left( -e^{(y)} + e^{(x)} \right)}{x-y} \]  

(18)

**Solving for \( P_3(s) \)**

Re-arranging Equation (8), and substituting the value of \( P_0(s) \) from equation (10), we get

\[ P_3(s) = \frac{\theta (s+B)}{(s^2 + s(B + A) + AB - \mu \lambda_2)s} \]  

(19)

Applying Heaviside’s rule for partial fraction to solve equation (19), we get

\[ P_3(s) := \frac{B \theta}{xy s} - \frac{\theta (B + x)}{x(x-y)(-s+x)} + \frac{\theta (y+B)}{y(x-y)(-s+y)} \]  

(20)
**Inverse Laplace**

Taking the inverse laplace of the above equation (20) we get the equation for state probability $P_3(s)$

$$P_3(t) = \theta \left( \frac{(-\lambda_3 - y - \mu) e^{(yt)}}{(x - y) y} + \frac{(-\lambda_3 + \mu + x) e^{(xt)}}{x (x - y)} + \frac{\mu + \lambda_3}{x y} \right)$$

(21)

Plots of Equations (12), (18) and (21) are shown in the Fig (B.10) –(B.12)

**Fig B.10** Probability plot for State 0
$\lambda_1 = 0.003, \lambda_2 = 0.02, \mu = 0.03, \theta = 0.04$

Fig B.11 Probability plot for State 2
$\lambda_1 = 0.003, \lambda_2 = 0.02, \lambda_2 = 0.03, \mu = 0.03,$

$\theta = 0.09$

$\theta = 0.07$

$\theta = 0.04$

Fig B.12  Probability plot for State 3