TOWARDS AN UNDERSTANDING OF GIRARD’S
TRANSCENDENTAL SYNTAX: SYNTAX BY TESTING

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Abstract

Through his work in ludics and Geometry of Interaction, Jean-Yves Girard invites us to a change of paradigm in the study of logic: the quest for a transcendental syntax, some kind of idealized language that emerges from the rules of logic. Amongst these rules, “testing” plays a leading role in defining a duality for the interpretation of negation.

The present work focuses on a notion of polarity which is a central technique used throughout Girard’s work to express linear negation. We describe some properties and illustrate them with examples with the purpose of getting acquainted with the technique. We also highlight how the classical connectives (conjunction and disjunction) arise from an interpretation based on testing. In a sense, this work is intended to provide an alternative introduction to Girard’s ideas and we hope it can have some pedagogical value.
To Philip Scott, Pieter Hofstra and Richard Blute,
for your source of inspiration and your dedicated work,
and for making this project possible;

À mes élèves qui, sans le savoir,
m’ont permis de vivre un certain idéal;

A Mariana, mi ángel trascendental.
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Chapter 1

Introduction

1.1 Testing in logic

“Logic is the study of reasoning” [32], and in the syntactical tradition, it is concerned with the possibilities of analyzing formulas, or propositions, within a formal system. Semantics gives a meaning to syntax and is often viewed as a way to test if a proposition can be formulated in the system; said differently, it evaluates the validity of statements in a language. Although syntax and semantics coexist, it is tempting to adopt an essentialist philosophical viewpoint\(^1\) and think of semantics as pre-existing syntax. For example, we may believe some absolute truth is inherent to our world, and formal systems (syntax) can be used to express what is true. In mathematics, we can think of the standard model \(\mathbb{N}\) as a way to test what propositions in arithmetics are valid, and any theory derived from a set of axioms (e.g. Peano’s) as a way to capture arithmetical truths. But this hypothesis of transparency [19] between syntax and semantics has been refuted, according to Girard, by Gödel’s incompleteness theorem.

Girard then proposed a change of paradigm in logic: instead of trying to validate or

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\(^1\)In philosophy, essentialism refers to the conception that there exists an abstract world of pure essences or ideas (in the sense of Plato), eternal and unchangeable, that give a meaning to existence (e.g. of a human being). By opposition, existentialism postulates that the (human) being is defined by its own actions and not by a predetermined essence.
evaluate statements, he suggested we study the *sufficient conditions*\(^2\) allowing the emergence of logical languages. In order to do so, we need a *transcendental syntax*\(^3\), e.g. *ludics*, *proof-nets* or *Geometry of Interaction*, where *testing* comes from *within* the syntax itself\(^4\).

Our starting point is to identify a proposition \(A\) with a set \(A\), where we can think of the elements as *observations, explanations, models, proofs*, etc., for the proposition. These elements are determined by a *testing process*: \(a \in A\) iff \(a\) successfully passes a set of *tests* for \(A\), noted \(Test(A)\). A useful metaphor here is to think of \(A\) as a set of strategies (*proofs*) for a “player-prover” to win a *game*, *i.e.* prove the proposition \(A\), and \(Test(A)\) as a set of counter-strategies (*counter-models*) for a “player-refuter” to prohibit the win, *i.e.* exhibit a model that invalidates \(A\). However, notice that this *duality* prover/refuter is somewhat *asymmetric*: the “burden of proof” rests upon the shoulders of the prover. Following Girard, we will ask that the duality “elements of \(A\) / tests for \(A\)” be *monist*, *i.e.* that elements and tests belong to the same *universe* \(E\). Also, we will ask that the testing process be *symmetric*, *i.e.* that tests for \(A\) are simultaneously tested by the elements of \(A\). Informally, we will have that

\[
A = \{a \in E \mid a \text{ passed all tests } b \in Test(A)\}
\]

and since by symmetry the elements of \(A\) are the tests for \(Test(A)\), we have

\[
A = Test(Test(A))
\]

In the analogy of a game prover *vs* refuter, if we let the strategies consist of *formal proofs* of \(A\) against *formal proofs* of *not* \(A\), we see that \(Test(A)\) corresponds to the

\(^2\)“*conditions de possibilité*” ([20], pp.1&4), which is inspired by Kant’s *transcendental* conditions, *i.e.* the hypotheses that make possible the coherence of perceptions ([21], p. 7).

\(^3\)The expression is the analogous of Kant’s *transcendental subject*, which accounts for the structure of knowledge ([20], p. 3).

\(^4\)Girard calls for “an *autonomous* approach to syntax” ([21], p. 2).
proofs of the *negation* of $A$. Therefore, in Girard’s transcendental syntax, a *proposition*, also called a “dichology”, is a set of elements, the “epistates”\(^5\), equal to its double negation; in other words, *negation* is an *involutive* testing process.

### 1.2 Overview of thesis

The present work is intended to provide an introduction to Girard’s ideas on the basis of *testing*. In Chapter 2, we define the *technique* used for testing, and discuss some properties and basic examples. In Chapter 3, we look at particular examples in logic: Section 3.1 exemplifies different dualities “models vs counter-models”, while Section 3.2 illustrates the idea in a categorical interpretation. Chapter 4 gives a glimpse of different examples of Girard’s transcendental syntax.

My main contribution to the subject would be to have brought together examples and properties, mostly from Girard’s work, under the central theme of *negation by testing*. In particular, the notion of *duality monomorphism* used in the interpretation of the classical connectives (see Section 3.2.1) allows for a different definition of the coherence spaces “$A \& B$” and “$A \oplus B$” (see Propositions 3.2.3 and 3.2.4), which highlights the common underlying structure used in other examples (see Section 3.1.1, 3.1.3 and Example A.0.4). I also personally contributed proofs of some propositions; these can be viewed as exercises for explaining the notion of *polarity* (*e.g.* Propositions 2.1.23, 2.1.24, 2.4.27, 2.4.17 and 2.5.5). In addition, I provided examples (*e.g.* Examples 3.1.21 and A.0.4) to illustrate some ideas, notably the use of Girard’s *locative product*. Finally, the views expressed here are my own interpretation of Girard’s work and may not represent his current views. Also, the appendices are *new* and represent different directions for future projects.

\(^5\)Epistates refers to some kind of judge or magistrate in Ancient Greece; as for dichology, it seems to be a *neologism* that could mean “discourse in two parts”.
Chapter 2

Polarity

Given a real vector space \((V, +, \cdot)\), we know that \(v, w \in V\) are orthogonal if 
\[
\langle v | w \rangle = 0,
\]
where \(\langle - | - \rangle : V^2 \to \mathbb{R}\) is the scalar product [33]. It is also common usage to write \(v \perp w\) to express the orthogonality of \(v\) and \(w\). Moreover, given \(W \subseteq V\), the set of elements of \(V\) that are orthogonal to every element of \(W\) is called the orthogonal of \(W\), and we denote it by \(W^\perp\).

In the following section, we will generalize this notion of orthogonality and call it polarity (or duality). For convenience, the notation used is similar to the one used in linear algebra. We will write \(x \perp y\) to express the polarity of \(x\) and \(y\), and \(\ll - | - \gg\) for the generalized “scalar product”.

2.1 General definitions and properties

Let \(E\) and \(F\) be arbitrary sets. Consider a set \(R\), called the scalar space [18], such that \(R = \emptyset \iff E = \emptyset\) or \(F = \emptyset\).

Given a binary function
\[
\ll - | - \gg : E \times F \to R
\]
with a distinguished subset $P \subseteq R$ called the pole, we define a duality relation $\perp \subseteq E \times F$ as follows:

$$x \perp y \overset{\text{def}}{\iff} \ll x \mid y \gg \in P$$

If $x \perp y$, we say that $x$ and $y$ are polar, or alternatively that they are dual (or in duality).

**Remark 2.1.1**

In some examples encountered in the present work, the duality relation $\perp \subseteq E \times F$ is not initially defined in terms of a binary function and a pole as above. In such situations, we let $\ll - \mid - \gg$ be the characteristic function in the following sense:

$$\ll - \mid - \gg : E \times F \rightarrow R = \{0, 1\}$$

$$(x, y) \mapsto \begin{cases} 
1 & \text{if } x \perp y \\
0 & \text{else}
\end{cases}$$

and let the pole be $P = \{1\}$.

**Definition 2.1.2 (Polar set)**

Given a duality relation $\perp \subseteq E \times F$, we define a function $Pol_E^r : \mathcal{P}(E) \rightarrow \mathcal{P}(F)$, where $\mathcal{P}(E)$ is the powerset of $E$, such that

$$Pol_E^r (X) \overset{\text{def}}{=} \{y \in F \mid \forall x \in X, x \perp y\}$$

for $X \subseteq E$.

Similarly, define a function $Pol_F^l : \mathcal{P}(F) \rightarrow \mathcal{P}(E)$

$$Pol_F^l (Y) \overset{\text{def}}{=} \{x \in E \mid \forall y \in Y, x \perp y\}$$

for $Y \subseteq F$. 
Pol\(_r\)(X) and Pol\(_F\)(Y) will be referred to as the \textit{polar sets} of X and Y respectively. We also say that \(X \subseteq E\) is a \textit{polar set} of Y if \(X = Pol\(_F\)(Y)\), and that X is a \textit{polar set} if X is a polar set of Y, for some \(Y \subseteq F\). If \(X = Pol\(_F\)(Y)\), Y may be called the \textit{pre-polar} of X (see [17]).

We can introduce the notation \(X \not\perp y\) to mean \(\forall x \in X,\ x \not\perp y\). In a similar way, \(X \not\perp Y\) means that for all \(x \in X\) and \(y \in Y\), \(x \not\perp y\), and we say that X and Y are \textit{polar}.

**Definition 2.1.3 (Duality)**

A \textit{duality} (or \textit{polarity}) between \(E\) and \(F\) consists of a map \(\ll - \mid - \gg : E \times F \rightarrow R\), a pole \(P \subseteq R\) and functions \(Pol\(_r\)\) and \(Pol\(_F\)\). If \(E = F\), we call it an \textit{R-duality on} E and we ask that \(Pol\(_r\) = Pol\(_F\)\). If \(E = F = R\), we simply say it is a \textit{duality on} E.

From now on, we assume a duality between \(E\) and \(F\), if \(E,F \neq \emptyset\).

**Remark 2.1.4**

If \(E\) or \(F\) is the empty set, then \(\ll - \mid - \gg : E \times F \rightarrow R\) is the \textit{empty map}, \textit{i.e.} \(\text{dom}(\ll - \mid - \gg) = \text{Im}(\ll - \mid - \gg) = \emptyset\). If \(E = \emptyset\), then

\[
Pol\(_F\)(\emptyset) = \{y \in F \mid \forall x \in \emptyset,\ \ll x \mid y \gg \in P\} = F
\]

and for any \(Y \subseteq F\), \(Pol\(_F\)(Y) = \emptyset\).

We can now take a look at some basic properties:

**Proposition 2.1.5** Given a duality between sets \(E\) and \(F\), we have:

(i) \(E\) and \(F\) are polar sets;

(ii) \(X \subseteq Pol\(_F\)(Pol\(_r\)(X))\) for all \(X \subseteq E\), and \(Y \subseteq Pol\(_F\)(Pol\(_r\)(Y))\) for all \(Y \subseteq F\);

(iii) \(X \subseteq X' \Rightarrow Pol\(_r\)(X') \subseteq Pol\(_r\)(X)\) for all \(X, X' \subseteq E\), and \(Y \subseteq Y' \Rightarrow Pol\(_F\)(Y') \subseteq Pol\(_F\)(Y)\) for all \(Y, Y' \subseteq F\).
**Chapter 2. Polarity**

*Proof*: Let \( \perp \subseteq E \times F \) be an arbitrary duality relation with respect to some pole \( P \subseteq \mathbb{R} \) and function \( \ll \mid \rightarrow : E \times F \rightarrow \mathbb{R} \).

For (i): We have \( \text{Pol}_E^\ell (\emptyset) \overset{\text{def}}{=} \{ x \in E \mid \forall y \in \emptyset, \ll x \mid y \gg \in P \} = E \).

Similarly, \( \text{Pol}_E^r (\emptyset) = F \) (cf. also Remark 2.1.4).

For (ii): Let \( x \in X \subseteq E \). Then for all \( y \in \text{Pol}_E^r (X) \), \( x \perp y \). Therefore \( x \in \text{Pol}_E^r (\text{Pol}_E^r (X)) \). Similarly, \( y \in Y \Rightarrow y \in \text{Pol}_E^r (\text{Pol}_E^r (Y)) \) for all \( Y \subseteq F \).

For (iii): Suppose \( X \subseteq X' \) with \( X, X' \subseteq E \). If \( y \in \text{Pol}_E^r (X') \), then for all \( x \in X' \), \( x \perp y \) (by definition). In particular, \( x \perp y \) for all \( x \in X \) (by hypothesis). Hence \( y \in \text{Pol}_E^r (X) \). Again, similar reasoning shows that \( Y \subseteq Y' \Rightarrow \text{Pol}_F^r (Y') \subseteq \text{Pol}_F^r (Y) \) for all \( Y, Y' \subseteq F \). \( \square \)

**Corollary 2.1.6** For the situation above, \( \text{Pol}_E^r (X) = \text{Pol}_E^r (\text{Pol}_E^r (\text{Pol}_E^r (X))) \) for all \( X \subseteq E \) and \( \text{Pol}_F^r (Y) = \text{Pol}_F^r (\text{Pol}_F^r (\text{Pol}_F^r (Y))) \) for all \( Y \subseteq F \).

*Proof*: Using Proposition 2.1.5 (ii), we obviously have that \( \text{Pol}_E^r (X) \subseteq \text{Pol}_E^r (\text{Pol}_E^r (\text{Pol}_E^r (X))) \) for all \( X \subseteq E \). By (ii) again, we have \( X \subseteq \text{Pol}_E^r (\text{Pol}_E^r (X)) \), and by (iii), this implies that \( \text{Pol}_E^r (\text{Pol}_E^r (\text{Pol}_E^r (X))) \subseteq \text{Pol}_E^r (X) \). Analogous reasoning shows the other equality. \( \square \)

We can make some immediate observations. In the following, suppose there is a duality between \( E \) and \( F \).

Let \( \mathcal{E} = (\mathcal{P}(E), \subseteq) \) and \( \mathcal{F} = (\mathcal{P}(F), \subseteq) \) be the usual partially ordered powersets, considered as categories.

We can define a functor from \( \mathcal{E} \) to \( \mathcal{F} \) such that on objects, it maps \( X \subseteq E \) to its polar set \( \text{Pol}_E^r (X) \). On arrows, by Proposition 2.1.5 (iii), we know it maps \( X \rightarrow X' \) to \( \text{Pol}_E^r (X') \rightarrow \text{Pol}_E^r (X) \), so it is contravariant, i.e. it is a (covariant) functor from \( \mathcal{E} \) to \( \mathcal{F}^{\text{op}} = (\mathcal{P}(F), \supseteq) \). We can also define an analogous functor from \( \mathcal{F}^{\text{op}} \) to \( \mathcal{E} \) that assigns \( Y \in \mathcal{P}(F) \) to \( \text{Pol}_F^r (Y) \). Such a situation is called a polarity in [29] (p.13).
Definition 2.1.7 (Closure operation)

Let $\mathcal{C} = (C, \preceq)$ be a preordered set. A map $cl : C \to C$ is called a closure operation (see [29], p.12) if it satisfies, for all $a, b \in C$:

(i) $a \preceq cl(a)$ (we say $cl$ is extensive)

(ii) $a \preceq b \Rightarrow cl(a) \preceq cl(b)$ (we say $cl$ is increasing)

(iii) $cl(cl(a)) \preceq cl(a)$

Remark 2.1.8

From (i), (ii) and (iii), we see that the map $cl$ is idempotent, i.e. $cl(cl(a)) = cl(a)$.

Proposition 2.1.9 The map $Pol^r_F \circ Pol^l_E : \mathcal{P}(E) \to \mathcal{P}(E)$ is a closure operation and similarly for $Pol^l_F \circ Pol^r_E : \mathcal{P}(F) \to \mathcal{P}(F)$.

Proof: Let $X, X' \subseteq E$. Then $X \subseteq Pol^l_F(Pol^r_E(X))$ and $X \subseteq X' \Rightarrow Pol^r_E(X') \subseteq Pol^l_F(Pol^r_E(X)) \subseteq Pol^l_F(Pol^r_E(X'))$ by Proposition 2.1.5 (ii) and (iii), so the map $Pol^l_F \circ Pol^r_E$ is extensive and increasing. By Corollary 2.1.6, we have $Pol^r_E(X) = Pol^r_E(Pol^l_F(Pol^r_E(X)))$. Using Proposition 2.1.5 (iii), we then get $Pol^l_F(Pol^r_E(Pol^l_F(Pol^r_E(X)))) \subseteq Pol^l_F(Pol^r_E(X))$. Similar reasoning shows that $Pol^r_E \circ Pol^l_F$ is a closure operation.

Proposition 2.1.10 $Pol^r_E : \mathcal{E} \to \mathcal{F}^{op}$ is left adjoint to $Pol^l_F : \mathcal{F}^{op} \to \mathcal{E}$. That is: $\mathcal{F}^{op}(Pol^r_E(X), Y) \cong \mathcal{E}(X, Pol^l_F(Y))$. This means $Pol^r_E(X) \supseteq Y$ iff $X \subseteq Pol^l_F(Y)$.

Proof: Suppose we have a morphism $Pol^r_E(X) \to Y$ in $\mathcal{F}^{op}$ for some $X \in \mathcal{P}(E)$ and $Y \in \mathcal{P}(F)$. Set theoretically, this simply means $Y \subseteq Pol^r_E(X)$. Hence for all $y \in Y$, we have $y \in Pol^r_E(X)$, i.e. $X \perp y$. Therefore $Pol^l_F(Y) \overset{def}{=} \{ x \in E \mid \forall y \in Y, x \perp y \}$ contains $X$, and we have a morphism $X \to Pol^l_F(Y)$ in $\mathcal{E}$. Conversely, in a similar way, if we have a morphism $X \to Pol^l_F(Y)$ in $\mathcal{E}$, then $X \subseteq Pol^l_F(Y)$, so $X \perp Y$. Clearly $Pol^r_E(X) \supseteq Y$, and we have an arrow $Pol^r_E(X) \to Y$ in $\mathcal{F}^{op}$.
Remark 2.1.11
Saying that \( \text{Pol}_E^r \) and \( \text{Pol}_F^l \) are adjoint functors between poset categories \( \mathcal{E} = (\mathcal{P}(E), \subseteq) \) and \( \mathcal{F}^{op} = (\mathcal{P}(F), \supseteq) \) is equivalent to saying that if either \( X \subseteq \text{Pol}_F^l(Y) \) or \( Y \subseteq \text{Pol}_E^r(X) \), then \( X \) and \( Y \) are polar, i.e. \( X \perp Y \).

Definition 2.1.12 (Fact)
We say that \( X \subseteq E \) is a biorthogonally closed set, or a fact, when
\[
X = \text{Pol}_F^l(\text{Pol}_E^r(X))
\]
Similarly, \( Y \subseteq F \) is a fact when \( Y = \text{Pol}_E^r(\text{Pol}_F^l(Y)) \).

Here are some properties about facts:

Proposition 2.1.13 Given a duality between sets \( E \) and \( F \):

(i) polar sets are exactly the facts;

(ii) there is a one-to-one correspondence between the set of facts of \( E \) and the set of facts of \( F \).

Proof: For (i) : Let \( X \subseteq E \) be a fact. Then \( X \) is the polar set of \( Y = \text{Pol}_E^r(X) \). Conversely, suppose \( X \subseteq E \) is a polar set, i.e. \( X = \text{Pol}_F^l(Y) \) for some \( Y \subseteq F \). By Corollary 2.1.6, we have \( \text{Pol}_F^l(Y) = \text{Pol}_F^l(\text{Pol}_F^r(\text{Pol}_E^r(X))) \), hence \( X = \text{Pol}_F^l(\text{Pol}_E^r(X)) \).

For (ii) : \( \text{Pol}_E^r \) is an injective map from the facts of \( E \) to the facts of \( F \). Indeed, suppose \( \text{Pol}_E^r(X) = \text{Pol}_E^r(X') \) for facts \( X, X' \subseteq E \). Then \( \text{Pol}_F^l(\text{Pol}_E^r(X)) = \text{Pol}_F^l(\text{Pol}_E^r(X')) \), which implies \( X = X' \) by definition. Analogously, \( \text{Pol}_F^l \) is an injection from the facts of \( F \) to the facts of \( E \), hence \( |E| = |F| \) by Schröder-Bernstein’s theorem [23]. \( \square \)
Now let’s consider the (bounded) lattices \( \mathcal{E} = (\mathcal{P}(E), \subseteq, \cap, \cup) \) and \( \mathcal{F} = (\mathcal{P}(F), \subseteq, \cap, \cup) \).

**Proposition 2.1.14** Given a duality between sets \( E \) and \( F \), we have, for all \( X_i \subseteq E \) and \( Y_i \subseteq F \):

(i) \( \bigcap_{i \in I} Pol^r_E (X_i) = Pol^r_E \left( \bigcup_{i \in I} X_i \right) \) and \( \bigcap_{i \in I} Pol^l_F (Y_i) = Pol^l_F \left( \bigcup_{i \in I} Y_i \right) \);

(ii) \( \bigcup_{i \in I} Pol^r_E (X_i) \subseteq Pol^r_E \left( \bigcap_{i \in I} X_i \right) \) and \( \bigcup_{i \in I} Pol^l_F (Y_i) \subseteq Pol^l_F \left( \bigcap_{i \in I} Y_i \right) \).

**Proof**: Let \( \bot \) be an arbitrary duality relation with respect to some pole \( P \).

(i) \[
y \in \bigcap_{i \in I} Pol^r_E (X_i) \iff \text{ for all } i \in I, y \in Pol^r_E (X_i) \\
\quad \iff \text{ for all } i \in I \text{ and for all } x \in X_i, \langle x, y \rangle \in P \\
\quad \iff \forall x \in \bigcup_{i \in I} X_i, \langle x, y \rangle \in P \\
\quad \iff y \in Pol^r_E \left( \bigcup_{i \in I} X_i \right)
\]

Similarly, \( x \in \bigcap_{i \in I} Pol^l_F (Y_i) \iff x \in Pol^l_F \left( \bigcup_{i \in I} Y_i \right) \) for all \( Y_i \subseteq F \).

(ii) Suppose \( \bigcap_{i \in I} X_i \neq \emptyset \) (otherwise, it is done).
Let \( y \in \bigcup_{i \in I} Pol^r_E (X_i) \). Then \( y \in Pol^r_E (X_i) \) for some \( i, i.e. X_i \perp y \). In particular, for all \( x \in \bigcap_{i \in I} X_i \), \( x \perp y \). Hence \( y \in Pol^r_E \left( \bigcap_{i \in I} X_i \right) \).

**Remark 2.1.15**

Proposition 2.1.14 simply tells us that the set of facts is closed under arbitrary intersection, but it may not be closed under union in general.
Proposition 2.1.16 Given a duality between sets $E$ and $F$, $\text{Pol}_F^r (\text{Pol}_E^r (X))$ is the smallest fact that contains $X$, for $X \subseteq E$; similarly, $\text{Pol}_E^r (\text{Pol}_F^r (Y))$ is the smallest fact that contains $Y$, for $Y \subseteq F$.

Proof: Let $S = \bigcap_{i \in I} X_i$, where $\{X_i \subseteq E\}$ is the set of all facts containing $X$.

By Proposition 2.1.13 (i), we can write $X_i = \text{Pol}_F^r (Y_i)$ for some $Y_i \subseteq F$, so $S = \bigcap_{i \in I} \text{Pol}_F^r (Y_i) = \text{Pol}_E^r (\bigcup_{i \in I} Y_i)$ by Proposition 2.1.14 (i).

This means $S$ is a polar set, hence a fact (containing $X$), and it is the smallest. We will show $S = \text{Pol}_F^l (\text{Pol}_E^r (X))$:

Clearly $S \subseteq \text{Pol}_F^l (\text{Pol}_E^r (X))$, since $\text{Pol}_F^l (\text{Pol}_E^r (X))$ is a fact containing $X$.

So by Proposition 2.1.5 (iii), we have

$$X \subseteq S \subseteq \text{Pol}_F^l (\text{Pol}_E^r (X))$$

$$\Rightarrow \text{Pol}_E^r (\text{Pol}_F^l (\text{Pol}_E^r (X))) \subseteq \text{Pol}_E^r (S) \subseteq \text{Pol}_E^r (X)$$

But $\text{Pol}_E^r (X) = \text{Pol}_E^r (\text{Pol}_F^l (\text{Pol}_E^r (X)))$ by Corollary 2.1.6, so $\text{Pol}_E^r (X) = \text{Pol}_E^r (S)$, and consequently $\text{Pol}_F^l (\text{Pol}_E^r (X)) = \text{Pol}_F^l (\text{Pol}_E^r (S)) = S$. 

Now by Remark 2.1.15, we may wonder under what conditions the union of facts is a fact.

Let $X, X' \subseteq E$ be facts. By Proposition 2.1.13 (ii), we know the map $\text{Pol}_E^r$ sets up a bijection between the facts of $E$ and the facts of $F$, so let $Y = \text{Pol}_E^r (X)$ and $Y' = \text{Pol}_E^r (X')$ be the corresponding facts. Then we have the following:

Proposition 2.1.17 $X \cup X'$ is a fact iff $\text{Pol}_F^l (Y) \cup \text{Pol}_F^l (Y') = \text{Pol}_F^l (Y \cap Y')$.

Proof:

$$X \cup X' = \text{Pol}_F^l (\text{Pol}_E^r (X \cup X'))$$

$$\overset{\text{Prop.} 2.1.14}{\iff} X \cup X' = \text{Pol}_F^l (\text{Pol}_E^r (X) \cap \text{Pol}_E^r (X'))$$

$$\iff \text{Pol}_F^l (Y) \cup \text{Pol}_F^l (Y') = \text{Pol}_F^l (\text{Pol}_E^r (Y) \cap \text{Pol}_E^r (\text{Pol}_F^l (Y'))$$

$$\iff \text{Pol}_F^l (Y) \cup \text{Pol}_F^l (Y') = \text{Pol}_F^l (Y \cap Y')$$

\qed
We may now wonder what are the conditions for every subset of $E$ and $F$ to be a fact. We already know that $|E| = |F|$ is a necessary condition by Proposition 2.1.13 (ii), but it is not sufficient.

Here is an adapted version of a criterion which was mentioned in a more general framework by Wright [35]. But first a lemma:

**Lemma 2.1.18** Let $\alpha : \mathcal{P}(E) \rightarrow \mathcal{P}(F)$ be a homomorphism of the usual Boolean algebras $\mathcal{P}(E)$ and $\mathcal{P}(F)$, i.e. $\alpha(\emptyset) = \emptyset$, $\alpha(E) = F$, $\alpha(X \cap X') = \alpha(X) \cap \alpha(X')$ and $\alpha(X \cup X') = \alpha(X) \cup \alpha(X')$ for all $X, X' \subseteq E$.

Then, for all $X \subseteq E$,

$$\overline{\alpha(X)} = \alpha(\overline{X})$$

where $\overline{X}$ is the complement of $X$.

**Proof**: $\alpha(\emptyset) = \alpha(X \cap \overline{X}) = \alpha(X) \cap \alpha(\overline{X}) = \emptyset$ and $\alpha(E) = \alpha(X \cup \overline{X}) = \alpha(X) \cup \alpha(\overline{X}) = F$.

**Proposition 2.1.19** Suppose we have a duality between sets $E$ and $F$. Then every $X \subseteq E$ and $Y \subseteq F$ is a fact if and only if the maps

$$\alpha : \mathcal{P}(E) \rightarrow \mathcal{P}(F) \quad X \mapsto Pol^r_E(X)$$

and

$$\beta : \mathcal{P}(F) \rightarrow \mathcal{P}(E) \quad Y \mapsto Pol^l_F(Y)$$

are reciprocal isomorphisms of the usual Boolean algebras $\mathcal{P}(E)$ and $\mathcal{P}(F)$.

**Proof** (Proposition 2.1.19): For ($\Rightarrow$), we first show $\alpha$ is an isomorphism:
• For all $X, X' \subseteq E$

\[
\alpha(X) = \alpha(X') \Rightarrow \overline{\text{Pol}^r_E(X)} = \overline{\text{Pol}^r_E(X')}
\]
\[
\Rightarrow \text{Pol}^r_E(X) = \text{Pol}^r_E(X')
\]
\[
\Rightarrow \text{Pol}^\ell_F(\text{Pol}^r_E(X)) = \text{Pol}^\ell_F(\text{Pol}^r_E(X'))
\]
\[
\Rightarrow X = X'
\]

hence $\alpha$ is injective.

• for surjectivity: for all $Y \subseteq F$, $\alpha(\underbrace{\text{Pol}^\ell_F(\overline{Y})}_{\in E}) = \overline{\text{Pol}^r_E(\text{Pol}^\ell_F(\overline{Y}))} = \overline{Y} = Y$

• $\alpha(\emptyset) = \overline{\text{Pol}^r_E(\emptyset)} = \overline{F} = \emptyset$

• $\alpha(E) = \overline{\text{Pol}^r_E(E)} = \overline{E} = F$ (for $\emptyset \subseteq F$)

• for all $X, X' \subseteq E$,

\[
\alpha(X \cup X') = \overline{\text{Pol}^r_E(X \cup X')}
\]
\[
= \overline{\text{Pol}^r_E(X) \cap \text{Pol}^r_E(X')}
\]
\[
= \overline{\text{Pol}^r_E(X) \cup \text{Pol}^r_E(X')}
\]
\[
= \alpha(X) \cup \alpha(X')
\]

• for all $X, X' \subseteq E$,

\[
\alpha(X \cap X') = \overline{\text{Pol}^r_E(X \cap X')}
\]
\[
= \overline{\text{Pol}^r_E(X) \cup \text{Pol}^r_E(X')}
\]
\[
= \overline{\text{Pol}^r_E(X) \cap \text{Pol}^r_E(X')}
\]
\[
= \alpha(X) \cap \alpha(X')
\]
Now we show \((\beta \circ \alpha)(X) = X\) for all \(X \subseteq E\):

\[
x \in \beta(\alpha(X)) \iff x \in \beta \left( Pol_E^r (X) \right) \\
\iff x \notin \beta \left( Pol_E^r (X) \right) \text{ by Lemma 2.1.18} \\
\iff x \notin Pol_F^l (Pol_E^r (X)) \\
\iff x \notin X \\
\iff x \in X
\]

Similarly, \((\alpha \circ \beta)(Y) = Y\) for all \(Y \subseteq F\), and we have shown that \(\alpha\) and \(\beta\) are reciprocal, i.e. \(\beta = \alpha^{-1}\). Hence \(\beta\) is also an isomorphism.

For \((\Leftarrow)\): To show that all \(X \subseteq E\) are facts, it suffices to show that \(Pol^r_F (Pol^r_E (X)) \subseteq X\). Let \(x \in Pol^r_F (Pol^r_E (X)) = Pol^r_F \left( \alpha(X) \right) = \beta(\alpha(X))\). Then \(x \notin \beta(\alpha(X))\), hence \(x \in \beta(\alpha(X)) = X\). Similarly, \(Pol^r_E (Pol^r_F (Y)) \subseteq Y\) for all \(Y \subseteq F\).

In what follows, we will determine another criterion for every subset of \(E\) and \(F\) to be a fact which is based on the duality relation.

**Proposition 2.1.20** Suppose we have a duality between sets \(E\) and \(F\). Then

(i) if \(\triangleleft = \emptyset\), the only facts are \(\emptyset, E\) and \(F\);

(ii) if \(\triangleleft = E \times F\), the only facts are \(E\) and \(F\).

**Proof**: For (i): For all \(X \subseteq E\), if \(X \neq \emptyset\), we have \(Pol^r_E (X) = \{ y \in F \mid X \perp y \} = \emptyset\), and since \(Pol^l_F (\emptyset) = E\), this means \(E\) is a fact. Similarly, \(F\) is a fact.

For (ii): For all \(X \subseteq E\), \(Pol^r_E (X) = F\), and for all \(Y \subseteq F\), \(Pol^l_F (Y) = E\).

**Definition 2.1.21**

Let \(E, F\) be non-empty sets such that \(|E| = |F|\).

Let \(\varphi : E \rightarrow F\) be a bijection.

Then

\[
\triangleleft_\varphi \overset{\text{def}}{=} \{ (x, y) \in E \times F \mid y \neq \varphi(x) \}
\]
will be referred to as a classical duality relation.

Remark 2.1.22

Notice that if \(|E| = |F| \in \{0, 1\}\), then \(\preceq_\varphi = \emptyset\) and every subset of \(E\) and \(F\) is a fact. Also, the choice of \(\preceq_\varphi\) may not always be appropriate to define an \(R\)-duality on \(E\): we must ensure that \(\varphi^2 = Id\), so that we fulfill the condition that \(\preceq_\varphi\) is a symmetric relation (see Proposition 2.4.2).

Proposition 2.1.23 Given a duality between sets \(E\) and \(F\), suppose every subset of \(E\) and every subset of \(F\) is a fact. Then the duality relation is classical, i.e. of the form \(\preceq_\varphi\) for some bijection \(\varphi : E \rightarrow F\).

Proof: By Proposition 2.1.13 (ii), we know \(|E| = |F|\). If \(E = F = \emptyset\), then the duality relation is clearly classical. If \(|E| = \{a\}\) and \(|F| = \{\alpha\}\), the only possible duality relation is \(\preceq_\varphi = \emptyset\); also, the only bijection is \(\varphi(a) = \alpha\), hence \(\preceq_\varphi = \preceq\).

So suppose \(|E| = |F| \geq 2\). Since every subset is a fact by hypothesis, we must have \(Pol_E^r (Pol_F^r (\emptyset)) = \emptyset\) and \(Pol_F^l (Pol_E^r (\emptyset)) = \emptyset\). Since \(Pol_F^r (\emptyset) = E\) and \(Pol_F^l (\emptyset) = F\), we conclude that \(Pol_E^r (E) = \emptyset\) and \(Pol_F^l (F) = \emptyset\). Therefore, for all \(y \in F\), there exists \(x \in E\) such that \(\langle x | y \rangle \notin P\), and similarly for all \(x \in E\), there exists \(y \in F\) such that \(\langle x | y \rangle \notin P\). Now let \(x \in E\) and consider \(X = E \setminus \{x\}\). Since \(X\) is a fact, we cannot have \(Pol_E^l (X) = \emptyset\), so there is some \(y \in F\) such that \(X \preceq y\). And since \(X \subseteq Pol_F^r (\{y\}) \neq E\) (\(\{y\}\) is also a fact), we have \(Pol_E^r (X) = \{y\}\). Therefore \(x\) is the only element in \(E\) such that \(x \not\preceq y\). Moreover, if \(\mathcal{R} = \{E \setminus \{x\} \subseteq E | x \in E\}\) and \(\mathcal{S} = \{\{y\} \subseteq F | y \in F\}\), then the following maps are clearly bijective:

\[
\begin{align*}
f : E & \rightarrow \mathcal{R} \\
x & \mapsto E \setminus \{x\}
\end{align*}
\]

\[
Pol_E^r |_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{S} \\
E \setminus \{x\} & \mapsto Pol_E^r ((E \setminus \{x\})) = \{y\}
\]
Therefore the map \( \varphi = g \circ Pol^r_E \mid_R \circ f \) is also a bijection, and it is such that \( \varphi(x) = y \) iff \( x \not\sim y \). We conclude that \( \mathcal{L} = \mathcal{L}_{\varphi} \).

We can also state the converse:

**Proposition 2.1.24** Let \( |E| = |F| \) and suppose we have a duality between \( E \) and \( F \) with a classical duality relation \( \mathcal{L}_{\varphi} \), for some bijection \( \varphi : E \rightarrow F \).

Then every subset of \( E \) and \( F \) is a fact (with respect to \( \mathcal{L}_{\varphi} \)).

**Proof**: Obviously, if \( |E| \in \{0, 1\} \), then every subset is a fact by Remark 2.1.22.

So let \( |E| \geq 2 \). We already know \( E \) and \( F \) are facts by Proposition 2.1.5 (i). Now consider any set \( X \subseteq E \). It suffices to show that \( Pol^r_F (Pol^r_E (X)) \subseteq X \):

Since \( \mathcal{L}_{\varphi} \neq E \times F \) by definition, then \( Pol^r_F (Pol^r_E (\emptyset)) = Pol^r_F (F) = \emptyset \). So suppose \( X \neq \emptyset \). By hypothesis, we have

\[
Pol^r_E (X) \overset{\text{def}}{=} \{ y \in F \mid X \mathcal{L}_{\varphi} y \} = F\backslash\{ \varphi(x) \mid x \in X \}
\]

Also,

\[
Pol^r_F (Pol^r_E (X)) \overset{\text{def}}{=} \{ x \in E \mid x \mathcal{L}_{\varphi} Pol^r_E (X) \} = E\backslash\{ \varphi^{-1}(y) \mid y \in Pol^r_E (X) \}
\]

Therefore, if \( x \notin X \), then \( \varphi(x) \in Pol^r_E (X) \). This means \( \varphi^{-1}(\varphi(x)) = x \notin Pol^r_F (Pol^r_E (X)) \). \( \square \)

**Example 2.1.25 (Subsets as facts)**

It might be useful to illustrate Propositions 2.1.23 and 2.1.24.

Consider \( E = \{a, b, c\} \) and \( F = \{\alpha, \beta, \gamma\} \). Let \( \varphi : E \rightarrow F \) be such that \( \varphi(a) = \alpha \),
\( \varphi(b) = \beta \) and \( \varphi(c) = \gamma \). Consider a duality between \( E \) and \( F \) such that the duality relation is

\[
\mathcal{L}_\varphi = \{(a, \beta), (a, \gamma), (b, \alpha), (b, \gamma), (c, \alpha), (c, \beta)\}.
\]

This ensures every subset of \( E \) and \( F \) is a fact. Indeed, \( Pol_E^r(X) = Pol_F^e(Y) = \emptyset \) and the following facts are in correspondence (\( X \bowtie Y \) means \( Pol_E^r(X) = Y \) and \( Pol_F^e(Y) = X \)):

\[
\begin{align*}
\{a\} & \bowtie \{\beta, \gamma\} \\
\{b\} & \bowtie \{\alpha, \gamma\} \\
\{c\} & \bowtie \{\beta, \gamma\} \\
\{a, b\} & \bowtie \{\gamma\} \\
\{a, c\} & \bowtie \{\beta\} \\
\{b, c\} & \bowtie \{\alpha\}
\end{align*}
\]

### 2.2 Basic examples of duality between distinct sets \( E \) and \( F \)

In the last section, while looking at posets \((\mathcal{P}(E), \subseteq)\) and \((\mathcal{P}(F), \subseteq)\) with the extra structure given by polar sets, we recovered the notion of a Galois correspondence, which is sometimes referred to as an antitone or order-reversing Galois connection.

Let’s take a look at the original Galois correspondence [3] in the light of our setting:

**Example 2.2.1 (Galois groups and subfields as facts)**

Let \( E \) be a field, and \( F = Aut(E) \) be the set of all automorphisms of \( E \), hence a group with respect to composition. Let \( R = \{0, 1\} \), and \( P = \{1\} \).
Define the map $\ll - \mid - \gg : E \times Aut(E) \rightarrow R$ such that

$$(x, \sigma) \mapsto \begin{cases} 1 & \text{if } \sigma(x) = x \\ 0 & \text{else} \end{cases}$$

For any $X \subseteq E$, the polar set is given by

$$Pol_E^r(X) = \{ \sigma \in Aut(E) \mid X \perp \sigma \}$$
$$= \{ \sigma \in Aut(E) \mid \forall x \in X (\ll x \mid \sigma \gg \in P) \}$$
$$= \{ \sigma \in Aut(E) \mid \forall x \in X (\sigma(x) = x) \}$$

Dually, for $Y \subseteq Aut(E)$, the polar set is given by

$$Pol_F^r(Y) = \{ x \in E \mid x \perp Y \}$$

An immediate observation is the following:

**Proposition 2.2.2** If $X \subseteq E$ is a fact, then $X$ is a subfield of $E$.

**Proof**: Suppose $X \subseteq E$ is a fact. Then by Proposition 2.1.13, we can write $X = Pol_F^r(Y) = \{ x \in E \mid \forall \sigma \in Y, \sigma(x) = x \}$ for some $Y \subseteq Aut(E)$. If $Y = \emptyset$, then $X = E$ and it is done, so suppose $Y$ contains at least one automorphism, say $\sigma$.

Obviously, 0 and 1 are in $X$. Suppose $x, x' \in X$. Then $\sigma(x + x') = \sigma(x) + \sigma(x') = x + x'$, so $x + x' \in X$. Similarly, $x \cdot x' \in X$. Also, $\sigma(-x) = \sigma(-1) \cdot \sigma(x) = -1 \cdot x = -x$, hence $-x \in X$, and if $x \neq 0$, then

$$\sigma(1) = 1 \iff \sigma(x^{-1} \cdot x) = 1$$
$$\iff \sigma(x^{-1}) \cdot \sigma(x) = 1$$
$$\iff \sigma(x^{-1}) = x^{-1}$$

Thus $x^{-1} \in X$. \qed
Proposition 2.2.3 If \( Y \subseteq Aut(E) \) is a fact, then \( Y \) is a subgroup of \( Aut(E) \).

Proof: Suppose \( Y \subseteq Aut(E) \) is a fact, then \( Y = Pol^r_E(X) \) for some \( X \subseteq E \) (by Prop. 2.1.13). If \( X \) is empty, then \( Y = Aut(E) \), so suppose \( X \neq \emptyset \).

Obviously, \( Pol^r_E(X) \) contains at least the identity automorphism, \( Id \), so \( Y \) is not empty. If \( \sigma, \tau \in Y \), then for all \( x \in X \), \( \sigma^{-1}(x) = \sigma^{-1}(\sigma(x)) = Id(x) = x \), so \( \sigma^{-1} \in Y \), and \( \tau(\sigma(x)) = \tau(x) = x \), so \( \tau \circ \sigma \in Y \).

The last two propositions give us necessary conditions for subsets of \( E \) and \( Aut(E) \) to be facts. The Fundamental Theorem of Galois Theory gives us sufficient conditions, as follows:

Assume that \( E \) is a finite, separable, normal extension of a field \( E' \).

The subfields \( K \) of \( E \) that contain \( E' \) are facts of \( E \), and the subgroups \( H \) of \( Pol^r_E(E') \), which are called the Galois groups of \( E \) over \( E' \), noted \( Gal(E/E') \), are facts of \( Aut(E) \). The Fundamental Theorem of Galois Theory says they are in a one-to-one correspondence (see [3]), i.e.

\[
K = Pol^l_E(H) \iff H = Pol^r_E(K)
\]

In our previous notation, this means \( K \triangleleft H \). For example, let \( E' = \mathbb{Q} \) and \( E = \mathbb{Q}(\sqrt{2}, \sqrt{3}) \). Then \( Pol^l_E(\mathbb{Q}) \) is of order 4, with elements \( \sigma_i \) such that

\[
\begin{align*}
\sigma_0(\sqrt{2}) &= \sqrt{2} & \sigma_0(\sqrt{3}) &= \sqrt{3} \\
\sigma_1(\sqrt{2}) &= \sqrt{2} & \sigma_1(\sqrt{3}) &= -\sqrt{3} \\
\sigma_2(\sqrt{2}) &= -\sqrt{2} & \sigma_2(\sqrt{3}) &= \sqrt{3} \\
\sigma_3(\sqrt{2}) &= -\sqrt{2} & \sigma_3(\sqrt{3}) &= -\sqrt{3}
\end{align*}
\]

The facts of \( Pol^r_E(\mathbb{Q}) \) are \( \{\sigma_0\}, \{\sigma_0, \sigma_1\}, \{\sigma_0, \sigma_2\}, \{\sigma_0, \sigma_3\} \) and \( Pol^r_E(\mathbb{Q}) \) itself, corresponding respectively to facts \( \mathbb{Q}(\sqrt{2}, \sqrt{3}), \mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{3}), \mathbb{Q}(\sqrt{6}) \) and \( \mathbb{Q} \), i.e. \( \{\sigma_0\} \triangleleft \mathbb{Q}(\sqrt{2}, \sqrt{3}) \), etc.
Since we are looking at situations where \( E \) and \( F \) may be distinct sets, many mathematical examples may arise. The last example had some historical value; the next two examples may be considered as pedagogical for the present work:

**Example 2.2.4 (Lines and points as facts)**

This one is really a toy example inspired by a metaphor used in [16] (p.8).

Let \( E \) be the set of points in the plane and \( F \) the set of (straight) lines.

For the moment, let \( E = \mathbb{R}^2 \) and let an element \( \ell \) of \( F \) be a set

\[
\ell = \{(x, y) \in \mathbb{R}^2 \mid Ax + By + C = 0, \text{ for } A, B, C \in \mathbb{R}\}
\]

Let \( R = \{0, 1\} \) and \( P = \{1\} \).

Define the map \( \langle - \mid - \rangle : E \times F \to R \) such that

\[
\langle (x, y) \mid \ell \rangle = \begin{cases} 
1 & \text{if } (x, y) \in \ell \\
0 & \text{else}
\end{cases}
\]

In words, we have stated that a point and a line are in duality if and only if “the point belongs to the line”. Now \( \text{Pol}_E^r (\{(x, y)\}) \) is the set of lines that “intersect at \((x, y)\)”, while \( \text{Pol}_E^l (\{(x_1, y_1), (x_2, y_2)\}) \), with \((x_1, y_1) \neq (x_2, y_2)\), is the singleton that contains the “only line that passes through \((x_1, y_1)\) and \((x_2, y_2)\)”. Dually, \( \text{Pol}_F^l (\{\ell\}) \) is the set of all points that belong to \( \ell \), i.e. \( \text{Pol}_F^r (\{\ell\}) = \ell \).

We observe that \( \text{Pol}_E^r (X) = \emptyset \) if the points in \( X \) are not aligned, and similarly \( \text{Pol}_F^l (Y) = \emptyset \) if the lines in \( Y \) do not intersect at a common point.

Therefore the only facts are \( E, F, \emptyset \), singletons and their polar set.

This said, we can abstract this polarity and see “points” and “lines” as primitive undefined terms. Our familiar definition of a line \( \ell \) as a set of points can be recovered by \( \text{Pol}_F^l (\ell) \); but dually, we have a definition of a point \( p \) as a set of lines (such that \( p \) is their intersection) given by \( \text{Pol}_E^r (p) \).
Example 2.2.5 (Convex sets as facts)

This one was pointed out in [29] (p.13). Let $E$ be the set of points of a plane, say $E = \mathbb{R}^2$, and $F$ the set of half planes, where $H \in F$ is given by $H = \{(x, y) \in \mathbb{R}^2 \mid Ax + By + C \geq 0$, for $A, B, C \in \mathbb{R}\}$.

Let $R = \{0, 1\}$ and $P = \{1\}$.

Define the map $\langle - \mid - \rangle : E \times F \rightarrow R$ such that

\[
\langle (x, y) \mid H \rangle = \begin{cases} 
1 & \text{if } (x, y) \in H \\
0 & \text{else}
\end{cases}
\]

Hence $(x, y) \perp H$ if and only if the point $(x, y)$ belongs to the half-plane $H$.

Then for $X \subseteq E$, $\mathsf{Pol}^l_F(\mathsf{Pol}^r_E(X))$ is the intersection of all half-planes containing $X$, i.e. the convex hull of $X$. Hence a convex set $X$ is a fact, and we can write $X = \mathsf{Pol}^l_F(Y)$ for some set of half-planes $Y \subseteq F$ such that $Y = \mathsf{Pol}^r_E(X)$.

Now we can abstract our notion of a convex set and describe it in two equivalent ways: as the usual set of points $\mathsf{Pol}^l_F(Y)$, or dually by the half-planes containing it, i.e. $\mathsf{Pol}^r_E(X)$.

Remark 2.2.6 In our setting of polarity, we also recognize the idea of a game in the sense of Lafont and Streicher [26]. By definition, a game over a set $R$ (whose elements are called scalars) is a pair of sets $E, F$, whose elements are respectively called the vectors and the forms, equipped with a map $\langle - \mid - \rangle : E \times F \rightarrow R$.

This is a generalization of the original description given by Von Neumann and Morgenstern, where $E$ and $F$ represented strategies for each player, and the map $E \times F \rightarrow \mathbb{R}$ was the payoff function as mentioned in [26]. As Lafont and Streicher point out, a game over $R$ is simply a special case of Chu space over $R$. 
2.3 Metamathematical examples

Example 2.3.1 (Propositional logic)

Let $E = \text{PROP}$, i.e. the usual set of well-formed formulas of propositional logic. Take $F$ to be the set of valuations $Val$, i.e. the set of maps $[-] : \text{PROP} \rightarrow \{0,1\}$ which are valuations. (Refer to [34] for the terminology.) Let $R = \{0,1\}$ and $P = \{1\}$. We define the map $\ll - \mid - \gg : \text{PROP} \times Val \rightarrow R$ such that $\ll A \mid [-] \gg = [A]$. Traditionally, $[A] = 1$ means that $A$ is a valid (or true) proposition, while $[A] = 0$ means it is not.

Now we can express some metamathematical theorems in terms of polar sets. Let $\Gamma \subseteq \text{PROP}$ be a set of formulas. Given a formal system, $\Gamma \vdash \varphi$ means that the formula $\varphi \in \text{PROP}$ is derivable from $\Gamma$ (using the deduction rules of the system). We say that $\Gamma$ is deductively closed if for any $\varphi \in \text{PROP}$, $\Gamma \vdash \varphi$ implies that $\varphi \in \Gamma$.

We also say that a formal system is sound (or adequate, correct) with respect to $Val$ if for any $\varphi \in \text{PROP}$,

$$\vdash \varphi \Rightarrow \models \varphi$$

where $\models \varphi$ means that for all valuations $[-] \in Val$, we have $[\varphi] = 1$.

Let $\Theta = \{\varphi \in \text{PROP} \mid \vdash \varphi\}$. $\Theta$ is the set of all provable formulas in the system, i.e. the theorems. Obviously, the set of theorems is deductively closed.

Proposition 2.3.2 For a given formal system, the following are equivalent:

(i) The system is sound (with respect to $Val$);

(ii) $\Theta \subseteq Pol^E_F(Val)$;

(iii) $Val = Pol^E_F(\Theta)$.

Proof: For (i)$\Rightarrow$(ii): Suppose soundness. If $\theta \in \Theta$, then $\vdash \theta$, which implies $\models \theta$ by hypothesis. This means for all valuations $[-] \in Val$, $[\theta] = 1$, hence $\theta \in Pol^E_F(Val)$.
For (ii)⇒(iii): By Proposition 2.1.5, we know $Val$ is a polar set, i.e. a fact (Prop. 2.1.13), so

$$\Theta \subseteq Pol^r_E (Val) \Rightarrow Pol^r_E (Pol^l_E (Val)) \subseteq Pol^r_E (\Theta) \Rightarrow Val \subseteq Pol^r_E (\Theta)$$

For (iii)⇒(i): Suppose $Val \subseteq Pol^r_E (\Theta)$. Take any $\varphi \in PROP$ and suppose $\vdash \varphi$. Then $\varphi \in \Theta$, and since $Pol^r_E (\Theta)$ contains every valuation by hypothesis, we conclude by definition that $\varphi \perp Val$, hence $\models \varphi$.

We know that if

$$\Theta_c \overset{def}{=} \{ \varphi \in PROP \mid \vdash_c \varphi \}$$

where $\vdash_c \varphi$ means that $\varphi$ is derivable from classical rules, and if

$$\Theta_i \overset{def}{=} \{ \varphi \in PROP \mid \vdash_i \varphi \}$$

where $\vdash_i \varphi$ means that $\varphi$ is derivable from intuitionistic rules, then

$$\Theta_i \subseteq \Theta_c$$

since the deductive rules of intuitionistic logic are included in those of classical logic (indeed, intuitionistic logic is simply classical logic without the reductio ad absurdum rule).

Using Proposition 2.1.5, we get

$$Pol^r_E (\Theta_c) \subseteq Pol^r_E (\Theta_i)$$

Hence soundness of classical logic implies soundness of intuitionistic logic (with respect to $Val$) by Proposition 2.3.2.

Now we say that a formal system is complete with respect to $Val$ if for any $\varphi \in PROP$,

$$\models \varphi \Rightarrow \vdash \varphi$$
Proposition 2.3.3  For a given formal system, the following are equivalent:

(i) The system is complete (with respect to \( \text{Val} \));

(ii) \( \text{Pol}^F(\text{Val}) \subseteq \Theta \).

Proof: For (i)⇒(ii): If \( \varphi \in \text{Pol}^F(\text{Val}) \), then by definition \( \varphi \perp \text{Val} \), hence \( \models \varphi \). By completeness, \( \vdash \varphi \), so \( \varphi \in \Theta \). For (ii)⇒(i): Clear by definition.

It is well-known that classical logic is complete with respect to \( \text{Val} \), which can be expressed by \( \text{Pol}^c(\text{Val}) \subseteq \Theta_c \). And since it is also sound, we can see that \( \Theta_c \) is a fact corresponding to \( \text{Val} \), i.e. \( \Theta_c \triangleright \text{Val} \).

We can generalize the last example to (first order) predicate logic, which was first pointed out by Lawvere in 1969 (reprinted in [30]):

Example 2.3.4 (Predicate logic)

Consider a language \( \mathcal{L} \) and let \( E \) be the set of all well-formed \( \mathcal{L} \)-sentences. Let \( F \) be the set of \( \mathcal{L} \)-structures, i.e. the structures that can give a meaning to the language \( \mathcal{L} \). Let \( R = \{0, 1\} \) and \( P = \{1\} \).

We define the map \( \ll - \mid - \gg : E \times F \rightarrow R \) such that:

\[
\ll \sigma \mid M \gg = \begin{cases} 
1 & \text{if } M \models \sigma \quad (M \text{ is a model for } \sigma) \\
0 & \text{otherwise}
\end{cases}
\]

In words, we have simply defined a duality between syntax and semantics.

We recall that a theory \( \Theta \subseteq E \) is closed under derivability, i.e. \( \Theta \vdash \sigma \) implies \( \sigma \in \Theta \) ([34], p.104).

Also, a set \( \Pi \subseteq E \) such that \( \Theta = \{ \sigma \mid \Pi \vdash \sigma \} \) is called an axiom set of the theory \( \Theta \).

We call such a theory \( \Theta \) a deductive theory.

Moreover, we can define the theory of (a structure) \( M \in F \), noted \( \text{Th}(M) \) (see
by $Th(M) \overset{def}{=} Pol^t_F(\{M\})$.

Notice that in a sense, $Th(M)$ is *induced* from a structure $M$, so we may refer to it as an *induced theory*.

Let’s take a concrete example. Let $\mathcal{L}$ be the usual language of *Peano arithmetic*. Let $PA \subseteq E$ be a set of *axioms* [32]. We expect that $Pol^t_E(PA)$ contains $\mathbb{N}$, the *standard* model. Now using Proposition 2.1.9, we know $PA \subseteq Pol^t_E(Pol^t_E(PA)) \subseteq Pol^t_E(\{\mathbb{N}\}) = Th(\mathbb{N})$. And we know from Gödel that if we consider the theory $\hat{PA} = \{\sigma \in E \mid PA \vdash \sigma\}$, then $\hat{PA} \subset Th(\mathbb{N})$.

**Proposition 2.3.5** If $\hat{PA}$ is a fact, then $\mathbb{N}$ is not the only model for $\hat{PA}$.

**Proof:**

\[
\hat{PA} \text{ is a fact} \iff \hat{PA} = Pol^t_E(Pol^t_E(\hat{PA})) \\
\implies Pol^t_E(\hat{PA}) \neq \{\mathbb{N}\} \text{ since } Pol^t_F(\{\mathbb{N}\}) = Th(\mathbb{N}) \\
\implies \{\mathbb{N}\} \subsetneq Pol^t_E(\hat{PA})
\]

\[
\square
\]

**Remark 2.3.6**

We know there are *non-standard* models for arithmetic, but do we know if there is such a set of models in correspondence with $\hat{PA}$ so that $\hat{PA}$ is a fact?

Let’s try to understand Popper’s philosophy in the light of *polarity*:

**Example 2.3.7 (Popper’s refutationism)**

Popper’s *logic* of scientific discovery is based on the idea of *conjectures* and *refutations*. In order to *refute* a *conjecture*, we put it to the *test*: if it holds, we say it has not been refuted (yet); if it fails, we simply reject it (see [31]).
Let $E$ be a set of $\mathcal{L}$-sentences with respect to a language $\mathcal{L}$. Let $F$ be a set of tests for $E$. Let $R = \{0, 1\}$ and $P = \{1\}$. We define the map $\langle - | - \rangle : E \times F \rightarrow R$ such that

$$\gamma \perp \mathcal{T} \iff \langle \gamma | \mathcal{T} \rangle = 1 \iff \text{"conjecture } \gamma \text{ is not refuted by the test } \mathcal{T} \"$$

We define a theory as a subset $\Gamma \subseteq E$ of conjectures. Notice that we do not require a theory to be deductive.

We say that a theory $\Gamma \subseteq E$ is contingent, i.e. has not been refuted yet, when $Pol^r_E (\Gamma) = F$. We might say that it is "valid" in some sense (indeed, in some fields like medicine, propositions are tested by experimental means, hence truth is given by "so far so good"). Notice that it looks like a generalization of soundness, so we may then say that $\Gamma$ is sound with respect to the tests in $F$.

In the same line of thought, we may say that $\Gamma$ is complete (with respect to $F$) when $Pol^l_F (F) \subseteq \Gamma$. This simply means that the theory can explain everything that has been tested successfully.

Consider some cases:

- If $F = \emptyset$, then no test is associated to what can be said in $E$ (with respect to a language $\mathcal{L}$). Then a theory $\Gamma \subseteq E$ is not testable. We can say it is sound, but so it is for any theory! Moreover, anything can be induced from $F$, even contradictory statements! Therefore $E$ is the only fact (of $E$) and it corresponds to $F$, i.e. $E \models F$. In words, $E$ is the largest valid theory, so everything is true.

  We retrieve here Popper’ criterion of testability (or refutability) for a theory to be called “scientific”: it has to be exposed to a non-empty set of tests. Otherwise, it is called “metaphysics” (see [31]).

- Since a theory $\Gamma \subseteq E$ is true when $Pol^r_E (\Gamma) = F$, absolute truth would mean
that $Pol^r_E(\Gamma)$ contains every possible test. Obviously, the question is: how can we tell $F$ is sufficiently large and would $\Gamma$ be testable in practice?

So instead of talking of absolute truth (which either does not exist or is not achievable), we will say a theory is less false the more $F$ contains (severe) tests that “do not invalidate” the theory.

- The leading example in science is Newton’s laws of physics, say $N \subseteq E$, where the elements of $E$ are expressed in the language of Euclidian geometry. This theory is sound (true) with respect to some set $F$ of tests, i.e. $Pol^r_E(N) = F$. It may be convenient here to think of tests as observations in some sense. Now $N$ contains a prediction that “light travels in a straight line” which has been tested with success in $F$. But if we consider $F' = F \cup \{T\}$, where $T$ is a test at a cosmic scale, then the prediction fails. Einstein’s relativity is a theory that is less false in this sense, since its polar set contains $F'$.

- The logic of scientific discovery from Popper’s viewpoint lies in the search of an increasing sequence of testing sets $F \subseteq F' \subseteq F'' \subseteq \ldots$ until a theory $\Gamma \subseteq E$ fails; we have then made progress towards (absolute) truth, since at least we have discovered that some conjecture $\gamma \in \Gamma$ was not true!

The main critique offered by Girard ([13],[15]) to Popper’s philosophy is that it gives the questionable status of “science” to some fields like medicine (where theories tend to reduce the more they are tested), while it ignores the scientific value of Gödel’s incompleteness theorem since it cannot be “tested” in Popper’s sense.

**Example 2.3.8 (Lakatos’ Proofs and Refutations)**

Lakatos [27] had an analogous viewpoint for the logic of mathematical discoveries. A theory $\Gamma \subseteq E$ for Lakatos is a set of conjectures (which are tentatively or informally “proven”, i.e. that were not deduced from a set of axioms) which may be sound for a set of mathematical structures $F$, until we consider more structures that falsify an element of $\Gamma$. For example, the conjecture “Euler’s formula $V - E + F = 2$ applies
to any polyhedron” may be true when tested with the set of “convex polyhedra”, but fails otherwise.

2.4 \textit{R-Duality on a set $E$}

We recall that when we have a duality between a set $E$ and itself, we call it an \textit{R-duality on $E$} and ask that $\text{Pol}_E^r = \text{Pol}_E^\ell$.

\textbf{Definition 2.4.1 (Cyclic pole)}

Given a map $\ll - \mid - \gg : E^2 \to R$, a pole $P \subseteq R$ is cyclic if for all $x, y \in E$, $\ll x \mid y \gg \in P \Rightarrow \ll y \mid x \gg \in P$.

Equivalently, we can say that $\perp \subseteq E^2$ is a symmetric relation, i.e. $x \perp y \Rightarrow y \perp x$.

What follows is a necessary and sufficient condition to define an \textit{R-duality on $E$}:

\textbf{Proposition 2.4.2} We have an \textit{R-duality on $E$} iff the pole $P$ is cyclic.

\textbf{Proof}: For necessity:

Obviously, if the pole is empty, then it is cyclic, so suppose $P \neq \emptyset$. Let $a, b \in E$ and suppose $\ll a \mid b \gg \in P$, i.e. $a \perp b$. Consider $\text{Pol}_E^r (\{a\}) \overset{\text{def}}{=} \{y \in E \mid a \perp y\}$. Then $b \in \text{Pol}_E^r (\{a\})$. And since $\text{Pol}_E^r (\{a\}) = \text{Pol}_E^\ell (\{a\})$ by hypothesis, we have $b \in \text{Pol}_E^\ell (\{a\}) \overset{\text{def}}{=} \{x \in E \mid x \perp a\}$, which means $b \perp a$. Hence $\ll b \mid a \gg \in P$. For sufficiency: clear.

We introduce the notation $A ^\perp$ for $\text{Pol}_E^r (A)$ (and $\text{Pol}_E^\ell (A)$). Now it might be convenient to recall some properties of Section 2.1:

\textbf{Proposition 2.4.3} Given an \textit{R-duality on $E$}, we have, for all $A, A_i, B \subseteq E$,

(i) $A \subseteq A ^{\perp \perp}$;
(ii) $A \subseteq B \Rightarrow B ^\perp \subseteq A ^\perp$;
(iii) $A^\perp = A^\perp \perp \perp$;
(iv) $\bigcap_{i \in I} A_i^\perp = (\bigcup_{i \in I} A_i)^\perp$;
(iv) $\bigcup_{i \in I} A_i^\perp \subseteq (\bigcap_{i \in I} A_i)^\perp$.

Proof: See Propositions 2.1.5, 2.1.14 and Corollary 2.1.6.

Remark 2.4.4

It is interesting to observe that if we think of $A^\perp$ as the intuitionistic negation “$\neg A := A \to \bot$”, intersection, union and inclusion respectively as “$\land, \lor, \vdash$”, then the formulas given by Proposition 2.4.3 are provable intuitionistically. In fact, when the inclusion is not given in Proposition 2.4.3, the associated sequent is not provable. For example, $\neg (A \land B) \vdash \neg A \lor \neg B$ is not provable, while $\neg A \lor \neg B \vdash \neg (A \land B)$ is:

$$
\frac{A \vdash A}{A \to \bot, A \vdash \bot}
\frac{A \rightarrow \bot, A \land B \vdash \bot}{(A \to \bot) \lor (B \rightarrow \bot) \vdash (A \land B) \rightarrow \bot}
$$

Besides being a good mnemonic way to remember which De Morgan law is provable and which one is not in intuitionistic calculus, maybe we can hope that this interpretation of negation of $A$ is an alternative for the traditional “$A$ implies absurdity”.

Now let’s look at some algebraic properties of $\left( \mathcal{P}(E), \cup, \cap, (\neg)^\perp \right)$ with the ordering given by “$\subseteq$”. Set theoretically, we obviously know that for any $A, B, C \in \mathcal{P}(E)$:

$$
A \cup B = B \cup A
$$

$$
A \cup (B \cup C) = (A \cup B) \cup C
$$

$$
A \cup (B \cap C) = (A \cup B) \cap (A \cup C)
$$

$$
A \cup A = A
$$
\( A \cap (A \cup B) = A \)

These also hold \emph{dually} by replacing “\( \cup \)” by “\( \cap \)” (and vice versa). We also have neutral elements \( \emptyset \) and \( E \); so we may ask what other properties we get with respect to \((-)^\perp\), for any pole \( P \).

By Proposition 2.4.3, we automatically have

\[ A^\perp \cap B^\perp = (A \cup B)^\perp \]

If we consider a classical relation \( \perp := \perp_{\varphi} \) for some bijection \( \varphi : E \rightarrow E \) (see Definition 2.1.21), then every subset is a fact, so moreover

\[ A^\perp \cup B^\perp = (A \cap B)^\perp \]

by Proposition 2.1.17.

We may wonder if \((\mathcal{P}(E), \cup, \cap, (-)^\perp, \emptyset, E)\) is a Boolean algebra (where \( \perp \) is classical). Not quite: \( A \cup A^\perp = E \) and \( A \cap A^\perp = \emptyset \) fail in general. We then say it is simply a \emph{De Morgan algebra} ([25]).

In fact, it is Boolean for exactly one duality relation:

**Proposition 2.4.5** \((\mathcal{P}(E), \cup, \cap, (-)^\perp, \emptyset, E)\) is a Boolean algebra iff the duality relation is classical of the form \( \perp_{\text{Id}} \), where \( \text{Id} : E \rightarrow E \) is the identity map.

\textit{Proof} : \((\Rightarrow)\): Take \( \{a\} \) for some \( a \in E \). By hypothesis, \( \{a\} \cap \{a\}^\perp = \emptyset \), so \( a \notin \{a\}^\perp \).

But also \( \{a\} \cup \{a\}^\perp = E \), thus \( \{a\}^\perp = E \setminus \{a\} \). This means \( a \perp x \) for all \( x \in E \), except for \( x = a \). But since \( a \) was arbitrary, \( \perp = \{(x, y) \in E^2 \mid x \neq y\} \), which is simply a classical duality relation \( \perp_{\varphi} \) where \( \varphi(x) = x \).

\((\Leftarrow)\): We already know that “De Morgan laws” hold, so all we need to verify is that \( A \cup A_{\text{Id}}^\perp = E \) and \( A \cap A_{\text{Id}}^\perp = \emptyset \) for all \( A \subseteq E \). But this is obvious since \( A_{\text{Id}}^\perp = E \setminus A \). \( \square \)
Remark 2.4.6

Notice that we retrieved the notion of the complement of $A \subseteq E$ as a particular case of a classical duality relation. Indeed, $A^{\bot_{Id}} = E \setminus A$.

Now consider $\mathcal{H} \subseteq \mathcal{P}(E)$ such that $(\mathcal{H}, \subseteq, \cap, \cup, N, T, \Rightarrow)$ is a Heyting algebra, where $N$ and $T$ are respectively the least and the greatest elements, and for all $A, B \in \mathcal{H}$ the Heyting implication is defined as:

$$A \Rightarrow B \overset{\text{def}}{=} \bigcup\{X \in \mathcal{H} | A \cap X \subseteq B\}$$

i.e. $A \Rightarrow B$ is the largest set (in $\mathcal{H}$) such that its intersection with $A$ is contained in $B$ (in other words, $X \cap A \subseteq B$ iff $X \subseteq A \Rightarrow B$).

The pseudo-complement of $A \in \mathcal{H}$ is given by:

$$\neg_N A \overset{\text{def}}{=} A \Rightarrow N$$

We may wonder how the pseudo-complement is related to our notion of polar set.

**Proposition 2.4.7** Let $\mathcal{H} \subseteq \mathcal{P}(E)$ be a filter, where $E$ is a non-empty set. Given an $R$-duality on $T$, we have $\neg_N A = A^{\bot}$ for all $A \in \mathcal{H}$ if and only if $\bot = \bot_{Id} \cup N^2$.

*Proof*: For all $A \in \mathcal{H}$, we can easily see that $X = (T \setminus A) \cup N$ is the biggest set in the filter such that $A \cap X$ is in $N$. Therefore $\neg_N A = X$. Suppose $\neg_N A = A^{\bot}$. We have previously observed that when the duality relation is of the form $\bot_{Id}$, $A^{\bot_{Id}} = T \setminus A$. So all we have to add to $\bot_{Id}$ are the couples of the form $(x, x)$ such that $x \in N$, since $N \subseteq A$ for any $A$. Conversely, if $\bot = \bot_{Id} \cup N^2$, then $A^{\bot} := \{x \in T \mid A \perp x\} = (T \setminus A) \cup N \overset{\text{def}}{=} (A \Rightarrow N)$.

**Proposition 2.4.8** Let $\mathcal{H} \subseteq \mathcal{P}(E)$ be a total order, i.e. a chain, where $E$ is a non-empty set. Given an $R$-duality on $T$, we have $\neg_N A = A^{\bot}$ for all $A \in \mathcal{H}$ if and only if $\perp = (N \times T) \cup (T \times N)$. 

\[\square\]
Proof: $(\Rightarrow)$: By definition,

$$\neg_N A := (A \Rightarrow N) = \begin{cases} \top & \text{if } A = N \\ N & \text{else} \end{cases}$$

Since by hypothesis $\neg_N A = A^\perp$, we consider two cases: if $A = N$, then $N^\perp := \{x \in E \mid N \perp x\} = \top$, which means $N \perp \top$ (and $\top \perp N$ since $\perp$ is symmetric). And if $A \neq N$, $A^\perp := \{x \in E \mid A \perp x\} = N$; since every $A \in \mathcal{H}$ is contained or equal to $\top$, we have $\top \perp N$ (and $N \perp \top$).

$(\Leftarrow)$: For any $\mathcal{H} \ni A \neq N$, $A^\perp = N$; and clearly $N^\perp = \top$. Therefore $A^\perp = \neg_N A$ as given above.

\[\square\]

Examples 2.4.9 (Polarity in filters and chains)

Let’s illustrate this:

- When $\emptyset \in \mathcal{H}$, we have $N = \emptyset$. If $\mathcal{H}$ is a filter, then it is improper and we retrieve the notion of complement. If $\mathcal{H}$ is a chain, then $\perp = \emptyset$.

- Consider $E = \{a, b, c, d\}$ and take the following sublattice $\mathcal{H}$ of $\mathcal{P}(E)$:

Using Proposition 2.4.7, if the pole is of the form

$$\perp = \{(a,a), (a,b), (b,a), (a,c), (c,a), (a,d), (d,a), (b,b), (b,c), (c,b),$$

$$(b,d), (d,b), (c,d), (d,c)\}$$

then the pseudo-complement and the polar set of $A$ coincide.
Now consider the chain $\mathcal{C}$

\[
\begin{array}{c}
E \\
\{a, b, c\} \\
\{a, b\} \\
\{a\}
\end{array}
\]

Using Proposition 2.4.8, if the duality relation is of the form

\[
\perp = \{(a, a), (a, b), (b, a), (a, c), (c, a), (a, d), (d, a)\}
\]

then for all $A \subseteq \mathcal{C}$, $\neg_{\{a\}}A = A^\perp$.

Now let $E$ be a non-empty set and take any $R$-duality on $E$, with duality relation $\perp$. Define

\[
N := \{x \in E \mid x \perp E\}
\]

and

\[
I := \{x \in E \mid x \perp x\}
\]

In general, $N \subseteq I$, and the following properties hold:

**Proposition 2.4.10** For all $A \subseteq E$, we have

(i) $A \subseteq N \Rightarrow A^\perp = E$;

(ii) $E^\perp = N$;

(iii) $N \subseteq A^\perp$;

(iv) $A \cap A^\perp \subseteq I$;

(v) If $A$ is a fact, then $(A \cup A^\perp)^\perp = E \iff A \cap A^\perp = N$. 
Proof: (i) to (iv) are obvious by definition. For (v), we know by Proposition 2.4.3 that \((A \cup A^\perp)^\perp = (A \cap A^\perp)^\perp\), which is equal to \((A^\perp \cap A)^\perp\) since \(A\) is a fact. So \((A \cup A^\perp)^\perp = E \Rightarrow (A \cup A^\perp)^\perp = E \Rightarrow (A^\perp \cap A)^\perp = E^\perp\), hence \(A^\perp \cap A = N\) by (ii). The converse is obvious since \(N^\perp = E\) by (i).

Here are some examples of \(R\)-dualities on a set.

**Example 2.4.11 (Vector spaces as facts)**

This example motivates the notation used by Girard ([15],[17]):

Let \((E,+,\cdot)\) be a vector space over \(\mathbb{R}\). Consider an \(\mathbb{R}\)-duality on \(E\) with \(P = \{0\}\) as the pole, and define a symmetric bilinear map \(\ll - | - \gg : E^2 \rightarrow \mathbb{R}\). In particular, when \(\ll - | - \gg\) is *positive definite*, i.e. \(\ll v | v \gg \in [0,\infty]\) and \(\ll v | v \gg = 0\) if and only if \(v = 0\) for all \(v \in E\), we retrieve the notion of *scalar product* \((\ll v | v \gg = 0\) iff \(v = 0\) simply means \(N = I = \{0\}\)). Notice that the pole is cyclic and for any \(A \subseteq E\), we say \(A^\perp\) is the set of vectors *orthogonal* to the ones in \(A\) [22]. Here are some other interesting observations:

**Proposition 2.4.12** In vector spaces \(E\), every fact is a subspace of \(E\).

Proof: By Proposition 2.1.13 (i), facts are polar sets, so let \(A^\perp\) be a polar set for some \(A \subseteq E\). Clearly \(0 \in A^\perp\), since we have \(\ll a | 0 \gg = 0\) for all \(a \in A\) (by bilinearity), so \(A^\perp \neq \emptyset\). Also, let \(v, w \in A^\perp\). Then for any \(\lambda, \mu \in \mathbb{R}, \lambda v + \mu w \in A^\perp\) : for all \(a \in A, \ll a | \lambda v + \mu w \gg = \ll a | \lambda v \gg + \ll a | \mu v \gg = \lambda \ll a | v \gg + \mu \ll a | w \gg = \lambda \cdot 0 + \mu \cdot 0 = 0\).

**Definition 2.4.13**

Let \(E\) be a vector space and \(A, B \subseteq E\) be two subspaces. The *sum* of \(A\) and \(B\) is given by

\[
A + B \overset{\text{def}}{=} \{ v \in E \mid \exists a \in A, \exists b \in B \text{ such that } v = a + b \}
\]
The following is well-known (see [22]):

**Proposition 2.4.14** For all subspaces $A, B$ of $E$, $A + B$ is a subspace of $E$.

We also know from [22] (p.325):

**Proposition 2.4.15** Let $E$ be of finite dimension. Then for any subspace $A$ of $E$,

(i) $\dim E = \dim A + \dim A^\perp - \dim(A \cap N)$;

(ii) $A^\perp \perp = A + N$.

In particular, we have

**Proposition 2.4.16** If $E$ is a vector space of finite dimension and $N = \{0\}$, then every subspace $A$ of $E$ is a fact, i.e. $A^\perp \perp = A$.

Therefore by Propositions 2.4.12 and 2.4.16, we know that for any vector space of finite dimension, facts are exactly the subspaces. Moreover,

**Proposition 2.4.17** Let $E$ be of finite dimension. Then for all subspaces $A, B$ of $E$, $A + B = (A \cup B)^\perp \perp$.

**Proof**: Clearly $A \cup B \subseteq A + B$, and since $(A \cup B)^\perp \perp$ is the smallest fact containing $A \cup B$ by Proposition 2.1.16, it is the smallest subspace containing it (Proposition 2.4.12 and 2.4.16). All we need to show is that $A + B$, which is a subspace by Proposition 2.4.14, is contained in $(A \cup B)^\perp \perp$. Suppose $v \in A + B$, i.e. $v = a + b$ for $a \in A$, $b \in B$. Take any $w \in (A \cup B)^\perp : w \perp v'$ for all $v' \in A \cup B$, and in particular $w \perp a$ and $w \perp b$. Therefore $\ll w \mid v \rr = \ll w \mid a + b \rr = \ll w \mid a \rr + \ll w \mid b \rr = 0$. Since $w$ is arbitrary, this means $w \perp v$ for all $w \in (A \cup B)^\perp$, hence $v \in (A \cup B)^\perp \perp$. \qed
Definition 2.4.18

Let $E$ be a vector space and $A, B \subseteq E$ be two subspaces. The direct sum of $A$ and $B$ is given by

$$A \oplus B \overset{\text{def}}{=} \{ v \in E \mid \exists! a \in A, \exists! b \in B \text{ such that } v = a + b \}$$

The following is well-known (see [22]):

**Proposition 2.4.19** For all subspaces $A, B$ of $E$, $v \in A \oplus B$ if and only if $v \in A + B$ and $A \cap B = \{0\}$.

**Proposition 2.4.20** Let $A$ be a subspace of a vector space $E$ of finite dimension. Then $A \oplus A^\perp = E \iff A \cap A^\perp = \{0\} = N$.

**Proof:**

$$A \oplus A^\perp = E \iff A + A^\perp = E \text{ and } A \cap A^\perp = \{0\} \text{ by Proposition 2.4.19}$$

$$\iff (A \cup A^\perp)^\perp = E \text{ and } A \cap A^\perp = \{0\} \text{ by Proposition 2.4.17}$$

$$\iff A \cap A^\perp = N \text{ and } A \cap A^\perp = \{0\} \text{ by Proposition 2.4.10 (v)}$$

\[ \square \]

**Remark 2.4.21**

Notice that if $E$ is a complex vector space (of finite dimension), we have analogous results when $\ll - | - \gg : E^2 \rightarrow \mathbb{C}$ is an hermitian form, i.e. it is a sesquilinear form:

- $\ll x + x' | y \gg = \ll x | y \gg + \ll x' | y \gg$ and $\ll x | y + y' \gg = \ll x | y \gg + \ll x | y' \gg$;

- $\ll x | \lambda y \gg = \lambda \ll x | y \gg$ and $\ll \lambda x | y \gg = \overline{\lambda} \ll x | y \gg$. 

and

\[ \langle x \mid y \rangle = \langle y \mid x \rangle \]

Indeed, if the pole is \( \{0\} \), then it is cyclic: \( \langle x \mid y \rangle = 0 \iff \langle y \mid x \rangle = 0 \iff \langle y \mid x \rangle = 0 \) since \( 0 = \bar{0} \).

**Example 2.4.22 (Coherence spaces as facts)**

Let \( X \) be a set and consider \( \mathcal{E} = \mathcal{P}(X) \).

Define an \( \mathbb{N} \)-duality on \( \mathcal{E} \) as follows: take the map

\[ \langle - \mid - \rangle : \mathcal{E} \times \mathcal{E} \to \mathbb{N} \]

\[ (A, B) \mapsto |A \cap B| \]

and let the pole be \( P = \{0, 1\} \).

Here are some examples:

- Let \( X = \emptyset \), and \( \mathcal{E} = \{\emptyset\} \). Then \( \perp = \{(\emptyset, \emptyset)\} \), and by Proposition 2.1.19, the only fact is \( \mathcal{E} \).

- Let \( X = \{\ast\} \), and \( \mathcal{E} = \{\emptyset, \{\ast\}\} \). Then again \( \perp = \mathcal{E} \times \mathcal{E} \), and by Proposition 2.1.19, the only fact is \( \mathcal{E} \).

- Let \( X = \{t, f\} \). Then \( \perp = (\mathcal{E} \times \mathcal{E}) \setminus \{(X, X)\} \). The only facts are \( \mathcal{E} \) and \( \{\emptyset, \{t\}, \{f\}\} \).

- Let \( X = \{a, b, c\} \). Here are some corresponding facts:
Definition 2.4.23 A coherence space [10] is a subset $\mathcal{A} \subseteq \mathcal{P}(X)$, where $X$ is a set, which satisfies:

(i) Down-closure: if $A \in \mathcal{A}$ and $A' \subseteq A$, then $A' \in \mathcal{A}$.

(ii) Binary completeness: if $\mathcal{M} \subseteq \mathcal{A}$ and if for all $A_1, A_2 \in \mathcal{M}$, $A_1 \cup A_2 \in \mathcal{A}$, then $\bigcup \mathcal{M} \in \mathcal{A}$. In particular (if $\mathcal{M} = \emptyset$), we have $\emptyset \in \mathcal{A}$.

It will be convenient to write $|\mathcal{A}|$ for $\{a \in X \mid \{a\} \in \mathcal{A}\}$. The elements of $|\mathcal{A}|$ are called the tokens by Girard [10].

Moreover, we can define a coherence relation modulo $\mathcal{A}$ between tokens by

$$a \bowtie b \mod \mathcal{A} \iff \{a, b\} \in \mathcal{A}$$

$|\mathcal{A}|$ equipped with $\bowtie$ is a graph, called the web of $\mathcal{A}$: the vertices are the tokens, and there is an edge between $a$ and $b$ iff $a \bowtie \mathcal{A} b$, where $a \bowtie \mathcal{A} b$ means $a \bowtie b \mod \mathcal{A}$. The cliques are exactly the elements of $\mathcal{A}$ (see [10]).

In what follows, given a coherence space $\mathcal{A}$, we define an $\mathbb{N}$-duality on $\mathcal{P}(|\mathcal{A}|)$ as
discussed previously:

\[
\preceq - | - \succ : \mathcal{P}(|\mathcal{A}|) \times \mathcal{P}(|\mathcal{A}|) \rightarrow \mathbb{N} \\
(A, B) \mapsto |A \cap B|
\]

with \( P = \{0, 1\} \).

**Proposition 2.4.24** Let \( \mathcal{A} \) be a coherence space. Then \( \{a, b\} \in \mathcal{A} \iff \{a, b\} \notin \mathcal{A}^\perp \) for \( a, b \in |\mathcal{A}|, a \neq b \).

**Proof** : For \((\Rightarrow)\) : obvious by definition of the duality.
For \((\Leftarrow)\) : Since \( \{a, b\} \notin \mathcal{A}^\perp \), then there exists \( B \in \mathcal{A} \) such that \(|\{a, b\} \cap B| > 1\). Since \( \{a, b\} \subseteq B \), we conclude \( \{a, b\} \in \mathcal{A} \) by (i) of Definition 2.4.23.

**Remark 2.4.25**

Proposition 2.4.24 tells us that the web of \( \mathcal{A} \) is the complement of the web of \( \mathcal{A}^\perp \) (if we ignore the loops). Also, if \( a \sim_\mathcal{A} b \), then \( a \sim_\mathcal{A}^\perp b \), which means \( a \) and \( b \) are incoherent in \( \mathcal{A}^\perp \), i.e. \( \{a, b\} \notin \mathcal{A}^\perp \) or \( a = b \). In other words, the coherent subsets of \( \mathcal{A} \) correspond to the incoherent subsets of \( \mathcal{A}^\perp \), and vice versa.

The following equivalence can be found in [15] (p.194):

**Proposition 2.4.26** \( \mathcal{A} \) is a coherence space if and only if \( \mathcal{A}^{\perp\perp} = \mathcal{A} \).

**Proof** : For \((\Rightarrow)\) : it suffices to show \( \mathcal{A}^{\perp\perp} \subseteq \mathcal{A} \). Let \( A \subseteq |\mathcal{A}| \) and suppose \( A \notin \mathcal{A} \).
Since \( \mathcal{A} \) is neither \( \emptyset \) nor a singleton, we have \(|A| > 1\). Consider two cases:
If \( A \in \mathcal{A}^\perp \), then clearly \( A \notin \mathcal{A}^{\perp\perp} \) by definition;
if \( A \notin \mathcal{A}^\perp \), then \(|A| > 2\) by Proposition 2.4.24. Consider \( \mathcal{M} = \{\{x\} \mid x \in A\} \).
Clearly \( \bigcup \mathcal{M} = A \notin \mathcal{A} \). Hence by Definition 2.4.23 (ii), since \( \mathcal{M} \subseteq \mathcal{A} \), there exists \( A_1, A_2 \in \mathcal{M} \) such that \( A_1 \cup A_2 \notin \mathcal{A} \). Said differently, this means there exists \( \{x, y\} \subseteq A \) such that \( \{x, y\} \notin \mathcal{A} \). Therefore \( \{x, y\} \in \mathcal{A}^\perp \), and \( A \notin \mathcal{A}^{\perp\perp} \) by definition of the duality.
Now for \((\Leftrightarrow)\): Suppose \(A \in \mathcal{A}\) and \(A' \subseteq A\). We know \(A \in \mathcal{A}^{\perp}\), so \textit{a fortiori} \(A' \in \mathcal{A}^{\perp} = \mathcal{A}\), which satisfies condition (i) of Definition 2.4.23. For (ii), suppose \(\mathcal{M} \subseteq \mathcal{A}\) and \(\forall A, B \in \mathcal{M}, A \cup B \in \mathcal{A}\). Consider \(\bigcup \mathcal{M}\): we have to show it belongs to \(\mathcal{A}\).

If \(\mathcal{M} = \{\emptyset\}\) or \(\mathcal{M} = \{\emptyset, \{a\}\}\) (for any \(a \in |\mathcal{A}|\)), then clearly \(\bigcup \mathcal{M} \in \mathcal{A}\). Otherwise, \(\mathcal{M}\) contains two distinct non-empty subsets, say \(A, B\), and since by hypothesis \(A \cup B \in \mathcal{A}\), then \(\bigcup \mathcal{M} \notin \mathcal{A}^{\perp}\) (indeed, \(A \cup B \subseteq \bigcup \mathcal{M}\) and \(|A \cup B| > 1\)).

Let’s proceed by contradiction: suppose \(\bigcup \mathcal{M} \notin \mathcal{A}^{\perp}\). Then there exists \(B' \in \mathcal{A}^{\perp}\) such that \(|B' \cap \bigcup \mathcal{M}| > 1\), i.e. there are at least two elements \(a, b\) in \(B' \cap \bigcup \mathcal{M}\). But \(a, b\) belong to some sets in \(\mathcal{M}\), say \(a \in A\) and \(b \in B\) (\(A, B\) may not be distinct). So by hypothesis, \(A \cup B \in \mathcal{A}\). Therefore \(B' \notin \mathcal{A}^{\perp}\), contradiction. \(\square\)

Before moving to the next example, consider the following:

Let \(X\) be a \textit{finite} set and define an \(\mathbb{N}\)-duality on \(\mathcal{P}(X)\) as in the previous example, \textit{i.e.} \(A \perp B \iff \ll A \mid B \gg \in \{0, 1\} \iff |A \cap B| \in \{0, 1\}\). Also, define an \(\mathbb{N}\)-duality on \(\{0, 1\}^X\), the set of maps from \(X\) to \(\{0, 1\}\), such that \(\ll f \mid g \gg := \sum_{x \in X} f(x)g(x)\), with the duality relation given by \(f \circledast g \iff \ll f \mid g \gg \in \{0, 1\}\).

Now consider the bijection

\[
\varphi: \mathcal{P}(X) \longrightarrow \{0, 1\}^X
\]

\[
A \mapsto (X \xrightarrow{\chi_A} \{0, 1\})
\]

such that \(\chi_A\) is the \textit{characteristic function} associated to \(A\):\[
\chi_A(x) = \begin{cases} 
1 & \text{if } x \in A \\
0 & \text{else}
\end{cases}
\]

We can show the following:

\textbf{Proposition 2.4.27} \(A \perp B \iff \varphi(A) \not\perp \varphi(B)\)
Proof: Indeed, \( A \perp B \iff A \cap B = \emptyset \) or \( A \cap B = \{a\} \) for some \( a \in A, B \). The first case is equivalent to saying \( \ll \varphi(A) \mid \varphi(B) \gg = 0 \) since \( \varphi(A)(x) \neq \varphi(B)(x) \) for all \( x \in X \). The second case is equivalent to \( \ll \varphi(A) \mid \varphi(B) \gg = 1 \) since

\[
\varphi(A)(x)\varphi(B)(x) = \begin{cases} 
1 & \text{if } x = a \\
0 & \text{else}
\end{cases}
\]

In both cases, \( \varphi(A) \not\perp \varphi(B) \).

This suggests the notion of some kind of "duality morphism":

\[\text{Definition 2.4.28 (duality morphism)}\]

Given an \( R \)-duality on \( E \) and an \( R' \)-duality on \( E' \), with duality relations respectively \( \perp \) and \( \not\perp \), a map \( \varphi : E \to E' \) is a morphism of dualities if \( x \perp y \iff \varphi(x) \not\perp \varphi(y) \). We say it is a monomorphism when \( \varphi \) is injective, and an isomorphism when the map is bijective.

In our example, since \( \varphi \) is a bijection by Proposition 2.4.27, this means the facts in \( \{0,1\}^X \) correspond exactly to the facts of \( \mathcal{P}(X) \), i.e. the coherence spaces (Proposition 2.4.26). Therefore we have yet another alternative definition for coherence spaces (for finite \( X \)), which will be generalized in the following example.

\[\text{Example 2.4.29 (Probabilistic coherence spaces as facts)}\]

Let \( X \) be a finite set and define an \( \mathbb{R}_+ \)-duality on \( \mathbb{R}_+^X \), the set of functions from \( X \) to the positive real numbers, such that \( \ll f \mid g \gg = \sum_{x \in X} f(x)g(x) \) with \( \mathcal{P} = [0,1] \), i.e.

\[ f \perp g \iff \ll f \mid g \gg \in [0,1] \]

In this duality, facts are called probabilistic coherence spaces and are characterized by the following theorem proved by Girard in [17] (p.410):

\[\text{Theorem 2.4.30 A subset } \mathcal{A} \subseteq \mathbb{R}_+^X \text{, where } X \text{ is a finite set, is a probabilistic coherence space, i.e. a fact, if and only if it satisfies:}\]
(i) $A \neq \emptyset$ (in particular, it contains the “null” function $0(x) = 0$ for all $x \in X$).

(ii) $A$ is a closed convex.

(iii) If $f \leq g \in A$, i.e. $f(x) \leq g(x)$ for all $x \in X$, then $f \in A$.

This idea can be generalized by the following example of Ehrhard [6]:

Let $X$ be an arbitrary at most countable set, and consider an $R$-duality on $\mathbb{R}^X$, the set of functions from $X$ to $\mathbb{R}$, by letting

$$\langle f \mid g \rangle = \sum_{x \in X} |f(x)g(x)|$$

Here $R$ is the rig $\mathbb{R} \cup \{\infty\}$ and $P = \mathbb{R}$.

In other words, $f \perp g \iff \langle f \mid g \rangle < \infty$, i.e. iff the sum converges.

As pointed out by Ehrhard, facts are called (real) Köthe spaces (of web $X$).

Notice that in the previous example, if $X = \{x_1, x_2, \ldots, x_n\}$, we can represent $f \in \mathbb{R}_+^X$ by the following diagonal matrix:

$$M(f) = \begin{pmatrix}
    f(x_1) & 0 & \ldots & 0 \\
    0 & f(x_2) & \ddots & \vdots \\
    \vdots & 0 & \ddots & 0 \\
    0 & \ldots & 0 & f(x_n)
\end{pmatrix}$$

so that given another $g \in \mathbb{R}_+^X$, we have $\langle f \mid g \rangle = Tr (M(f)M(g))$. This remark leads naturally to quantum coherence spaces.

**Example 2.4.31 (Quantum coherence spaces as facts)**

In this example [17], we consider $X$ with the structure of a (complex) Hilbert space of finite dimension. Let $E$ be the space of hermitian operators over $X$, i.e. such that $< h(x) \mid y >_X = < x \mid h(y) >_X$ for all $h \in E$, where $< - \mid - >_X$ is the (hermitian) scalar product.
CHAPTER 2. POLARITY

$E$ is an euclidian space, i.e. a real hermitian space (or a real prehilbertian space of finite dimension), with $\langle h \mid k \rangle \overset{\text{def}}{=} \text{Tr}(hk)$ for all $h, k \in E$.

Now we define a $\mathbb{C}$-duality on $E$ by

$$h \perp k \leftrightarrow \langle h \mid k \rangle \in [0, 1]$$

Here, facts are called quantum coherence spaces and are characterized by the following theorem proved by Girard in [17] (p.414):

**Theorem 2.4.32** A subset $\mathcal{A} \subseteq E$, with $E$ as above, is a quantum coherence space, i.e. a fact, iff it satisfies:

(i) $0 \in \mathcal{A}$.

(ii) $\mathcal{A}$ is a closed convex.

(iii) If $nf \in \mathcal{A}$ for all $n \in \mathbb{N}$, then $-f \in \mathcal{A}$.

(iv) If $f, g \in \mathcal{A}$, $\lambda, \mu \geq 0$ and $\lambda f + \mu g \in \mathcal{A}$, then $\lambda f \in \mathcal{A}$.


2.5 Duality on a set $E$

We recall that by a duality on $E$, we mean an $E$-duality on $E$.

In other words, the map $\langle - \mid - \rangle : E^2 \to E$ plays the role of an internal product.

Notice that the map $\langle - \mid - \rangle$ may be defined using the structure on $E$ if there is any. For example, if $(E, \cdot, 1)$ is a monoid, then we may let $\langle a \mid b \rangle \overset{\text{def}}{=} a \cdot b$. But then we must be careful when choosing the pole $P \subseteq E$. Indeed, for the free monoid, say $\Sigma^*$, if the word $abc \in P$, then we must also have $bca \in P$ and $cab \in P$, since the pole has to be cyclic by Proposition 2.4.2.

Obviously, $P \in \{\emptyset, E\}$ are always possible poles, and if $E$ is a commutative monoid, then any subset of $E$ can be a pole.
The next lemmas and propositions will show how a duality on $E$ can serve to characterize additional structure on $E$. Let $(E, \cdot, 1)$ be a monoid. We define $\langle a \mid b \rangle \overset{df}{=} a \cdot b$.

**Notation:** In the following propositions referring to group theory, in order to avoid confusion with the notion of cyclic group, we write “cyclic” (in quotes) to denote cyclicity of the pole in a polarity.

**Lemma 2.5.1** Let $(G, \cdot, 1)$ be a group with $g \in G$ a given element. Then the pole $P = \{g\}$ is “cyclic” iff $g$ is in the center of $G$.

**Proof**: ($\Rightarrow$): Suppose $P$ is “cyclic” and take $a \in G$. Since $\langle a \mid a^{-1} \cdot g \rangle$ and $\langle g \cdot a^{-1} \mid a \rangle$ are in $P$, then $\langle a^{-1} \cdot g \mid a \rangle$ and $\langle a \mid g \cdot a^{-1} \rangle$ are also in $P$ (by hypothesis). Therefore $a \cdot a^{-1} \cdot g = a \cdot g \cdot a^{-1}$, which means $g \cdot a = a \cdot g$.

($\Leftarrow$): Suppose for all $a \in G$, we have $a \cdot g = g \cdot a$. Then take $a, b \in G$ and suppose $\langle a \mid b \rangle \in P$, i.e. $a \cdot b = g$. Then $b = a^{-1} \cdot g \overset{hyp}{\Rightarrow} b = g \cdot a^{-1} \Rightarrow b \cdot a = g$. Thus $\langle b \mid a \rangle \in P$.

**Proposition 2.5.2** Let $(G, \cdot, 1)$ be a group, $|G| \geq 2$ and $g \in G$. Given a duality on $G$ such that $P = \{g\}$, the only facts are $\emptyset$, $G$ and every singleton.

**Proof**: Since $G^\perp = \{b \in G \mid \forall a \in G, a \cdot b = b \cdot a = g\} = \emptyset$, then $G^{\perp \perp} = G$. Now take any non-empty set $A$ different than $G$ and consider $A^\perp = \{b \in G \mid \forall a \in A, a \cdot b = b \cdot a = g\}$. If $|A| \geq 2$, then $A^\perp = \emptyset$, so $A$ is not a fact. If $A$ is a singleton, say $\{a\}$, then $A^\perp = \{b \in G \mid a \cdot b = b \cdot a = g\}$. But $A^\perp$ is not empty iff $a^{-1} \cdot g = g \cdot a^{-1}$, which is the case by Lemma 2.5.1. Hence $A^{\perp \perp} = A$.

**Lemma 2.5.3** Let $(G, \cdot, 1)$ be a group. Then the pole $P = G \setminus \{g\}$, for some $g \in G$, is “cyclic” iff $g$ is in $Z(G)$, the center of $G$. 
Proof: $(\Rightarrow)$

\[ g \notin P \implies g \cdot h \cdot h^{-1} \notin P \text{ for all } h \in G \]
\[ \implies h^{-1} \cdot g \cdot h \notin P \text{ for all } h \in G \text{ (since } P \text{ is “cyclic”)} \]
\[ \implies g = h^{-1} \cdot g \cdot h \text{ for all } h \in G \text{ (since there is only one element not in } P) \]
\[ \implies h \cdot g = g \cdot h \text{ for all } h \in G, \text{ i.e. } g \in Z(G). \]

Conversely, let \( P = G \setminus \{g\} \) with \( g \in Z(G) \). Suppose \( \langle a \mid b \rangle \in P \), i.e. \( a \cdot b \neq g \).

Now suppose \( \langle b \mid a \rangle \notin P \). This means

\[ b \cdot a = g \implies b = g \cdot a^{-1} \]
\[ \implies b = a^{-1} \cdot g \text{ (since } g \in Z(G)) \]
\[ \implies a \cdot b = g \]

which is a contradiction. \( \square \)

**Proposition 2.5.4** Let \((G, \cdot, 1)\) be a group and suppose there is a duality on \( G \).
Then the pole is of the form \( P = G \setminus \{g\} \) for some \( g \in G \) iff for all \( A \subseteq G \), \( A \) is a fact with respect to \( P \).

**Proof**: Notice that if \( G = \{1\} \), then it’s clear since \( \emptyset \) and \( G \) are the only subsets of \( G \). If \( |G| = 2 \), i.e. \( G = \{1, \sigma\} \), with \( \sigma^2 = 1 \), then for \( P = \emptyset \), singletons are not facts; for \( P \in \{\{1\}, \{\sigma\}\} \), every subset is a fact.

So consider when \( |G| > 2 \).

$(\Rightarrow)$ Clearly \( \emptyset \) and \( G \) are facts, since \( \emptyset \downarrow = G \) and \( G \uparrow = \emptyset \) (indeed, for any \( h \in G \), \( (g \cdot h^{-1}) \cdot h \notin P \)).

Now consider any non-empty set \( A \subseteq G \). By Proposition 2.4.3, we know \( A \subseteq A^{\downarrow \uparrow} \).

Let’s proceed by contraposition:

Take \( h \in G \) and suppose \( h \notin A \). Since \( A^{\downarrow} = \{b \in G \mid \forall a \in A, a \cdot b \in P\} \),
then \( h^{-1} \cdot g \in A^\perp \) (inverses are unique). Hence \( g^{-1} \cdot (h \cdot g) \notin A^{\perp\perp} \) since \( A^{\perp\perp} = \{a \in G \mid \forall b \in A^\perp, b \cdot a \in P\} \). But \( g^{-1} \cdot (h \cdot g) = g^{-1} \cdot (g \cdot h) = (g^{-1} \cdot g) \cdot h = h \) since \( g \in Z(G) \) by Lemma 2.5.3.

Therefore \( h \notin A^{\perp\perp} \), and \( A^{\perp\perp} \subseteq A \).

\((\Leftarrow)\) Suppose for all \( A \subseteq G \), we have \( A = A^{\perp\perp} \).

By Proposition 2.1.23, we know the duality relation is classical of the form \( \perp \phi \), where \( \phi : G \rightarrow G \) is a bijection.

Said differently, this means for all \( a \in G \), \( \ll a \mid b \gg = a \cdot b \in P \) for all but one \( b \in G \).

The Cayley table of \( (G, \cdot, 1) \) tells us that for any given \( a \), every element of \( G \) is in \( P \) except for some \( g_a \) (depending on \( a \)). And for every other distinct \( a' \), there is also only one \( g_{a'} \notin P \). Hence \( g_a \) must be equal to \( g_{a'} \). Therefore the pole contains every element of \( G \) except for one.

\(\square\)

**Proposition 2.5.5**  Let \( (M, \cdot, 1) \) be a monoid. Suppose we have a duality on \( M \) given by \( P = M \setminus \{1\} \).

Then \( (M, \cdot, 1) \) is a group iff \( A^{\perp\perp} = A \) for all \( A \subseteq M \).

**Proof**: \((\Rightarrow)\) : Clear by Proposition 2.5.4.

\((\Leftarrow)\) : Suppose every subset of \( M \) is a fact. By Proposition 2.1.23, the duality relation is classical, hence any \( a \in M \) is polar to any other element except for some \( m_a \in M \), where \( \ll a \mid m_a \gg = a \cdot m_a = 1 \). And since the pole is "cyclic", we also have \( m_a \cdot a = 1 \), which means \( m_a \) is the inverse of \( a \).

\(\square\)

### 2.6 Alternative notations

In Girard’s original paper on linear logic [8], given a phase space \( (M, \perp) \), i.e. a commutative monoid \( M \) and \( \perp \subseteq M \), the dual of \( X \subseteq M \) was defined as follows:

\[ X^\perp \overset{def}{=} \{y \in M \mid \forall x(x \in X \rightarrow yx \in \perp)\} \]
Linear implication was then defined by

\[ X \rightarrow Y \overset{\text{def}}{=} (XY)^\perp \]

where \( XY = \{ xy \mid x \in X \text{ and } y \in Y \} \).

This led to the alternative definition

\[ X \rightarrow Y \overset{\text{def}}{=} \{ y \in M \mid \forall x \in X, yx \in Y \} \]

The notation was followed later by [24].

Later in [15], Girard introduced \( \vdash \) for \( \perp \) and wrote \( \sim X \) for \( X^\perp \), called the negation of \( X \). He also introduced \( x \vdash y \) for \( xy \in \vdash \) and \( X \vdash Y \) for \( x \vdash y \ (x \in X, y \in Y) \).

It is interesting to notice that Girard then defined \( \sim X \) as a particular case of linear implication, i.e. \( \sim X \overset{\text{def}}{=} X \rightarrow \vdash \).

That idea was generalized in [15]. The subset \( \perp \subseteq M \) could be any pole \( P \subseteq R \), \( R \) being some set called the scalar space (see [18]). The product \( yx \) could be any binary map \( \langle x \mid y \rangle : E \times F \rightarrow R \), and the polar set of \( X \subseteq E \) (and \( Y \subseteq F \)) would be given by:

\[ X^P \overset{\text{def}}{=} \{ y \in F \mid \forall x \in X, \langle x \mid y \rangle \in P \} \]

and

\[ Y^P \overset{\text{def}}{=} \{ x \in E \mid \forall y \in Y, \langle x \mid y \rangle \in P \} \]

Notice that Girard’s notation of \( x \vdash y \) suggests that \( \vdash \) is a binary relation \( \vdash \subseteq M \times M \), but clearly it is not since \( \vdash \subseteq M \). Hence the definitions given in Section 2.1 of the present work seem less confusing. The introduction of the notation “\( A^\perp \)” for polar sets also tries to unify previous notations used by Girard and its followers, namely “\( A^\perp \)” and “\( \sim A \)”.
Chapter 3

Examples in logic

3.1 Tarskian semantics

3.1.1 Classical Logic revisited

In the light of Section 2, we can explore further the ideas mentioned in the introduction. Given an $R$-duality on $E$, we can see the map $\langle - \mid - \rangle : E^2 \rightarrow R$ as the testing process and the pole $P \subseteq R$ as the “positive results”, i.e. $\mu \perp \nu$ means “$\mu$ passed test $\nu$” (and vice versa by symmetry), so that $A^\perp$ are the tests for $A$. Therefore, since a proposition is the set $A$ of all elements that passed the tests for $A$, we have

$$A := \{ \mu \in E \mid \mu \perp A^\perp \}$$

$$= A^{\perp \perp}$$

We can now give a meaning to the traditional connectives. Suppose $A, B \subseteq E$ are sets defining two propositions, i.e. $A, B$ are facts. Set theoretically, we can imagine
the set representing the proposition “$A$ and $B$” as given by

\[
“A \text{ and } B” = \{ \mu \in E \mid \mu \perp A^{\perp} \text{ and } \mu \perp B^{\perp} \} \\
= \{ \mu \in E \mid \mu \perp A^{\perp} \} \cap \{ \mu \in E \mid \mu \perp B^{\perp} \} \\
= A^{\perp \perp} \cap B^{\perp \perp} \\
= A \cap B
\]

By De Morgan’s law, we can give a meaning to “$A$ or $B$”:

\[
“A \text{ or } B” := \text{ “not (not } A \text{ and not } B”)” \\
= (A^{\perp} \cap B^{\perp})^{\perp} \\
= (A \cup B)^{\perp \perp} \text{ by Proposition 2.4.3}
\]

We need to check that the previous meanings of conjunction and disjunction fit our definition by testing:

\[
“A \text{ or } B” := \{ \mu \in E \mid \mu \perp (“A \text{ or } B”)^{\perp} \} \\
= \{ \mu \in E \mid \mu \perp (A \cup B)^{\perp \perp} \} \\
= (A \cup B)^{\perp \perp \perp \perp} \\
= (A \cup B)^{\perp \perp}
\]

and

\[
“A \text{ and } B” := \{ \mu \in E \mid \mu \perp (“A \text{ and } B”)^{\perp} \} \\
= \{ \mu \in E \mid \mu \perp (A \cap B)^{\perp} \} \\
= (A \cap B)^{\perp \perp} \\
= A \cap B \text{ since the intersection of facts is a fact (see Remark 2.1.15).}
\]

Heuristically, if we think of $A$ as a set of observations or models associated to some proposition, then we expect that $A^{\perp}$ may contain observations of the negation or counter-models. But it may also happen that an element belongs to both a set and
its polar. This may look odd at first sight, but it occurs in many common situations. For example, if the proposition “this house is inhabited” is interpreted by $A$, it may contain irrelevant observations, e.g. $x \in A$ that says “when we look through the window, we can’t see anyone”. But then $A^\perp$, corresponding to the assertion that “this house is not inhabited”, will also contain $x$.

Now to give a semantical interpretation for classical propositional logic, $[-] : PROP_{CL} \rightarrow \mathcal{P}(E)$, we need a notion of truth or validity. Since classical logic is based on the principle that a proposition is true if and only if its negation is false, we consider a distinguished element $v \in E$ and say a proposition $A \in PROP_{CL}$ is true when $v \in \llbracket A \rrbracket$. Then we define an $R$-duality on $E$ such that given the map $\lll - | - \llrrr : E^2 \rightarrow R$, the pole $P \subseteq R$ is chosen in a way that $\lll v \mid x \llrrr \in P$ for all $x \neq v$. In other words, we have

$$(v, x) \in \perp \iff v \neq x$$

for all $x \in E$. We can see $v$ as a critical observation that is not in contradiction with any other observation of $E$ except for itself.

Here is how the interpretation goes for $PROP_{CL}$: atomic propositions are mapped to facts of $E$, with the usual constants interpreted as follows:

$$\top \mapsto E$$
$$\bot \mapsto E^\perp$$

Other formulas are defined inductively:

$$\neg A \mapsto \llbracket A \rrbracket^\perp$$
$$A \land B \mapsto \llbracket A \rrbracket \cap \llbracket B \rrbracket$$
$$A \lor B := \neg (\neg A \land \neg B) \mapsto \left( \llbracket A \rrbracket^\perp \cap \llbracket B \rrbracket^\perp \right)^\perp = \left( \llbracket A \rrbracket \cup \llbracket B \rrbracket \right)^\perp$$
$$A \rightarrow B := \neg A \lor B \mapsto \left( \llbracket A \rrbracket^\perp \cup \llbracket B \rrbracket^\perp \right)^\perp = \left( \llbracket A \rrbracket^\perp \cup \llbracket B \rrbracket \right)^\perp$$
**Proposition 3.1.1** All formulas of \( \text{PROP}_{\text{CL}} \) are mapped to facts under the above interpretation.

*Proof*: Clear, since \( E \) is a fact by Proposition 2.1.5, polar sets are facts by Proposition 2.1.13, and the intersection of polar sets is a polar set by Proposition 2.4.3. \( \Box \)

**Remark 3.1.2**

Notice that using Proposition 2.4.3, \( A \lor B \) is mapped to \( \left( [A] \cup [B] \right)^\bot \), which is the smallest fact that contains \( [A] \cup [B] \) by Proposition 2.1.16. Also, \( A \rightarrow \perp \) is mapped to \( \left( [A] \downarrow \cup E \downarrow \right)^\bot \), and by Proposition 2.4.3, \( \emptyset \subseteq [A] \downarrow \Rightarrow [A] \downarrow \subseteq \emptyset \downarrow \Rightarrow \emptyset \downarrow \subseteq [A] \downarrow \downarrow \), hence \( E \downarrow \subseteq [A] \downarrow \). Therefore \( \left( [A] \downarrow \cup E \downarrow \right)^\bot = \left( [A] \downarrow \right)^\bot = [A] \downarrow \), which is the interpretation of \( \neg A \). Hence we retrieve the traditional equivalence of propositions: \( \neg A = A \rightarrow \perp \).

We will show that (any) formal system used to derive classical formulas is both *sound* and *complete* with respect to this interpretation.

We say that an *inference rule* is valid (in the formal system) if whenever the (interpreted) premises contain \( v \), then so does the conclusion.

We also say that \( A \) is a *tautology* (\( \models A \)) if and only if \( v \in [A] \) for any assignment \( [-] : \text{PROP}_{\text{CL}} \rightarrow \mathcal{P}(E) \).

**Lemma 3.1.3** Given an \( R \)-duality on \( E \) as above, we have \( v \in A \iff v \notin A^\downarrow \) for any \( A \subseteq E \).

*Proof*: Clear by the definition of the duality relation. \( \Box \)

Obviously, it follows that

**Corollary 3.1.4** \( v \in A \cup A^\downarrow \)

**Lemma 3.1.5** For any fact \( A, B, C \subseteq E \), we have
CHAPTER 3. EXAMPLES IN LOGIC

(i) $v \in \left[ [A]^\perp \cup ( [B]^\perp \cup [A] \right)^\perp\perp$

(ii) $v \in \left( \left( [A]^\perp \cup ( [B]^\perp \cup [C] \right)^\perp\perp \right) \cup \left( \left( [A]^\perp \cup [B] \right)^\perp\perp \cup \left( [A]^\perp \cup [C] \right)^\perp\perp \right)^\perp\perp$

(iii) $v \in \left( \left( [B]^\perp \cup [A]^\perp \right)^\perp\perp \right) \cup \left( \left( [A]^\perp \cup [B] \right)^\perp\perp \right)^\perp\perp$

Proof: For (i): clear by Lemma 3.1.3 and Corollary 3.1.4.

For (ii): If $v \in [A]^\perp$ or $v \in [C]$, this is done, so suppose it is not the case. Then suppose $v \not\in [B]$. Thus $v \not\in [A]^\perp \cup [B]$, which means $v \in \left( [A]^\perp \cup [B] \right)^\perp\perp \perp$ by Lemma 3.1.3. And if $v \in [B]$, then $v \not\in [B]^\perp$, hence $v \not\in [A]^\perp \cup [B]^\perp \cup [C]^\perp$ and consequently $v \in \left( [A]^\perp \cup ( [B]^\perp \cup [C] \right)^\perp\perp$, which concludes the proof.

For (iii): If $v \in [A]^\perp$ or $v \in [B]$, then $v \in \left( [A]^\perp \cup [B] \right)^\perp\perp$ and it’s done, so suppose it is not the case. Then $v \in \left( [B]^\perp \cup [A]^\perp \right)^\perp\perp$. □

Theorem 3.1.6 (Soundness) Given an $R$-duality on $E \neq \emptyset$ as described above, if $\vdash_c A$ for $A \in \text{PROP}_{CL}$ then for all assignments $[-] : \text{PROP}_{CL} \rightarrow \mathcal{P}(E)$, we have $v \in [A]$.

Proof: We need to check these Hilbert system’s axioms and inference rule Modus ponens:

- For $A \rightarrow (B \rightarrow A)$, we have $[A \rightarrow (B \rightarrow A)] = [\neg A \vee (\neg B \vee A)] \ni v$ by Lemma 3.1.5 (i).

- For $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))$, we have $v \in \left[ \neg (\neg A \vee (\neg B \vee C)) \vee (\neg (\neg A \vee B) \vee (\neg A \vee C)) \right]$ by Lemma 3.1.5 (ii).
• For \((\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)\), we have \(v \in [\neg (\neg B \lor \neg A) \lor (\neg A \lor B)]\) by Lemma 3.1.5 (iii).

• For the inference rule \(A \quad \frac{A \rightarrow B}{B}\) (Modus ponens):
  If \(v \in [A]\) and \(v \in [A \rightarrow B] = ([A] \cup [B])^{\perp \perp}\), then \(v \notin [A]^{\perp}\) and \(v \notin [B]\) by Lemma 3.1.3.

\[\square\]

**Theorem 3.1.7 (Completeness)** Given an \(R\)-duality on \(E \neq \emptyset\) as described above and \(A \in \text{PROP}_{\text{CL}}\), if \(\models A\), then \(\vdash_{e} A\).

**Proof**: Indeed, suppose \(\models A\). This means for any assignment \([-\cdot] : \text{PROP}_{\text{CL}} \rightarrow \mathcal{P}(E)\), we have \(v \in [A]\). We know \(A\) is equivalent to a conjunctive normal form \(D_1 \land ... \land D_n\) where \(D_i = l_1 \lor ... \lor l_k\) (\(l_j\) are literals, \(i.e.\) atoms or negation of atoms). Since \(v \in [A]\), then \(v \in [D_1 \land ... \land D_n] = [D_1] \cap ... \cap [D_n]\). So \(v \in [D_i]\) for every \(i\). But since \([D_i] = ([l_1] \cup ... \cup [l_k])^{\perp \perp}\), we have \(v \in [l_1] \cup ... \cup [l_k]\) by Lemma 3.1.3. This means every \([D_i]\) is of the form \(([l_1] \cup ... \cup [p] \cup ... \cup [p]^{\perp} \cup ... \cup [l_k])^{\perp \perp}\) for some atomic proposition \(p\), since it is the only way to have \(v \in [D_i]\) for any interpretation, guaranteed by Corollary 3.1.4 (indeed, if it were not of this form, we could have an interpretation such that for every atomic proposition \(q\), \(v \notin [q]\), and consequently \(v \notin [D_i]\)).

Since we know \(p \lor \neg p\) is provable classically, it follows that every \(D_i\) is provable, and consequently \(D_1 \land ... \land D_n\).

\[\square\]

**Remark 3.1.8**

We observe that choosing a *classical* duality relation that is *not* reflexive is a sufficient condition to interpret classical logic. Moreover, the interpretation of “\(A \lor B\)” becomes simply “\([A] \cup [B]\)”. To make things even simpler, we see that if \(E\) is a singleton, then any formula \(A\) is assigned to one of the only two facts, \(E\) or \(\emptyset\). We then retrieve a semantics *isomorphic* to the traditional *true/false* semantics. However,
this last interpretation has the structure of a Boolean algebra, while we see that in
general, our interpretations are De Morgan algebras (with classical duality relations,
see Section 2.4).

Example 3.1.9 (Classical duality on a group)

Here’s a very concrete particular case of the general setting described above. Let
\((G, \cdot, 1)\) be a group. Define a duality on \(G\) by letting \(\ll a \mid b \gg = a \cdot b\) and taking
the pole \(P = G\setminus\{1\}\). Since every subset of \(G\) is a fact by Proposition 2.5.4, we can
have the following interpretation (where \(A\) is valid if \(1 \in [A])\):

\[
\begin{align*}
\text{atomic proposition } p &\mapsto \text{some subset of } E \\
\top &\mapsto E \\
\bot &\mapsto \emptyset \\
\neg A &\mapsto [A]^\perp \\
A \land B &\mapsto [A] \cap [B] \\
A \lor B &\mapsto [A] \cup [B] \\
A \rightarrow B &\mapsto [A]^\perp \cup [B]
\end{align*}
\]

Observe that since we never used \(\text{associativity}\) of “\(\cdot\)”, our interpretation holds for
any magma \((G, \cdot, 1)\) with unity where every element has an \(\text{inverse}\).

Remark 3.1.10

Notice that when \([A \rightarrow B] = E\), we have \([A]^{-1} \subseteq [B]\), where \([A]^{-1} := \{a^{-1} \mid a \in [A]\}\). Indeed, for all \(x \in E\), if \(x \in [A]^\perp \cup [B]\), then \(x \in [A]^{-1}\) or \(x \in [B]\).
Since \([A]^\perp = E \setminus [A]^{-1}\), this means either \(x \notin [A]^{-1}\) or \(x \in [B]\). Classically
speaking, this is equivalent to saying if \(x \in [A]^{-1}\), then \(x \in [B]\).

In the next interpretation, things will be made so that the \(\text{validity}\) of a proposition
\(A\) will be given when \([A] = E\).
Let $E$ be a non-empty set and take any $R$-duality on $E$, with duality relation $\perp$. As in Section 2.4, let $N := \{ x \in E \mid x \perp E \}$ and $I := \{ x \in E \mid x \perp x \}$.

**Lemma 3.1.11** If $N = I$, then for all $A, B \subseteq E$ we have

(i) \( (A \cap (B \cap A^\perp)^\perp)^\perp = E \);

(ii) \( (B^\perp \cap A^\perp \perp \cap (A \cap B^\perp)^\perp \perp) \perp = E \).

**Proof**: For (i), we have \( (B \cap A^\perp) \subseteq A^\perp \Rightarrow A^\perp \subseteq (B \cap A^\perp)^\perp \Rightarrow (B \cap A^\perp)^\perp \subseteq A^\perp \). Since $A \cap A^\perp \subseteq N$ by Proposition 2.4.10 (iv), a fortiori $A \cap (B \cap A^\perp)^\perp \subseteq N$. Thus the result follows by Proposition 2.4.10 (i).

For (ii), since $A \subseteq A^\perp$, we have $A \cap B^\perp \subseteq A^\perp \perp \cap B^\perp \Rightarrow (A^\perp \cap B^\perp)^\perp \subseteq (A \cap B^\perp)^\perp \Rightarrow (A \cap B^\perp)^\perp \subseteq (A^\perp \cap B^\perp)^\perp = A^\perp \cap B^\perp$ since $A^\perp \cap B^\perp$ is a fact. The result follows by Proposition 2.4.10 (iv) and (i).

**Lemma 3.1.12** Suppose we have an $R$-duality on $E$ such that $N = I$ and for all facts $X, Y, Z \subseteq E$, if $x \in X \cap Y$ and $X \neq Y$, then $x \in Z$ or $x \in Z^\perp$. Then for all facts $A, B, C \subseteq E$, we have

\[ \left( (A \cap (B \cap C^\perp)^\perp \perp \cap (A \cap B^\perp)^\perp \perp \cap (A \cap C^\perp)^\perp \perp ) \right)^\perp = E \]

**Proof**: Let $x \in A \cap C^\perp$, $x \notin N$. If $x \in B$, then $x \notin (A \cap (B \cap C^\perp)^\perp \perp)$ or else we have $x \in B^\perp$, which implies $x \notin (A \cap B^\perp)^\perp$. The result follows using Proposition 2.4.10 (i).
The interpretation of classical logic is given as follows: we have an $R$-duality on $E$ such that

(i) $N = I$ and
(ii) for all facts $X, Y, Z \subseteq E$ if $x \in X \cap Y$ and $X \neq Y$, then $x \in Z$ or $x \in Z^\perp$.

As before, we assign atomic propositions to facts, and

\[
\begin{align*}
\top & \mapsto E \\
\bot & \mapsto E^\perp \\
\neg A & \mapsto [A]^\perp \\
A \land B & \mapsto [A] \cap [B] \\
A \lor B & \mapsto [A]^\perp \cup [B]^\perp = ([A] \cup [B])^\perp \\
A \rightarrow B & \mapsto \neg A \lor B \mapsto ([A] \cap [B]^\perp)^\perp
\end{align*}
\]

By Proposition 3.1.1, every proposition is assigned to a fact, so we can prove soundness:

**Theorem 3.1.13 (Soundness)** Given an $R$-duality on $E \neq \emptyset$ as described by $(\ast)$, if $\vdash_c A$ for $A \in PROP_{CL}$ then for all assignments $[-] : PROP_{CL} \rightarrow \mathcal{P}(E)$, we have $[A] = E$.

**Proof**: We check these axioms as before:

- $[A \rightarrow (B \rightarrow A)] = E$ by Lemma 3.1.11 (i).
- $[(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))] = E$ by Lemma 3.1.12.
- $[(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)] = E$ by Lemma 3.1.12 (ii).

For *modus ponens*, we clearly see that if $[A] = E$ and $[A \rightarrow B] = ([A]^\perp \cup [B])^\perp = E$, then $[B] = E$. Indeed, $([A]^\perp \cup [B])^\perp = E$ is equivalent to saying $([A]^\perp \cap [B]^\perp)^\perp = E$, which means $[A] \cap [B]^\perp = N$. Hence $[B]^\perp = N$, and $[B] = E$. 

\[\square\]
Remark 3.1.14

The previous conditions on the duality given by (∗) seem too strong to ensure the soundness theorem for classical logic. Indeed, in example 3.1.21, the condition (i) \( N = I \) holds, but not (ii). Therefore examples 3.1.16 and 3.1.17 of interpretations of classical propositional calculus illustrate Theorem 3.1.13 but are rather “specialized”.

Example 3.1.15 (Subsets as propositions)

Consider an \( R \)-duality on \( E \neq \emptyset \) with classical duality relation \( \bot_{Id} \). We have \( N = I = \emptyset \) and every \( x \in E \) is either in \( X \subseteq E \) or in its complement. We retrieve the traditional Boolean algebra \( \mathcal{P}(E) \) interpretation where conjunction, disjunction and negation are simply the set theoretic intersection, union and complement.

Example 3.1.16 (Subspaces as propositions)

The \( \mathbb{R} \)-duality on \( \mathbb{R}^2 \) with the scalar product \(< \vec{v} \mid \vec{w} > = v_1w_1 + v_2w_2\), where \( \vec{v} = (v_1, v_2) \), \( \vec{w} = (w_1, w_2) \) and \( v \perp w \iff < v \mid w > = 0 \), also fulfills the requirements for our interpretation: \( N = I = \{ \vec{0} \} \) and for any facts \( V, W \), i.e. subspaces (see Example 2.4.11), if \( V \neq W \), we have \( V \cap W = \{ \vec{0} \} \), and obviously \( \vec{0} = (0, 0) \) is in any subspace. Using Proposition 2.4.17, we get the following interpretation for the disjunction:

\[
[A \vee B] := ([A] \cup [B])_{\bot_{Id}} = [A] + [B]
\]

Example 3.1.17 (Coherence spaces as propositions)

Let \( X = \{a, b, c\} \) be a set and consider an \( \mathbb{N} \)-duality on \( \mathcal{P}(X) \) such that facts are coherence spaces (see Section 2.4). Here \( N \) consists of all singletons and the empty set, and clearly \( N = I \). Suppose \( A \in \mathcal{A} \cap \mathcal{B} \) for distinct facts \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X) \). Notice that \( A \neq X \), otherwise \( A = \mathcal{B} = \mathcal{P}(X) \). If \( A \in N \), then it belongs to every fact. If \( |A| = 2 \), then \( A \in \mathcal{C} \) or \( A \in \mathcal{C}^\perp \) for all \( \mathcal{C} \subseteq \mathcal{P}(X) \) by Proposition 2.4.24. Let’s mention that this also works \( a \ a \ fortiori \) for coherence spaces representing the Booleans, i.e. where \( X = \{t, f\} \) (see Example 2.4.22).
3.1.2 Intuitionistic Logic

Let \( E \) be a non-empty set. Consider \( \mathcal{H} \subseteq \mathcal{P}(E) \) such that \( (\mathcal{H}, \subseteq, \cap, \cup, N, \top, \Rightarrow) \) is a Heyting algebra (see section 2.4). We interpret the formulas of propositional intuitionistic calculus, noted \( PROPIL \), in \( \mathcal{H} \) as follows:

Define \( [-] : PROPIL \rightarrow \mathcal{H} \) such that atomic propositions are mapped to any subset of \( \mathcal{H} \) and other formulas are assigned inductively:

\[
\begin{align*}
\bot & \mapsto N \\
A \land B & \mapsto \llbracket A \rrbracket \cap \llbracket B \rrbracket \\
A \lor B & \mapsto \llbracket A \rrbracket \cup \llbracket B \rrbracket \\
A \rightarrow B & \mapsto \bigcup \{ X \in \mathcal{H} \mid \llbracket A \rrbracket \cap X \subseteq \llbracket B \rrbracket \} \\
\neg A := A \rightarrow \bot & \mapsto \bigcup \{ X \in \mathcal{H} \mid \llbracket A \rrbracket \cap X \subseteq N \}
\end{align*}
\]

In this interpretation, the validity of \( A \) is given when \( \llbracket A \rrbracket = \top \). Heuristically, if we think of \( E \) as a set of explanations, we can see \( \top \) as a sufficient set of elements that can explain or justify the truth of a proposition.

In Section 2.4, we have seen that in some cases we can give a meaning to \( \neg A \) in terms of a polar set \( \llbracket A \rrbracket^\perp \) by defining an appropriate R-duality on \( \top \) (see Propositions 2.4.7 and 2.4.8). An immediate observation is that in our examples of Heyting algebras, the least element \( N \) coincides exactly with our previous definition \( N := \{ x \in E \mid x \bot E \} \). Notice also that when the interpretation of \( PROPIL \) is given on \( \mathcal{H} = \mathcal{P}(E) \), \( \llbracket A \rightarrow B \rrbracket = (E \setminus \llbracket A \rrbracket) \cup \llbracket B \rrbracket \), so the interpretation of \( \llbracket \neg A \rrbracket \) is the complement of \( \llbracket A \rrbracket \), and \( A \lor \neg A \) is always valid. This is the usual Boolean algebra interpretation (since \( \mathcal{H} = \mathcal{P}(E) \) is then a Boolean algebra).

Remark 3.1.18

In our previous interpretation for intuitionistic propositional logic, formulas are not necessarily mapped to facts, e.g. when the Heyting algebra is an arbitrary filter.
or chain. Intuitively, this suggests that in the framework of polarity, intuitionistic propositions may be represented by sets that were not “properly tested”, i.e. that are not biorthogonally closed (see also Remark 2.4.4).

### 3.1.3 Linear Logic

It is a good thing to recall Girard’s *phase semantics* (see [15]) as an historical example of the use of polar sets to express the idea of *negation* in logic.

Basically, the idea is to define a duality on a monoid \((M, \cdot, 1)\) by letting

\[
\lll - \mid - \rrr : M \times M \to M
\]

\[
(m, n) \mapsto m \cdot n
\]

and choose a cyclic pole \(P \subseteq M\). Also, for all \(A, B \subseteq M\), we define

\[
A \cdot B \overset{\text{def}}{=} \{a \cdot b \mid a \in A \text{ and } b \in B\}
\]

Such a structure is called a *phase space* by Girard [8] (when the monoid is commutative).

Originally, Girard’s definition of a polar set (see Section 2.6) didn’t imply that the pole was cyclic in general, so the monoid had to be commutative to ensure this condition. Without a cyclic pole, we would have lost associativity of \(\otimes\), but the drawback of having a commutative monoid is that we restricted phase semantics to the interpretation of commutative linear logic only.

Later, even with a generalized definition of polar sets (for \(E \neq F\)), Girard stuck to the original definition and used a commutative monoid for phase semantics. He simply mentioned that if we wanted to have associativity of \(\otimes\) without commutativity (and retrieve things like Lambek’s syntactic calculus [28]), we could take any non-commutative monoid but choose a cyclic pole (see [15]). Notice that from our definition and the discussion in Section 2.4, an \(R\)-duality on \(E\) forces the pole to be
cyclic (or equivalently that $\perp$ be a symmetric relation).

We define the interpretation $\llbracket - \rrbracket : PROP_{LL} \to \mathcal{P}(M)$ such that constants (i.e. neutral elements respectively for $\&$, $\oplus$, $\otimes$ and $\exists$) are given by

$$
\begin{align*}
\top & \mapsto M \\
0 & \mapsto M^\perp \\
1 & \mapsto P^\perp \\
\bot & \mapsto P
\end{align*}
$$

**Remark 3.1.19**

It is interesting to notice that $P^\perp$ is a submonoid of $M$: clearly $1 \in P^\perp := \{ m \in M \mid \forall p \in P, m \cdot p, p \cdot m \in P \}$. And if $m, n \in P^\perp$, i.e. $m \cdot p, p \cdot m, n \cdot p, p \cdot n \in P$ for all $p \in P$, then $(m \cdot n) \cdot p = m \cdot (n \cdot p) \in P$ and similarly $p \cdot (m \cdot n) \in P$, which means that $m \cdot n \in P^\perp$. Also, $M$ is a fact by Proposition 2.1.5 (i), and so is $P$: Indeed, suppose $m \in P^\perp$. By definition, this means $m \perp P^\perp$, and in particular, $m \perp 1$. Therefore $m \cdot 1 = m \in P$.

Now since we want negation to be involutive, we assign every atomic proposition to facts, i.e. subsets $X \subseteq M$ such that $X^\perp = X$ (with respect to the choice of $P$), and define other formulas of $PROP_{LL}$ inductively:

$$
\begin{align*}
\sim A & \mapsto \llbracket A \rrbracket^\perp \\
A \& B & \mapsto \llbracket A \rrbracket \cap \llbracket B \rrbracket \\
A \oplus B & \mapsto (\llbracket A \rrbracket \cup \llbracket B \rrbracket)^\perp \\
A \otimes B & \mapsto (\llbracket A \rrbracket \cdot \llbracket B \rrbracket)^\perp \\
A \exists B & \mapsto (\llbracket A \rrbracket^\perp \cdot \llbracket B \rrbracket)^\perp \\
!A & \mapsto (\llbracket A \rrbracket \cap I)^\perp \\
?A & \mapsto (\llbracket A \rrbracket^\perp \cap I)^\perp
\end{align*}
$$
where \( \mathcal{I} \) is the set of \textit{idempotent} elements of \( M \) that are in \( P^\perp \).

Define \( A \circ B := \sim A \forall B \). In particular, we have \( A \circ P = \sim A \).

We observe the use of \textit{double negation} in order to ensure closure of facts.

Also, we can show \( \otimes \) is associative (using the property \( [A]^\perp \cdot [B]^\perp \subseteq ([A] \cdot [B])^\perp \), see [8]).

Defining \( A \) to be \textit{valid} when \( 1 \in [A] \), we can show \textit{soundness} and \textit{completeness} in the sense that \( A \) is provable in linear logic if and only if \( 1 \in [A] \) for any interpretation in any phase space (see [8]). The proof of \textit{completeness} is instructive and here is the general idea: if we suppose a proposition \( A \) is valid in any phase space, then in particular consider the \textit{free} commutative monoid \((M, \cdot, \emptyset)\) of \textit{contexts} (multisets of formulas) with “ \( \cdot \)” defined by \textit{concatenation}. We choose the pole \( P = \{ \Gamma | \vdash \Gamma \text{ is provable} \} \). The interpretation goes as follows:

\[
A \mapsto [A] = \{ \Gamma | \vdash \Gamma, A \text{ is provable} \}
\]

It can be shown that \([A]\) is indeed a fact, and since \( \emptyset \in [A] \) by hypothesis, then we have \( \vdash A \).

**Remark 3.1.20**

Notice that if we define a duality on the monoid \((\mathcal{P}(X), \cup, \emptyset)\) with the pole \( P = \{ X \} \), where \( X \) is a non-empty set, the two negations coincide and \( !A = A \), so we retrieve an interpretation of classical logic where facts are \textit{filters}. This comment will be made more explicit in the following example.

**Example 3.1.21 (Classical logic in a phase semantics)**

Let \( X \) be a non-empty set and define a phase space as follows: consider the monoid \((\mathcal{P}(X), \cup, \emptyset)\) and define a duality on \( \mathcal{P}(X) \) such that

\[
A \perp B \iff A \cup B = X
\]
Notice that the set of idempotent elements of the monoid \((\mathcal{P}(X), \cup, \emptyset)\) is \(\mathcal{I} = \mathcal{P}(X)\), hence \([!A] = [?A] = [A]\) in the above interpretation of linear logic.

**Definition 3.1.22** Let \(A \in \mathcal{P}(X)\) for some non-empty set \(X\). The set \(\uparrow(A) = \{A' \in \mathcal{P}(X) \mid A \subseteq A'\}\) is called the principal filter generated by \(A \subseteq X\) (in the poset \((\mathcal{P}(X), \subseteq)\)).

**Proposition 3.1.23** With respect to the above duality on \(\mathcal{P}(X)\), the facts are of the form \(\uparrow(A)\) for \(A \subseteq X\). Moreover, for any \(A \subseteq X\), \(\uparrow(A)^\perp = \uparrow(\overline{A})\).

**Proof**: Clear by definition.

We define the locative product (see [17]) of \(A, B \subseteq \mathcal{P}(X)\) by

\[ A \mathbin{\Box} B \overset{\text{def}}{=} \begin{cases} \{ \Gamma \cup \Delta \mid \Gamma \in A, \Delta \in B \} & \text{if } A, B \neq \emptyset \\
\emptyset & \text{else} \end{cases} \]

**Proposition 3.1.24** Let \(A, B \in \mathcal{P}(X)\) and let \(A = \uparrow(A)\) and \(B = \uparrow(B)\). Then \(A \mathbin{\Box} B = A \cap B\).

**Proof**: Clearly \(A \mathbin{\Box} B = \uparrow(A \cup B)\), and \(A \cap B\) contains every element of \(\mathcal{P}(X)\) that contains both \(A\) and \(B\), i.e., \(A \cap B = \{C \in \mathcal{P}(X) \mid A \cup B \subseteq C\}\).

If we interpret linear logic as above, atomic propositions are filters, and using Proposition 3.1.24 we easily see that:

\[ [1] = \{X\}^\perp = \uparrow(\emptyset) = \mathcal{P}(X) = [\top] \]
\[ [0] = \mathcal{P}(X)^\perp = \uparrow(\emptyset)^\perp = \uparrow(\emptyset) = \{X\} = [\bot] \]
\[ [A \otimes B] = \left( [A] \mathbin{\Box} [B] \right)^\perp = ([A] \cap [B])^\perp = [A] \cap [B] = [A \& B] \]
\[ [A \triangledown B] = \left( [A]^\perp \mathbin{\Box} [B]^\perp \right)^\perp = ([A]^\perp \cap [B]^\perp)^\perp = ([A] \cup [B])^\perp = [A \oplus B] \]

Also, \([A \Rightarrow B] = [A \rightarrow B] = \left( [A] \cap [B]^\perp \right)^\perp\), and we can prove the soundness theorem:
Theorem 3.1.25 (Soundness) Given a duality on $E = \mathcal{P}(X)$ as above, where $X$ is a non-empty set, if $\varphi \in \text{PROP}_{CL}$ is provable (classically) then for all assignments $[\cdot] : \text{PROP}_{CL} \rightarrow \mathcal{P}(E)$ as defined above, we have $[[\varphi]] = E$.

Proof: Since $N = I = \{X\}$, we can use Lemma 3.1.11 and show $[a \to (b \to a)] = E$ and $[(-b \to -a) \to (a \to b)] = E$.

Modus ponens is clear and for $\varphi = (a \to (b \to c)) \to ((a \to b) \to (a \to c))$, we can see that $[[\varphi]] = E$. Indeed, suppose $[[a]] = \uparrow(A)$, $[[b]] = \uparrow(B)$ and $[[c]] = \uparrow(C)$ for some $A, B, C \subseteq X$. Using Proposition 3.1.23, we can show $[[a \to (b \to c)]] = (\uparrow(A) \cap (\uparrow(B) \cap \uparrow(C))^{\perp})^{\perp} = \uparrow(A \cap B \cap C)$ and $[[((a \to b) \to (a \to c))] = \uparrow((A \cup B) \cap \overline{A} \cap C)$. It follows that

$$[[\varphi]] = (\uparrow(A) \cap (\uparrow(B) \cap \uparrow(C))^{\perp})^{\perp}$$

$$= (\uparrow(A \cap B \cap C) \cap \uparrow((\overline{A} \cap B) \cup A \cup \overline{C}))^{\perp}$$

$$= \uparrow((\overline{A} \cap B \cap C) \cup (\overline{A} \cap B) \cup A \cup \overline{C})^{\perp}$$

$$= \uparrow(X)^{\perp}$$

$$= \uparrow(\emptyset) = E$$

Remark 3.1.26

The interpretations presented in this Section were meant to illustrate the idea that propositions in logic may be defined in terms of sets of elements with a notion of validity, i.e. provability, and a meaning of negation as the result of some testing. However, none of these examples seems appropriate to define a proposition as a set of its formal proofs.

3.2 Heyting interpretation

This section illustrates how the notion of polarity as negation arose in Girard’s work in categorical logic, where proofs are morphisms (in a Cartesian closed category).
But before, we present a generalized version for the interpretation of the classical
connectives “and”, “or” as discussed at the beginning of Section 3.1.1.

### 3.2.1 Interpretation of the connectives by testing

In the proposition “A and B”, the universe and testing process for which A has
been defined may be different from B. Hence “A and B” comes from some other
testing that should preserve the previous testings in a sense. Let’s try to make it
clear. Consider an \( R \)-duality on \( E \), with testing map \( \ll - \mid - \gg_E : E^2 \to R \),
pole \( P \subseteq R \) and duality relation \( \perp \subseteq E^2 \), and an \( R' \)-duality on \( E' \) with testing map
\( \ll - \mid - \gg_{E'} : E'^2 \to R' \), pole \( P' \subseteq R' \) and duality relation \( \perp' \subseteq E'^2 \). Next
we consider an \( R'' \)-duality on \( E'' \), with \( \ll - \mid - \gg_{E''} : E''^2 \to R'' \), \( P'' \subseteq R'' \) and
\( \perp'' \subseteq E''^2 \), and two duality monomorphisms \( \iota_E : E \to E'' \) and \( \iota_{E'} : E' \to E'' \).
The interpretation of the connectives is then given by

\[
\text{“A and B”} := \{ \mu \in E'' \mid \mu \perp \iota_E (A \perp) \text{ and } \mu \perp \iota_{E'} (B \perp) \}\]

\[
= \{ \mu \in E'' \mid \mu \perp \iota_E (A \perp) \} \cap \{ \mu \in E'' \mid \mu \perp \iota_{E'} (B \perp) \}
\]

\[
= \iota_E (A \perp) \perp \cap \iota_{E'} (B \perp) \perp
\]

and

\[
\text{“A or B”} := \text{“not ( not A and not B)”}
\]

\[
= (\iota_E (A \perp) \perp \cap \iota_{E'} (B \perp) \perp) \perp
\]

\[
= (\iota_E (A) \cup \iota_{E'} (B)) \perp \perp
\]

where \( \iota_E (A) := \{ \iota_E (a) \mid a \in A \} \).

### 3.2.2 A new look at coherence spaces

Initially motivated by a will to find a categorical interpretation of intuitionistic logic
(inspired by Scott’s domains), the following example led to the discovery of linear
logic.
Example 3.2.1 (Category of coherence spaces)

Consider a category where objects are coherence spaces and morphisms are linear stable maps in the following sense:

Definition 3.2.2 A linear stable map $f$ from $A$ to $B$, where $A, B$ are coherence spaces, satisfies the following properties:

(i) If $A \in A$, then $f(A) \in B$, where $A$ is any element of $A$. In words, it simply says that a clique is mapped to another clique.
(ii) If $A \subseteq A' \in A$, then $f(A) \subseteq f(A')$.
(iii) $f \left( \bigcup_{i \in I} A_i \right) = \bigcup_{i \in I} f(A_i)$, i.e. $f$ preserves union.
(iv) If $A \cup A' \in A$, then $f(A \cap A') = f(A) \cap f(A')$ (stability).

Negation plays a central role in this interpretation and is defined as in section 2.4: given a coherence space $A$, the $N$-duality is defined on $\mathcal{P}(|A|)$, with duality relation $\perp \subseteq \mathcal{P}(|A|)^2$, by

$$A \perp A' \iff |A \cap A'| \leq 1$$

and the polar set of $A$ given by

$$A^\perp = \{A' \subseteq |A| \mid \forall A \in A, A \perp A' \}.$$ 

Two conjunctions coexist in this interpretation. The first one, noted “&” (with), which is related to the usual intuitionistic conjunction “$\land$”, is given as follows, where $A, B$ are coherence spaces:

- $|A \& B| \text{ def } = |A| \times \{1\} \cup |B| \times \{2\}$, i.e. the tokens of $A \& B$ are in the disjoint union of tokens of $A$ and $B$. In other words, the tokens of $A \& B$ are the tokens of both $A$ and $B$, and we have a way to distinguish them.

- The coherence is given by

$$ (a, 1) \bowtie_{A \& B} (b, 2) \text{ for all } a \in |A|, b \in |B| $$
and

\[(a, 1) \trianglerighteq_{\mathcal{A} \& \mathcal{B}} (a', 1) \iff a \trianglerighteq_{\mathcal{A}} a'\]
\[(b, 2) \trianglerighteq_{\mathcal{A} \& \mathcal{B}} (b', 2) \iff b \trianglerighteq_{\mathcal{B}} b'\]

In terms of polarity, we can give an alternative definition of the coherence space \(\mathcal{A} \& \mathcal{B}\). Indeed, consider an \(\mathbb{N}\)-duality on \(E = \mathcal{P}(\lvert \mathcal{A} \rvert)\), an \(\mathbb{N}\)-duality on \(E' = \mathcal{P}(\lvert \mathcal{B} \rvert)\) and an \(\mathbb{N}\)-duality on \(E'' = \mathcal{P}(\lvert \mathcal{A} \& \mathcal{B} \rvert)\), with duality relations \(\perp\), \(\prec\) and \(\approx\) respectively (defined as above). We then consider the two duality embeddings

\[\iota_E : E \rightarrow E''\]
\[A \mapsto A \times \{1\}\]
\[\iota_{E'} : E' \rightarrow E''\]
\[B \mapsto B \times \{2\}\]

Indeed, if \(A \perp A'\), then \(A \cap A'\) contains at most one element; and consequently, \(A \times \{1\} \cap A' \times \{1\}\) contains also at most one element (and similarly for \(B \not\sim B'\)).

We can show

**Proposition 3.2.3** \(\mathcal{A} \& \mathcal{B} = \iota_E (\mathcal{A}_\perp) \triangleq \cap \iota_{E'} (\mathcal{B}_\approx) \triangleq\)

**Proof**: The key idea is to use Proposition 2.4.24: \(\{a, a'\} \in \mathcal{A} \iff \{a, a'\} \notin \mathcal{A}_\perp\), for \(a \neq a'\), which simply means \(a \trianglerighteq_{\mathcal{A}} a'\) iff \(a \not\prec_{\mathcal{A}} a'\) (for \(a \neq a'\)). This means the only two-element sets that are contained in \(\iota_E (\mathcal{A}_\perp)\) are of the form \(\{(a, 1), (a', 1)\}\), where \(a \trianglerighteq_{\mathcal{A}_\perp} a'\), i.e. \(a \not\prec_{\mathcal{A}_\perp} a'\). Hence \(\iota_E (\mathcal{A}_\perp) \triangleq\) contains every two-element set (in \(E''\)) except for those of the form \(\{(a, 1), (a', 1)\}\), where \(a \not\prec_{\mathcal{A}} a'\). A similar reasoning shows that \(\iota_{E'} (\mathcal{B}_\approx) \triangleq\) contains every two-element set (in \(E''\)) except for those of the form \(\{(b, 2), (b', 2)\}\), where \(b \not\sim_{\mathcal{B}} b'\). Therefore \(\iota_E (\mathcal{A}_\perp) \triangleq \cap \iota_{E'} (\mathcal{B}_\approx) \triangleq\) contains all two-element sets in \(E''\) of the form

\[\{(a, 1), (b, 2)\} \text{ for all } a \in \lvert \mathcal{A} \rvert, \ b \in \lvert \mathcal{B} \rvert\]
\[\{(a, 1), (a', 1)\} \text{ for } a \trianglerighteq_{\mathcal{A}} a'\]
The second conjunction, noted "⊗" (times), is closer in spirit to the set theoretic cartesian product:

- $|A \otimes B| \overset{\text{def}}{=} |A| \times |B|.$
- The coherence is given by
  \[(a, b) \succ_{A \otimes B} (a', b') \iff a \prec_A a' \text{ and } b \prec_B b'.\]

Alternatively, given an $N$-duality on $P(|A| \times |B|)$ such that $C \perp C' \iff |C \cap C'| \leq 1,$ we have

**Proposition 3.2.4** $A \otimes B = \{ A \times B \mid A \in A, B \in B \} \overset{\text{def}}{\equiv} \overset{\text{def}}{\equiv}$

**Proof:** It suffices to show that $\{ A \times B \mid A \in A, B \in B \} \overset{\text{def}}{\equiv} \overset{\text{def}}{\equiv}$ is the smallest fact, i.e. the smallest coherence space, that contains exactly the two-element sets of the form $\{(a, b), (a', b')\}$, where $\{a, a'\} \in A$ and $\{b, b'\} \in B$. For all two-element sets $\{a, a'\} \in A, \{b, b'\} \in B$, the set $\{ A \times B \mid A \in A, B \in B \}$ contains sets of the form $\{(a, b), (a, b'), (a', b), (a', b')\}$. Therefore $\{ A \times B \mid A \in A, B \in B \} \overset{\text{def}}{\equiv}$ doesn’t contain the two-element sets that are contained in $\{(a, b), (a, b'), (a', b), (a', b')\}$. By Proposition 2.4.24, this means they are contained in $\{ A \times B \mid A \in A, B \in B \} \overset{\text{def}}{\equiv} \overset{\text{def}}{\equiv}$. Also, by construction, $\{ A \times B \mid A \in A, B \in B \}$ doesn’t contain a set $A \times B$ that has pairs $(a, b), (a', b')$ where $\{a, a'\} \notin A$ or $\{b, b'\} \notin B$. Therefore the two-element sets of the form $\{(a, b), (a', b')\}$, where $\{a, a'\} \notin A$ or $\{b, b'\} \notin B$, are in the polar set, and consequently they are not in $\{ A \times B \mid A \in A, B \in B \} \overset{\text{def}}{\equiv} \overset{\text{def}}{\equiv}$ by Proposition 2.4.24. □
According to our two conjunctions, we can then define two disjunctions \( \oplus \) and \( \neg \) by De Morgan’s property:

\[
\mathcal{A} \oplus \mathcal{B} \overset{\text{def}}{=} (\mathcal{A} \perp \& \mathcal{B} \perp) = (\iota_E \mathcal{A}) \cup (\iota_{E'} \mathcal{B})
\]

\[
\mathcal{A} \neg \mathcal{B} \overset{\text{def}}{=} (\mathcal{A} \perp \otimes \mathcal{B} \perp) = (\iota_E \mathcal{A}) \cup (\iota_{E'} \mathcal{B})
\]

And a linear implication \( \rightarrow \):

\[
\mathcal{A} \rightarrow \mathcal{B} \overset{\text{def}}{=} \mathcal{A} \neg \mathcal{B}
\]

We can retrieve the usual intuitionistic implication in terms of linear implication:

\[
\mathcal{A} \rightarrow \mathcal{B} \overset{\text{def}}{=} \mathcal{A} \neg \mathcal{B} = (\mathcal{A} \perp \otimes \mathcal{B} \perp)
\]

where the coherence space \( \mathcal{A} \) is given by

- \( \mathcal{A} \) is a set of types.
- \( \mathcal{A} = \{ A \in \mathcal{A} \mid \text{A is finite} \} \)

**Remark 3.2.5**

It is important to observe that at the categorical level, coherence spaces (or probabilistic or quantum coherence spaces) were originally meant to describe types. For example, for \( X = \{ t, f \} \), the coherence space \( \{ \emptyset, \{ t \}, \{ f \} \} \) represents the Booleans \([10]\) (p.55). Again, it is not clear whether these objects are appropriate candidates to define propositions as a set of proofs.
Chapter 4

Girard’s hell

In [15], Girard introduced three “enfers”\(^1\) \((-1, -2, -3)\) where logics can be studied. Tarskian semantics (see Section 3.1) exemplified the first underground \((-1)\), where we can at most tell the provability of a proposition, \(i.e.\) if it is provable or not. According to Girard [20], if we can say that the proposition “\(A\) or \(B\)” is provable in the first underground, we should not be able to tell which one of \(A\) or \(B\) is. Said differently, if an element is meant to represent the interpretation of “\(A\) or \(B\)”, then we have lost the information whether it comes from the set \(A\) or \(B\). Therefore in this underground it is reasonable to think of the set of elements identified with a proposition as a set of models or observations, but not as formal proofs.

Heyting interpretations characterize the second underground \((-2)\). At this level, we can “dig” further towards a more accurate description of logic by interpreting the structure of proofs, rather than provability, which allows to distinguish different proofs of a same formula. Categories offer examples that live in this underground, but we must not think categorical logic is exclusive to this level. Indeed, although we may view the Heyting level as given by a category where arrows are proofs, the poset reflection of this category lives at the Tarski level, \(i.e.\) the first underground.

\(^1\)Undergrounds in [15], or infernos in [21]
CHAPTER 4. GIRARD’S HELL

The third underground is characterized by a change of paradigm in the study of logic where, according to Girard [14], instead of “studying the rules of logic, we study the logic of the rules”. This level focuses on the interaction of proofs (e.g. cut-elimination). In [20] and [21], Girard split this third underground in two - one renamed the “third” underground (−3) and the other one the “fourth” (−4). For example, in this new third underground, our interpretation of logic can be seen as some sort of game “proofs of A vs proofs of not A” and we are interested in evaluating the result of the interaction, i.e. in knowing which one is a winner (a correct proof). In the fourth underground, we focus on how the rules are established. At this level, transcendental syntax\(^2\) refers to an idealized language that can explain the structure of logic. Proofs are elements of a universe \(E\), called epistates. Epistates correspond to some strategies that interact with each other in a “game” without rule or winner. Now some of these strategies are not really trying to prove anything, but only to forbid others, so the notion of proof is somewhat liberalized (we can call them simply tests or paraproofs). Therefore epistates play a more general normative (or deontic [21]) role, i.e. they establish which strategies are allowed and which ones are not, based on a norm or standard given by the definition of the testing process. This said, given an \(R\)-duality on \(E\), we call dichology ("dichologie" in French) a set \(A\) of epistates that is a fact. The dichology \(A^\perp\) plays the role of some forbidden space for \(A\), and vice versa. In other words, we can say that \(A^\perp\) and \(A\) define what strategies their dual is allowed to play. Intuitively, a dichology \(A\) corresponds to a proposition, while \(A^\perp\) is its negation. Table 1 summarizes the terminology used by Girard.

\(^2\)"To find the - modestly some - hypotheses making logic possible, this is transcendental syntax.” ([11], p. 7)
CHAPTER 4. GIRARD’S HELL

4.1 Towards existentialism

What follows is inspired by the philosophical considerations of Girard [20].

In philosophy, essentialism suggests that there is a pre-existing truth that can judge the validity of our statements or reasoning (this is sometimes called pure essence). This notion is omnipresent in most situations of duality (refer to the previous examples in Section 2.3):

- For propositional logics, theories are judged by truth assigments (valuations);
- for scientific theories (in the sense of Popper), observations or experiments/tests act as judges;
- for mathematics, theories are judged by mathematical objects, i.e. models along with Tarski’s definition of truth.

Even if Popper offered a severe critique to essentialism in metaphysics (by imposing that the laws of a theory had to be tested), there is still a predominance of truth (given by the tests) over theories. But as Girard points out, what if we put the quality of the test in perspective ([15], p.31), i.e. its validity, then who’s judging the judge? In logic, this suggests seeking a way to define a testing process that is not one-way only, where semantics (essence) tests and syntax (existence) is tested. According to Girard, we need a symmetric process, where the “judge is judged in return”.

<table>
<thead>
<tr>
<th>Epistates</th>
<th>Intuitive meaning</th>
<th>In Ludics</th>
<th>In GoI</th>
</tr>
</thead>
<tbody>
<tr>
<td>Judge, test, trial, épreuve, paraproof, strategy</td>
<td>Design</td>
<td>Project</td>
<td></td>
</tr>
<tr>
<td>Dichology</td>
<td>Set of epistates equal to its biorthogonal (i.e. a fact)</td>
<td>Behaviour</td>
<td>Conduct</td>
</tr>
</tbody>
</table>

Table 1: Terminology of Girard’s transcendental syntax
Examples presented in this work showed us that such a viewpoint is possible in the framework of polarity, if we abstract to a testing process $\ll x \mid y \gg$ where we can’t distinguish which one of $x$ or $y$ should be predominant, i.e. where syntax and semantics live side by side according to Girard. This said, we could ask for more: that all tests live in the same universe, i.e. that the $R$-duality is defined on some set $E$. Girard [12] calls such a duality between homogeneous objects a monist duality. Now we could have that everything lives in semantics (see Section 3.1), but following Girard, logic is about syntax [20], hence we should think of our universe $E$ as a set of proofs. So instead of having the traditional paradigm where a proof of $A$ is tested by a model for $\neg A$ (and vice versa), we aim for a monist duality where a proof of $A$ is tested by a proof of $\neg A$. Hence proofs are not tested with a notion of pre-existing semantics, e.g. truth values, but with other (para-) proofs (those of the negation).

We see that this last approach leads us, philosophically speaking, to existentialism. By analogy, an existentialist viewpoint sees the human being as being defined by interaction with his environment, as opposed to a predetermined meaning (the essentialist viewpoint). Therefore we want a notion of proof which is defined by some internalized interactions, and this can be achieved technically using polarity.

In [15] and [17], Girard mentioned that some definitions can be desessentialized, in the sense, it seems, that they can be expressed equivalently by a fact $A \subseteq E$, i.e. $A^{\downarrow \downarrow} = A$, given by an $R$-duality on $E$. Let's call “desessentialized objects” such objects that can be expressed as facts in Girard’s sense.

### 4.1.1 Examples of desessentialized objects

Here are some leading examples given by Girard:

- Coherence spaces, probabilistic coherence spaces and quantum coherence spaces (see Examples 2.4.22, 2.4.29 and 2.4.31).
This one is mentioned in [17] (p. 413): Given the real vector space \( \mathbb{R}^X \) with scalar product \( \langle f \mid g \rangle = \sum_{x \in X} f(x)g(x) \), the positive maps \( \mathbb{R}^+_X \) can be desessentialized by the following \( R \)-duality on \( \mathbb{R}^X \):

\[
f \perp g \iff \langle f \mid g \rangle \in [0, +\infty]
\]

\( \mathbb{R}^X \) is a fact; moreover, it is self-polar, i.e. \( (\mathbb{R}^X)_+^\perp = \mathbb{R}^X \).

Moreover, according to Girard ([17], p. 304), the following definition cannot be desessentialized:

**Definition 4.1.1 (Atomic coherence)**

Let \( X \) be an enumerable set. The atomic coherence \( \mathcal{A} \) (of a hypercoherence in [5], or \( \omega \)-coherence in [1]) is a subset of \( \mathcal{P}^*_\text{fin}(X) \) such that \( \{x\} \in \mathcal{A} \) for all \( x \in X \), where \( \mathcal{P}^*_\text{fin}(X) \) contains every finite subset of \( X \) except for \( \varnothing \).

The linear negation is defined by \( \sim \mathcal{A} := (\mathcal{P}^*_\text{fin}(X) \setminus \mathcal{A}) \cup \{ \{x\} \mid x \in X \} \).

But atomic coherences can be expressed as facts given an appropriate \( R \)-duality on \( \mathcal{P}^*_\text{fin}(X) \):

**Proposition 4.1.2** Given a \( \{0,1\} \)-duality on \( \mathcal{P}^*_\text{fin}(X) \) with testing map

\[
\ll A \mid B \gg := \begin{cases} 
1 & \text{if } |A \cap B| = 1 \\
1 & \text{if } |A \cap B| < |A \cup B| \\
0 & \text{else}
\end{cases}
\]

and pole \( P = \{1\} \), then the facts are exactly the atomic coherences.

**Proof**: Notice that given any set \( \mathcal{S} \subseteq \mathcal{P}^*_\text{fin}(X) \), the polar set \( \mathcal{S}^\perp \) contains every singleton, since for any \( Y \in \mathcal{S} \) and \( x \in X \), either \( |Y \cap \{x\}| = 1 \) or \( |Y \cap \{x\}| = 0 < |Y \cup \{x\}| \). Therefore if \( \mathcal{A} \subseteq \mathcal{P}^*_\text{fin}(X) \) is a fact, it is also an atomic coherence by definition. Conversely, if \( \mathcal{A} \) is an atomic coherence, then clearly it can be expressed by the following polar set : \( (\mathcal{P}^*_\text{fin}(X) \setminus \mathcal{A})^\perp \). \( \square \)
This said, it seems that facts can be used to express more definitions than simply those given by Girard. Let’s call “objects by testing” such objects that are exactly the facts $A \subseteq E$ given an $R$-duality on $E$.

### 4.1.2 Examples of objects by testing

- Let $E$ be a set. Then a set $X$ is a subset of $E$ if and only if $X$ is a fact given an $R$-duality on $E$ with any classical duality relation $\sqsubset_{\varphi}$.

- In Example 2.4.11, given a vector space of finite dimension, subspaces are exactly the facts when $N = \{0\}$.

- For a finite cyclic group, subgroups are exactly the facts with the following $R$-duality on the group:

Let $(G, \cdot, 1)$ be a finite cyclic group of order $n$. Define an $R$-duality on $G$ such that

$$g \sqsubset h \iff n \equiv 0 \mod (o(g) o(h))$$

where $o(g)$ is the order of $g \in G$. Notice that the duality relation is symmetric, since $o(g) o(h) = o(h) o(g)$. We recall that $o(g)$ is simply $|\langle g \rangle|$, the cardinal of the subgroup generated by $g$ (see [3]). Also, the following results are well-known [3]:

**Lemma 4.1.3 (Lagrange’s Theorem)** The order of any subgroup of a finite group $G$ is a divisor of the order of $G$.

**Lemma 4.1.4 (Fundamental Theorem of Finite Cyclic Groups)** Let $G$ be a cyclic group of finite order $n$.

(i) Every subgroup of $G$ is cyclic;

(ii) For each positive divisor $d$ of $n$, $G$ has exactly one subgroup of order $d$.

**Proposition 4.1.5** Given the duality above, the facts are exactly the subgroups of $G$. 
Proof: Suppose $H' \subseteq G$ is a fact. Then $H' = H^\perp$ for some subset $H \subseteq G$. If $H = \emptyset$, then $H^{\perp \perp} = G$, so $H'$ is a subgroup. Otherwise, we have to check the following:

- $H' \neq \emptyset$, since $1 \in H'$. Indeed, $1 \perp g$ for all $g \in G$: clearly $o(1) = |\langle 1 \rangle| = 1$, and since $\langle g \rangle$ is a subgroup, it follows from Lagrange’s theorem that $|\langle g \rangle| = o(g)$ divides $n$; therefore $n \equiv 0 \mod (o(1) o(g))$.

- Suppose $g \in H'$. Then for all $h \in H$, $g \perp h$, i.e. $n \equiv 0 \mod (o(g) o(h))$. But $o(g) = o(g^{-1})$: indeed, $g^{o(g)} = 1$ is equivalent to $1 = (g^{-1})^{o(g)}$.

- Suppose $g, g' \in H'$. Then for all $h \in H$, $g \perp h$ and $g' \perp h$, i.e. $o(g) o(h)$ and $o(g') o(h)$ both divide $n$. Suppose that $m$ is the least common multiple of the orders of every element of $H$. Then if $m \nmid n$, clearly $o(x) = 1$ for all $x \in H'$, which means $x = 1$ and $H'$ is the trivial group. But if $m \mid n$, say $n = km$, then $o(x) \mid k$ for all $x \in G$. Moreover, we know from the previous lemma there exists a (unique) cyclic subgroup of order $k$, hence generated by some element $g'' \in G$. Therefore $g'' \in H'$, and since $o(g) \mid k$ and $o(g') \mid k$, it follows that $\langle g \rangle$ and $\langle g' \rangle$ are cyclic subgroups of $\langle g'' \rangle$. Hence $g \cdot g' \in \langle g'' \rangle$, which means $|\langle g \cdot g' \rangle|$ divides $k$. We conclude that $o(g \cdot g') o(h)$ divides $n$ for all $h \in H$, and consequently that $g \cdot g' \in H'$.

For the converse, suppose $H \subseteq G$ is a cyclic subgroup of $G$ such that $|H| = k$. Then $H^\perp$ is also a cyclic subgroup (since it is a fact), and it is such that $|H^\perp| |H| = |G|$. Now $H^{\perp \perp}$ is again a cyclic subgroup and $|H^{\perp \perp}| |H^\perp| = |G|$, which means $|H^{\perp \perp}| = |H| = k$. And since $G$ has a unique subgroup of order $k$ by our lemma, we conclude that $H^{\perp \perp} = H$. 

We may wonder to what extent we can define a duality relation so that the facts are exactly the objects we would like to describe. Here is an example that shows it is not always possible:

**Proposition 4.1.6** Let $(G, \cdot, 1)$ be the dihedral group of order 8. There exists no $R$-duality on $G$ such that the facts are exactly the subgroups of $G$. 
Proof: We recall the Cayley table for $G$ (where $ab$ means $a \cdot b$):

\[
\begin{array}{cccccccc}
\cdot & 1 & b & a & a^2 & a^3 & ab & a^2b & a^3b \\
1 & 1 & b & a & a^2 & a^3 & ab & a^2b & a^3b \\
b & b & 1 & a^3b & a^2b & ab & a^3 & a^2 & a \\
a & a & ab & a^2 & a^3 & 1 & a^2b & a^3b & b \\
a^2 & a^2 & a^2b & a^3 & 1 & a & a^3b & b & ab \\
a^3 & a^3 & a^3b & 1 & a & a^2 & b & ab & a^2b \\
ab & ab & a & b & a^3b & a^2b & 1 & a^3 & a^2 \\
a^2b & a^2b & a^2 & ab & b & a^3b & a & 1 & a^3 \\
a^3b & a^3b & a^3 & a^2b & ab & b & a^2 & a & 1 \\
\end{array}
\]

We proceed by contradiction: suppose there is a duality relation $\perp$ such that facts are exactly the subgroups of $G$. Here is the lattice of subgroups ordered by inclusion:

Since $\{1\}$ is contained in every subgroup, then $\{1\}^\perp$ must contain every subgroup by Proposition 2.4.3 (ii). Hence $\{1\} \triangleright G$. Now consider any subgroup of order 2. None can be self-dual. Indeed, without loss of generality, suppose $\{1, b\}^\perp = \{1, b\}$. Then since $\{1, b\} \subseteq \{1, a^2, b, a^2b\}$, we must have $\{1, a^2, b, a^2b\}^\perp \subseteq \{1, b\}^\perp = \{1, b\}$ by Proposition 2.4.3 again. This forces $\{1, a^2, b, a^2b\} = \{1\}$, hence $\{1, a^2, b, a^2b\} \triangleright \{1\}$, which contradicts what we said earlier. An analogous reasoning shows that the subgroups of order 4 are not self-dual. Moreover, two distinct subgroups of order 2 cannot be in correspondence, since they are each included in a subgroup of order 4, and the polar set of such group should then be included in a two-element set, which
leads to obvious contradictions. Therefore the five subgroups of order 2 should be in
correspondence with the three subgroups of order 4, which is not possible.

4.2 Examples of Transcendental syntaxes

4.2.1 Ludics

Here epistates are called designs (“desseins”) and dichologies are behaviours (“com-
portements”). We may think of a design as the locative structure of a proof presented
in sequent calculus ([14], p.6), where locative means that each formula is given a dis-

tinct “address”. Said differently, a design can be understood as a “skeleton of a
sequent calculus derivation, where we do not manipulate formulas, but their loca-
tions (the addresses where the formulas are stored)” ([7], p.713). Thus the syntax is
some abstracted form of sequent calculus that is closer to Böhm trees [7].

Let $E$ be the set of designs. Define an $R$-duality on $E$ by letting

$$\ll< - | - \gg : E \times E \rightarrow R = \{ \text{Dai}, \text{Fid} \}$$

such that $\ll\mathcal{D} | \mathcal{E}\gg$ is the “result” of a normalisation procedure, i.e. some kind of
cut-elimination for ludics, given by the interaction between the rules (or actions) of
$\mathcal{D}$ and $\mathcal{E}$. Designs are like programs that test other behaviours’ designs, but can be
tested in return [13]. By analogy, we may view this reciprocal testing as some kind
of game [7]. When $\ll\mathcal{D} | \mathcal{E}\gg = \text{Dai}$, this means the last rule played (by either $\mathcal{D}$
or $\mathcal{E}$) is the daimon:

$$\begin{array}{c}
\Gamma \vdash \Lambda \\
\end{array}$$

This rule corresponds to “immediate terminaison” [14], which means the normalisa-
tion converges. When the normalisation process diverges, i.e. never ends (like an
infinite loop), we have \( \llcorner \mathcal{D} \mid \mathcal{E} \rrcorner = \mathfrak{F} \mathfrak{D} \) which means we admit the rule “\( \Omega \)”: 
\[
\vdash _{\mathcal{A}} \Omega 
\]

Let \( P = \{ \mathfrak{D} \mathfrak{a} \} \) be the pole, \( i.e. \) \( \mathcal{D} \perp \mathcal{E} \iff \llcorner \mathcal{D} \mid \mathcal{E} \rrcorner = \mathfrak{D} \mathfrak{a} \). This said, when we get a situation such that for some \( \mathcal{D} \in \mathcal{A} \) and \( \mathcal{E} \in \mathcal{A}^\perp \), \( \llcorner \mathcal{D} \mid \mathcal{E} \rrcorner = \mathfrak{F} \mathfrak{D} \), this means that we “don’t know” if \( \mathcal{A} \) (or \( \mathcal{A}^\perp \)) contains a winning strategy. Also, when \( \llcorner \mathcal{D} \mid \mathcal{E} \rrcorner = \mathfrak{D} \mathfrak{a} \), we know at least one of \( \mathcal{A} \) or \( \mathcal{A}^\perp \) gave up in the testing process.

Therefore a fact \( \mathcal{A} \) and its polar \( \mathcal{A}^\perp \) are behaviours that interact in consensus, \( i.e. \) that contain designs that object to the possibility of an infinite dispute with their polar set’s designs. We see that \( \mathcal{A} \) defines the “rule for \( \mathcal{A}^\perp \)” and vice versa ([14], p.33).

As mentioned before, some epistates (designs) do not really prove anything: they only play a normative role. We define a “proof” as a winning design \( \mathcal{D} \), \( i.e. \) such that for all \( \mathcal{E} \in \mathcal{A}^\perp \), \( \llcorner \mathcal{D} \mid \mathcal{E} \rrcorner = \mathfrak{D} \mathfrak{a} \), and the \( \mathfrak{x} \)-rule is played by \( \mathcal{E} \). Also, we say \( \mathcal{A} \) is true if it contains such a winning design.

The interpretation of “&” is given by the intersection of negative and disjoint behaviours (in the sense of Definition 54, p. 335, and Proposition 26 in [17], p. 344), while “\( \oplus \)” was the biorthogonal of the union of positive disjoint behaviours. It is worth mentioning that this latter definition was generalized by Girard (see [17], p. 344):

\[
\text{“} \mathcal{A} \oplus \mathcal{B} \text{” := } (\varphi(\mathcal{A}) \cup \psi(\mathcal{B}))^{\perp \perp}
\]

where \( \varphi, \psi \) are delocations of loci, \( i.e. \) some kind of injective maps (see [14], p. 34). This idea seems to have been originally conceived by Girard to express a spiritual

---

3The notation comes from \( \lambda \)-calculus ([17], p.315). Indeed consider the \( \lambda \)-term \( \lambda x.xx \). When applied to itself, and by using \( \beta \)-reduction, we get \( \Omega \):

\[
\begin{align*}
\Omega &:= (\lambda x.xx)(\lambda x.xx) \\
&\xrightarrow{\beta} (xx)[(\lambda x.xx)/x] \\
&= (\lambda x.xx)(\lambda x.xx) = \Omega
\end{align*}
\]
interpretation of the connectives, \textit{i.e.} a construction that is closer to the (traditional) categorical interpretation. But it is also close in a sense to our general interpretation of Section 3.2.1; the interpretation in \textit{Ludics} may then be seen as a particular case when \( \varphi \) and \( \psi \) are the \textit{identity} map.

### 4.2.2 Towards Geometry of Interaction

**Example 4.2.1 (Partitions as epistates)**

Let \( I \) be a (non-empty) finite set.

Consider \( E \subseteq \mathcal{P}(\mathcal{P}(I)) \) the set of (non-empty) partitions of \( I \).

We define an \( R \)-duality on \( E \) such that, for all \( \mathcal{P}, \mathcal{Q} \in E \), we have

\[
\mathcal{P} \perp \mathcal{Q} \Leftrightarrow G[\mathcal{P}, \mathcal{Q}] \text{ is a tree}
\]

where \( G[\mathcal{P}, \mathcal{Q}] \) is the \textit{bipartite graph} such that \textit{vertices} are elements of \( \mathcal{P} \) and \( \mathcal{Q} \), and an \textit{edge} between \( X \in \mathcal{P} \) and \( Y \in \mathcal{Q} \) is given by an element of \( X \cap Y \).

The duality defined above can be used to interpret proof-nets for linear logic. Indeed, suppose every literal of a formula is assigned to a distinct \textit{location} \( i \in I = \{1, 2, 3, \ldots, n\} \). Moreover, different instances of the same literal are assigned to different locations, so \textit{resources} are taken into account.

The basic idea is to assign a partition of \( I \) to the \textit{proof} of a formula. Following Danos and Regnier \cite{DanosRegnier93}, a derivation of the shape

\[
\vdash A_1, A_2, \ldots, A_n \quad \vdash B_m, B_{m+1}, \ldots, B_{m+k} \\
\vdash D(A_1, A_2, \ldots, A_n, B_m, B_{m+1}, \ldots, B_{m+k})
\]

corresponds to the partition \( \{\{1, 2, \ldots, n\}, \{m, m+1, \ldots, m+k\}\} \), while the derivation

\[
\vdash A_1, A_2, \ldots, A_n \\
\vdash D'(A_1, A_2, \ldots, A_n)
\]
corresponds to \( \{1, 2, \ldots, n\} \). An element of the partition corresponds to the *linked* literals in the proof-net. Let’s take an example to illustrate the discussion that follows:

Let \( \varphi := A \otimes ((\sim B \otimes C) \triangledown D) \).

There are two sequent calculus *proofs*:

\[
\begin{align*}
\frac{\vdash \sim B \quad \vdash C, D}{\vdash A} & \quad \frac{\vdash \sim B \otimes C, D}{\vdash (\sim B \otimes C) \triangledown D} \quad \text{and} \quad \frac{\vdash \sim B, D \quad \vdash C}{\vdash A} & \quad \frac{\vdash \sim B \otimes C, D}{\vdash (\sim B \otimes C) \triangledown D}
\end{align*}
\]

Given that 1, 2, 3, 4 are the respective locations for \( A, \sim B, C, D \), the proofs above correspond respectively to the partitions \( P = \{\{1\}, \{2\}, \{3, 4\}\} \) and \( P' = \{\{1\}, \{2, 4\}, \{3\}\} \).

The proof-net for the first derivation is

![Proof-net](attachment:image.png)
and for the second one

\[
\begin{array}{c}
A \\
\sim B \\
\sim B \otimes C \\
\sim B \otimes C \\
A \otimes((\sim B \otimes C) \otimes D)
\end{array}
\]

where dot lines represent the choices for the switches ("interrupteurs").

Now the interesting feature of this interpretation comes when we look at the negation:

Let \( \sim \varphi := \sim A \otimes ((B \otimes C) \otimes \sim D) \).

The proofs are

\[
\begin{align*}
\vdash \sim A, B, \sim C & \quad \vdash D \\
\vdash \sim A, (B \otimes C) \otimes \sim D & \quad \vdash A, (B \otimes C) \otimes \sim D
\end{align*}
\]

with partitions \( Q = \{\{1, 2, 3\}, \{4\}\} \) and \( Q' = \{\{1, 4\}, \{2, 3\}\} \) which correspond to the proof-nets

\[
\begin{array}{c}
\sim A \\
B \\
B \otimes C \\
(B \otimes C) \otimes \sim D \\
\sim A \otimes ((B \otimes C) \otimes \sim D)
\end{array}
\]
and

\[
\sim A \bowtie (B \bowtie C) \otimes \sim D
\]

If we look at the possible choices of switches in the last proof-net, we get \{l_1, l_2\}, \{l_1, r_2\}, \{r_1, l_2\} and \{r_1, r_2\}. Forgetting the links between literals, we see that each choice divides \sim A, B, \sim C, \sim D in some connected components. For example, given \{l_1, l_2\}, we get the partition \{\sim A\}, \{\sim C\} and \{B, \sim D\} which means B and \sim D are connected:

\[
\sim A \bowtie (B \bowtie C) \otimes \sim D
\]

In terms of locations, this corresponds to the partition \mathcal{P}' above, which was a proof of \varphi. In general, we observe that a choice of switch in the proof-net of a formula corresponds to a proof of its negation [17].

Therefore, with the duality defined above, we have

\[
\{\mathcal{P}, \mathcal{P}'\} \cong \{\mathcal{Q}, \mathcal{Q}'\} \quad \text{and} \quad \{\mathcal{P}, \mathcal{P}'\} = \{\mathcal{Q}, \mathcal{Q}'\}^\perp
\]

which means the set of proofs of \varphi is a fact, corresponding to the proofs of \sim \varphi.

Now we can show the following “bipolar” theorem due to Girard in [17] (p.445):
Theorem 4.2.2 Let $I, J$ be non-empty sets, $I \cap J = \emptyset$. Given an $R$-duality on $E$ and an $R'$-duality on $E'$ as above, where $E, E'$ are the sets of partitions of $I$ and $J$ respectively (with duality relations $\perp$ and $\parallel$ respectively), let $X \subseteq E$ and $Y \subseteq E'$ be facts.

Then the locative product $X \uplus Y \stackrel{\text{def}}{=} \{P \cup Q \mid P \in X, Q \in Y\}$ is a fact with respect to the $R''$-duality on $E''$, where $E''$ is the set of partitions of $I \cup J$ (given by the duality relation $\approx$).

Proof: It suffices to show $(X \uplus Y) \parallel \subseteq X \uplus Y$. Let $R \in (X \uplus Y) \parallel$ be a partition of $I \cup J$. Consider arbitrary partitions $S \in X^\perp$ of $I$ and $T \in Y^\parallel$ of $J$. Let $A_i \in S$ and $B_j \in T$ be the sets that contain $i \in I$ and $j \in J$ respectively. Define $S \sqcup_{ij} T := (S \setminus \{A_i\}) \cup (T \setminus \{B_j\}) \cup \{A_i \cup B_j\}$. In words, $S \sqcup_{ij} T$ is a partition of $I \cup J$ that consists of elements of $S$ and $T$ except for $A_i$ and $B_j$ that are “glued” together. Clearly, $S \sqcup_{ij} T \subseteq (X \uplus Y) \parallel$. Also, for all $i \in I, j \in J$, we have $R \parallel S \sqcup_{ij} T$. Now $R$ cannot contain a set (a part) that contains elements both from $I$ and $J$, say $i \in I$ and $j \in J$, since it would create a cycle with the “glued” part of $S \sqcup_{ij} T$. Therefore $R = P \cup Q$, with $P \in X$ and $Q \in Y$. It follows that $R \in X \uplus Y$. \hfill \square

This result is very interesting: it allows for an interpretation of the multiplicatives for linear logic.

$$\leadsto A \iff \llbracket A \rrbracket^\perp$$

$$A \otimes B \iff \llbracket A \rrbracket \uplus \llbracket B \rrbracket$$

with $A \nabla B := \leadsto (A \otimes \leadsto B)$ and $A \leftarrow B := \leadsto (A \otimes \leadsto B)$.

However, we need to keep in mind that when we have $\llbracket A \rrbracket^\perp$, the $R$-duality is defined on the set of partitions of the same set $I$ as those of $\llbracket A \rrbracket$.

The last example gives us an important illustration of the notion of tests or trials (“épreuves”), also known as paraproofs, which are a generalization of proofs. Partitions are tests: in particular $\mathcal{P}$ and $\mathcal{P}'$ are trials that succeeded when tested with...
Q and Q′ (the choices of switches) with respect to the duality. Notice that none of the sequent calculus derivations is necessarily a proof; each branch starts with a rule ⊢ Γ which may not be of the axiom form ⊢ A, ∼ A. This liberalized sequent calculus makes use of a rule similar to the $\mathcal{K}$-rule in Ludics (see Section 4.2.1).

Example 4.2.3 (Permutations as epistates)

Epistates originally arose in the form of permutations in Girard’s work [9]. Let $I \neq \emptyset$ be a finite set. Define a duality on $\text{Sym}(I)$, the set of permutations of $I$, by letting

$$< - | - > : \text{Sym}(I) \times \text{Sym}(I) \rightarrow \text{Sym}(I)$$

$$(\sigma, \tau) \mapsto \sigma \tau$$

and the pole $P = \{\text{cyclic permutations}\}$. In other words, we have

$$\sigma \prec \tau \iff \sigma \tau \text{ is cyclic}$$

where $\prec \subseteq \text{Sym}(I)^2$ is the duality relation. But we need to check that $\prec$ is symmetric, so suppose $\sigma \prec \tau$. This means for any $i \in I$, $\sigma \tau(i), (\sigma \tau)^2(i), \ldots, (\sigma \tau)^{|I|}(i)$ are pairwise distinct. We conclude that $\tau \prec \sigma$, i.e. $\tau \sigma(i), (\tau \sigma)^2(i), \ldots, (\tau \sigma)^{|I|}(i)$ are also pairwise distinct. Indeed, suppose $(\tau \sigma)^k(i) = (\tau \sigma)^{k'}(i)$ for $k \neq k'$. Then $\sigma(\tau \sigma)^k(i) = \sigma(\tau \sigma)^{k'}(i)$, which implies $(\sigma \tau)^k \sigma(i) = (\sigma \tau)^{k'} \sigma(i)$, i.e. $(\sigma \tau)^k(j) = (\sigma \tau)^{k'}(j)$ for some $j \in I$, which contradicts our hypothesis.

In [9], a cyclic permutation, i.e. with exactly one cycle, was called a longtrip. This idea can be illustrated as follows: Consider the cut of the proof-nets for $\varphi$ (corresponding to the partition $\mathcal{P}$) and $\sim \varphi$ (corresponding to $Q'$) in the previous example. The normal form (see [15] for the procedure used for normalization) is then given by

$$A \sim A \sim D \sim C \sim C \sim B \sim B$$

If we let $\sigma = (1)(2)(3 4)$ and $\tau = (1 4)(2 3)$, and recall that $A, B, C, D$ (and $\sim A, \sim B, \sim C, \sim D$) have respective locations 1, 2, 3, 4, then we could imagine a
particle traveling throughout the proof-net and coming back to its starting location by following the cyclic permutation given by $\sigma \tau = (1 \ 3 \ 2 \ 4)$.

An interesting feature of the last example is that the duality on the set of permutations takes into consideration the normalization process. Now Geometry of Interaction (GoI) can be viewed as a generalization of the previous approach to full linear logic, \textit{i.e.} MALL with exponentials [21]. Epistates take the form of unitary operators on a Hilbert space. The definition of the duality evolved along with different versions of GoI. For example, the original definition

$$U \perp V \Leftrightarrow UV \text{ nilpotent}$$

later became

$$U \perp V \Leftrightarrow \det(I - UV) \neq 0, 1$$

As we end this Chapter, it is worth mentioning that proof-nets originally opened the way to GoI, and they seem to remain a good starting point for the quest of a transcendental syntax (see Girard’s latest considerations in [21]).
Chapter 5

Discussion

In this work, we have exemplified the use of polarity in logic, first introduced by Girard in phase semantics and in a desessentialized viewpoint on coherence spaces. We have focused on the idea that this technique not only provides an alternative approach to the meaning of negation in logic, but it can also lead to a change of paradigm in the study of logic.

For future work, it would be interesting to investigate how Girard’s transcendental syntax would fit in categorical logic. Also, the idea of testing as a fundamental meaning of negation invites us to the possibility of multiple negations emerging from different testing procedures (see Appendix B). Moreover, maybe we can hope that some of the notions we have introduced are related in a way to Peter Dybjer’s testing [4].

We have encountered some concepts that seem general enough to suggest the emergence of other kinds of “transcendental syntaxes”. Indeed, as Girard pointed out in [20] (p.18), the coexistence of Ludics and GoI opens the door to the possibility of many such syntaxes. Maybe some ideas related to testing - e.g. the interpretation of & and ⊕ of Section 3.2 - may even serve as a starting point (see Appendix A).
Appendix A

Filters as a set of epistates

This example tries to give a simplified version for the interpretation of the additive fragment of linear logic where epistates may represent proofs.

Example A.0.4
Consider \((X, (-)^\circ)\), where \(X\) is an arbitrary non-empty set and \((-)^\circ : X \to X\) is a bijection such that \(x^{\circ\circ} = x\) for all \(x \in X\), i.e., an involution. Define a duality on \(E = \mathcal{P}(X)\) such that \(\ll A \mid B \gg_E = A^\circ \cap B\), where \(A^\circ := \{a^\circ \mid a \in A\}\), and the pole is given by \(P = E \setminus \{\emptyset\}\). Said differently, \(A \perp B \iff A^\circ \cap B \neq \emptyset\). It is easy to see that \(\perp\) is symmetric: if \(A^\circ \cap B \neq \emptyset\), then there exists \(b \in B\) such that \(b = a^\circ\) for some \(a \in A\). Hence \(b^\circ = a^{\circ\circ} = a\), which means \(B^\circ \cap A \neq \emptyset\).

Proposition A.0.5 With respect to the above duality on \(\mathcal{P}(X)\), the facts are unions of filters, more precisely of the form \(\bigcup_{i \in I} (\uparrow (A_i))\), for some \(A_i \subseteq X\), where \(\uparrow (A_i)\) is the principal filter generated by \(A_i\) (see Definition 3.1.22).

Proof: Let \(A \subseteq \mathcal{P}(X)\). Since facts are polar sets, we will look at \(A^\perp\). Consider a distinguished element \(a_i \in A_i\) for every \(A_i \in A\). Then \(B = \{a_i^\circ \mid \text{for every } i\}\) is in \(A^\perp\). Moreover, every \(B' \supseteq B\) is also in \(A^\perp\). Therefore \(\uparrow (B) \in A^\perp\). \(\square\)
It follows that the union of facts is a fact, which is quite convenient.

Now consider \((X',(-)^\ast)\) with \((-)^\ast\) a bijection such that \(x^{\ast\ast} = x\) and a duality on \(E' = \mathcal{P}(X')\) as above, with duality relation \(\preceq\). Consider also a duality on \(E'' = \mathcal{P}(X \cup X')\) such that \(\ll A \mid B \gg_{E''} := A^\ominus \cap B\), where \(A^\ominus := \{a^\ominus \mid a \in A\}\) with \(a^\ominus := \begin{cases} a^\circ & \text{if } a \in X \\ a^\ast & \text{if } a \in X' \end{cases}\) with duality relation \(\preceq\). We define injections \(\iota_E : E \rightarrow E''\) and \(\iota_{E'} : E' \rightarrow E''\) such that \(\iota_E(A) = A\) and \(\iota_{E'}(A') = A'\). We can easily see they are duality monomorphisms: if \(A \perp B\), then \(A^\circ \cap B \neq \emptyset\), and since \(A^\ominus = A^\circ\), it follows that \(A \preceq B\). The definition of the additives goes as follows: let \(A \subseteq E\) and \(B \subseteq E'\) be facts for dualities on \(E\) and \(E'\) respectively. Then

\[
\begin{align*}
\text{"A & B"} & := \iota_E(A^\perp) \preceq \cap \iota_{E'}(B^\perp) \preceq \\
& = A^\perp \preceq \cap B^\perp \preceq \\
\text{"A ⊕ B"} & := (\iota_E(A) \cup \iota_{E'}(B)) \preceq \preceq \\
& = (A \cup B) \preceq \preceq
\end{align*}
\]

An interesting feature of this interpretation is given by the following property:

**Proposition A.0.6** Let \(A, B \subseteq \mathcal{P}(X)\) be facts in the duality defined above. Then \(A \uplus B = A \cap B\).

**Proof**: Let \(A = \bigcup_{i \in I} \uparrow(A_i)\) and \(B = \bigcup_{j \in J} \uparrow(B_j)\) where \(A_i, B_j \subseteq X\). Clearly \(A \uplus B = \bigcup_{i \in I, j \in J} \uparrow(A_i \cup B_j)\), and \(A \cap B = \{C \in \mathcal{P}(X) \mid A_i \cup B_j \subseteq C\ \text{for some } i, j\}\). \(\square\)

This result is related to Girard’s *Mystery of Incarnation* [17].

**Definition A.0.7 (Incarnation)** Given a set of sets \(\mathcal{A}\), the *incarnation* of \(\mathcal{A}\), noted \(\mathcal{A}^i\), is given by

\[
\mathcal{A}^i := \{A \in \mathcal{A} \mid A = \bigcap\{A' \in \mathcal{A} \mid A' \subseteq A\}\}
\]
For *Ludics*, Girard showed the following result ([17], p.346):

**Theorem A.0.8 (Mystery of Incarnation)** For “negative” and “disjoint” behaviours $A$ and $B$, we have

$$(A \cap B)^i = A^i \uplus B^i$$

We recall that in this situation, $A \cap B =: A \& B$ (see Section 4.2.1). An analogous result can be shown in our case:

**Corollary A.0.9 (of Proposition A.0.6)** Let $A, B \subseteq \mathcal{P}(X)$ be facts in the duality defined above. Then

$$(A \cap B)^i = A^i \cup B^i$$

**Proof**: By Proposition A.0.6, $(A \cap B)^i = (A \uplus B)^i$, and by definition of $A^i$, it follows that $(A \uplus B)^i = A^i \uplus B^i$.

Here *epistates* are elements of the filters contained in a dichology $\mathcal{A}$. In particular, the *proofs* of $\mathcal{A}$ consist of the elements of $\mathcal{A}^i$. For example, if we let $\mathcal{A} = \uparrow\{f\}$ and $\mathcal{B} = \uparrow\{g\}$ be dichologies, where $f \in X, g \in X'$ as above, then $\mathcal{A}^i = \{\{f\}\}$ and $\mathcal{B}^i = \{\{g\}\}$, which means the proofs are $\{f\}$ and $\{g\}$. Moreover, a proof of $\mathcal{A}$ and $\mathcal{B}$ would be $\{f, g\}$, since we have $\mathcal{A} \& \mathcal{B} := \uparrow\{f, g\}$, and the proofs of $\mathcal{A}$ or $\mathcal{B}$ would be $\{f\}$ and $\{g\}$, since $\mathcal{A} \oplus \mathcal{B} := \uparrow\{f\} \cup \uparrow\{g\}$. Notice also that

$$(\mathcal{A} \& \mathcal{B}) = \uparrow\{f^i\} \cup \uparrow\{g^i\} = \mathcal{A} \uplus \mathcal{B} = (\mathcal{A} \uplus \mathcal{B}) =: \mathcal{A} \uplus \mathcal{B}.$$
Appendix B

Coexistence of multiple negations

From a set theoretic viewpoint of mathematics, the negation of a statement is often perceived as the complement of a set. Indeed, in number theory, if “$n$ is not even”, the first thing that comes to mind is that it is odd. In the framework of polarity, this could be described as a $\{-1, 0, 1\}$-duality on $\mathbb{N}^* := \{1, 2, 3, \ldots\}$ where the duality relation is classical of the form $\ pergP_{Id}$. To be precise, we could take the map $\perp - | - \perp : \mathbb{N}^* \times \mathbb{N}^* \rightarrow \{-1, 0, 1\}$ such that

$$\perp m \ | \ n \perp = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m, n \neq 1, m \neq n \\ -1 & \text{if } m = 1 \text{ or } n = 1, \text{ but not both} \end{cases}$$

with the pole $P = \{-1, 0\}$, which simply says “$m \neq n$”.

The set of even positive integers $E \subseteq \mathbb{N}^*$ is a fact, since $E_{ \ pergP_{Id} \ pergP_{Id}} = E$. In the same line of thought, when we think of “$n$ is not prime”, we intuitively think of $n$ as a composite number (well, if it is not intuitive, at least it is convenient if we want the fundamental theorem of arithmetic to hold). But technically speaking, if $\mathbb{P}$ is the set of prime numbers, $\mathbb{P}_{ \ pergP_{Id} \ pergP_{Id}}$ are not composite numbers, since “$1$” is included in the set. A first solution is to keep our notion of “negation as complement” but restrict our universe: we consider the $\{-1, 0, 1\}$-duality on $\mathbb{N}^* \backslash \{1\}$. But a second approach
would be to define another \{-1, 0, 1\}-duality on \(\mathbb{N}^*\) by taking the pole \(P' = \{0\}\). With respect to this duality, \(P\) is a fact. We see that the choice of a different pole (or equivalently a duality relation) induces a different negation over the same universe.

It is not clear yet if this is a promising avenue or not, but the idea of multiple negations coexisting looks very natural. Here are some properties:

**Proposition B.0.10** Given two \(R\)-dualities on \(E\), with duality relations \(\dag_1\) and \(\dag_2\), we have for all \(A \subseteq E\):

(i) \(\dag_1 \subseteq \dag_2 \Rightarrow A^{\dag_1} \subseteq A^{\dag_2}\)

(ii) \(A^{\dag_1} \cap A^{\dag_2} = A^{\dag_1 \cap \dag_2}\)

(iii) \(A^{\dag_1} \cup A^{\dag_2} = A^{\dag_1 \cup \dag_2}\)

**Proof**: For (i): if \(x \in A^{\dag_1}\), then \(x \dag_1 a\) for all \(a \in A\) by definition. Hence \(x \dag_2 a\) for all \(a \in A\) by hypothesis, and \(x \in A^{\dag_2}\).
For (ii): \(x \in A^{\dag_1} \cap A^{\dag_2}\) iff \(x \dag_1 A\) and \(x \dag_2 A\), which means \((x, a) \in \dag_1\) and \((x, a) \in \dag_2\) for all \(a \in A\). This is equivalent to saying \((x, a) \in \dag_1 \cap \dag_2\) for all \(a \in A\), which means \(x \in A^{\dag_1 \cap \dag_2}\). A similar reasoning holds for (iii).

**Proposition B.0.11** Given two \(R\)-dualities on a set \(E\), with classical duality relations \(\dag_\varphi\) and \(\dag_{id}\), where \(\varphi : E \to E\) is any bijection, we have

\[X^{\dag_\varphi \dag_{id} \dag_\varphi \dag_{id}} = X\]

for all \(X \subseteq E\).

**Proof**: Use Proposition 2.1.19, combined to Propositions 2.1.23 and 2.1.24.
References


