TOWARDS AN UNDERSTANDING OF GIRARD'S TRANSCENDENTAL SYNTAX: SYNTAX BY TESTING

By

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Abstract

Through his work in *ludics* and *Geometry of Interaction*, Jean-Yves Girard invites us to a change of paradigm in the study of logic : the quest for a *transcendental syntax*, some kind of *idealized language* that emerges from the *rules of logic*. Amongst these rules, "*testing*" plays a leading role in defining a *duality* for the interpretation of *negation*.

The present work focuses on a notion of *polarity* which is a central technique used throughout Girard's work to express *linear negation*. We describe some properties and illustrate them with examples with the purpose of getting acquainted with the technique. We also highlight how the *classical* connectives (*conjunction* and *disjunction*) arise from an interpretation based on *testing*. In a sense, this work is intended to provide an alternative introduction to Girard's ideas and we hope it can have some pedagogical value.

To Philip Scott, Pieter Hofstra and Richard Blute,
for your source of inspiration and your dedicated work,
and for making this project possible;

À mes élèves qui, sans le savoir,
m'ont permis de vivre un certain idéal;

A Mariana, mi ángel trascendental.

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Chapter 1

Introduction

1.1 *Testing* in logic

"Logic is the study of reasoning" [32], and in the *syntactical* tradition, it is concerned with the possibilities of analyzing *formulas*, or *propositions*, within a *formal system*. *Semantics* gives a meaning to syntax and is often viewed as a way to *test* if a proposition can be formulated in the system; said differently, it evaluates the *validity* of statements in a language. Although syntax and semantics coexist, it is tempting to adopt an *essentialist* philosophical viewpoint[1] and think of semantics as *pre-existing* syntax. For example, we may believe some *absolute truth* is inherent to our world, and formal systems (syntax) can be used to express what is true. In mathematics, we can think of the *standard* model $\mathbb{N}$ as a way to *test* what propositions in arithmetics are valid, and any theory derived from a set of axioms (*e.g.* Peano's) as a way to capture arithmetical *truths*. But this hypothesis of *transparency* [19] between syntax and semantics has been refuted, according to Girard, by Gödel's incompleteness theorem. Girard then proposed a change of paradigm in logic: instead of trying to validate or

[1] In philosophy, *essentialism* refers to the conception that there exists an abstract world of *pure essences* or *ideas* (in the sense of Plato), eternal and unchangeable, that give a meaning to *existence* (*e.g.* of a human being). By opposition, *existentialism* postulates that the (human) being is defined by its own actions and not by a predetermined *essence*.

evaluate statements, he suggested we study the *sufficient conditions*[2] allowing the emergence of logical languages. In order to do so, we need a *transcendental syntax*[3], *e.g. ludics*, *proof-nets* or *Geometry of Interaction*, where *testing* comes from *within* the syntax itself[4].

Our starting point is to identify a proposition **A** with a set A, where we can think of the elements as *observations*, *explanations*, *models*, *proofs*, etc., for the proposition. These elements are determined by a *testing process*: $a \in A$ iff a successfully passes a set of *tests* for A, noted $Test(A)$. A useful metaphor here is to think of A as a set of strategies (*proofs*) for a "player-prover" to win a *game*, *i.e.* prove the proposition **A**, and $Test(A)$ as a set of counter-strategies (*counter-models*) for a "player-refuter" to prohibit the win, *i.e.* exhibit a model that invalidates **A**. However, notice that this *duality* prover/refuter is somewhat *asymmetric*: the "*burden of proof*" rests upon the shoulders of the prover. Following Girard, we will ask that the duality "elements of A / tests for A" be *monist*, *i.e.* that elements and tests belong to the same *universe* E. Also, we will ask that the testing process be *symmetric*, *i.e.* that tests for A are simultaneously tested by the elements of A. Informally, we will have that

$$A = \{a \in E \mid a \text{ passed all tests } b \in Test(A)\}$$

and since by symmetry the elements of A are the tests for $Test(A)$, we have

$$A = Test(Test(A))$$

In the analogy of a game prover *vs* refuter, if we let the strategies consist of *formal proofs* of **A** against *formal proofs* of *not* **A**, we see that $Test(A)$ corresponds to the

[2]"*conditions de possibilité*" ([20], pp.1&4), which is inspired by Kant's *transcendental* conditions, *i.e.* the hypotheses that make possible the coherence of perceptions ([21], p. 7).

[3]The expression is the analogous of Kant's *transcendental subject*, which accounts for the structure of knowledge ([20], p. 3).

[4]Girard calls for "an *autonomous* approach to syntax" ([21], p. 2).

proofs of the *negation* of A. Therefore, in Girard's transcendental syntax, a *proposition*, also called a "*dichology*", is a set of elements, the "*epistates*"[5], equal to its *double negation*; in other words, *negation* is an *involutive testing process*.

1.2 Overview of thesis

The present work is intended to provide an introduction to Girard's ideas on the basis of *testing*. In Chapter 2, we define the *technique* used for testing, and discuss some properties and basic examples. In Chapter 3, we look at particular examples in logic: Section 3.1 exemplifies different dualities "models *vs* counter-models", while Section 3.2 illustrates the idea in a categorical interpretation. Chapter 4 gives a glimpse of different examples of Girard's transcendental syntax.

My main contribution to the subject would be to have brought together examples and properties, mostly from Girard's work, under the central theme of *negation by testing*. In particular, the notion of *duality monomorphism* used in the interpretation of the classical connectives (see Section 3.2.1) allows for a different definition of the coherence spaces "$\mathcal{A} \,\&\, \mathcal{B}$" and "$\mathcal{A} \oplus \mathcal{B}$" (see Propositions 3.2.3 and 3.2.4), which highlights the common underlying structure used in other examples (see Section 3.1.1, 3.1.3 and Example A.0.4). I also personally contributed proofs of some propositions; these can be viewed as exercises for explaining the notion of *polarity* (*e.g.* Propositions 2.1.23, 2.1.24, 2.4.27, 2.4.17 and 2.5.5). In addition, I provided examples (*e.g.* Examples 3.1.21 and A.0.4) to illustrate some ideas, notably the use of Girard's *locative product*. Finally, the views expressed here are my own interpretation of Girard's work and may not represent his current views. Also, the appendices are *new* and represent different directions for future projects.

[5] *Epistates* refers to some kind of judge or magistrate in Ancient Greece; as for *dichology*, it seems to be a *neologism* that could mean "discourse in two parts".

Chapter 2

Polarity

Given a *real vector space* $(V, +, \cdot)$, we know that $v, w \in V$ are *orthogonal* if $< v \mid w >= 0$, where $< - \mid - >: V^2 \longrightarrow \mathbb{R}$ is the *scalar product* [33]. It is also common usage to write $v \perp w$ to express the orthogonality of v and w. Moreover, given $W \subseteq V$, the set of elements of V that are orthogonal to every element of W is called the *orthogonal* of W, and we denote it by $W^{\perp}$.

In the following section, we will generalize this notion of orthogonality and call it *polarity* (or *duality*). For convenience, the notation used is similar to the one used in linear algebra. We will write $x \perp y$ to express the *polarity* of x and y, and $\ll - \mid - \gg$ for the generalized "scalar product".

2.1 General definitions and properties

Let E and F be arbitrary sets. Consider a set R, called the *scalar space* [18], such that $R = \varnothing \Leftrightarrow E = \varnothing$ or $F = \varnothing$.

Given a binary function

$$\ll - \mid - \gg : E \times F \longrightarrow R$$

with a distinguished subset $P \subseteq R$ called the *pole*, we define a *duality* relation $\perp\!\!\!\!\!\!\smile \subseteq E \times F$ as follows:

$$x \perp\!\!\!\!\!\!\smile y \overset{def}{\Leftrightarrow} \ll x \mid y \gg \in P$$

If $x \perp\!\!\!\!\!\!\smile y$, we say that x and y are *polar*, or alternatively that they are *dual* (or in *duality*).

Remark 2.1.1

In some examples encountered in the present work, the duality relation $\perp\!\!\!\!\!\!\smile \subseteq E \times F$ is not initially defined in terms of a binary function and a pole as above. In such situations, we let $\ll - \mid - \gg$ be the *characteristic* function in the following sense:

$$\begin{aligned} \ll - \mid - \gg : E \times F &\longrightarrow R = \{0,1\} \\ (x,y) &\mapsto \begin{cases} 1 & \text{if } x \perp\!\!\!\!\!\!\smile y \\ 0 & \text{else} \end{cases} \end{aligned}$$

and let the pole be $P = \{1\}$.

Definition 2.1.2 (Polar set)

Given a duality relation $\perp\!\!\!\!\!\!\smile \subseteq E \times F$, we define a function $Pol_E^r : \mathcal{P}(E) \longrightarrow \mathcal{P}(F)$, where $\mathcal{P}(E)$ is the *powerset* of E, such that

$$Pol_E^r(X) \overset{def}{=} \{y \in F \mid \forall x \in X, x \perp\!\!\!\!\!\!\smile y\}$$

for $X \subseteq E$.

Similarly, define a function $Pol_F^\ell : \mathcal{P}(F) \longrightarrow \mathcal{P}(E)$

$$Pol_F^\ell(Y) \overset{def}{=} \{x \in E \mid \forall y \in Y, x \perp\!\!\!\!\!\!\smile y\}$$

for $Y \subseteq F$.

$Pol^r_E(X)$ and $Pol^\ell_F(Y)$ will be referred to as the *polar sets* of X and Y respectively. We also say that $X \subseteq E$ is a *polar set of* Y if $X = Pol^\ell_F(Y)$, and that X is a *polar set* if X is a polar set of Y, for some $Y \subseteq F$. If $X = Pol^\ell_F(Y)$, Y may be called the *pre-polar* of X (see [17]).

We can introduce the notation $X \perp y$ to mean $\forall x \in X,\ x \perp y$. In a similar way, $X \perp Y$ means that for all $x \in X$ and $y \in Y$, $x \perp y$, and we say that X and Y are *polar*.

Definition 2.1.3 (Duality)

A *duality* (or *polarity*) between E and F consists of a map $\ll - \mid - \gg : E \times F \longrightarrow R$, a pole $P \subseteq R$ and functions Pol^r_E and Pol^ℓ_F. If $E = F$, we call it an *R-duality on E* and we ask that $Pol^r_E = Pol^\ell_F$. If $E = F = R$, we simply say it is a *duality on E*.

From now on, we assume a duality between E and F, if $E, F \neq \varnothing$.

Remark 2.1.4

If E or F is the empty set, then $\ll - \mid - \gg : E \times F \longrightarrow R$ is the *empty* map, *i.e.* $dom(\ll - \mid - \gg) = Im(\ll - \mid - \gg) = \varnothing$. If $E = \varnothing$, then

$$Pol^r_E(\varnothing) = \{y \in F \mid \forall x \in \varnothing, \ll x \mid y \gg \in P)\} = F$$

and for any $Y \subseteq F$, $Pol^\ell_F(Y) = \varnothing$.

We can now take a look at some basic properties:

Proposition 2.1.5 *Given a duality between sets E and F, we have:*

(i) *E and F are polar sets;*

(ii) *$X \subseteq Pol^\ell_F(Pol^r_E(X))$ for all $X \subseteq E$, and $Y \subseteq Pol^r_E\left(Pol^\ell_F(Y)\right)$ for all $Y \subseteq F$;*

(iii) *$X \subseteq X' \Rightarrow Pol^r_E(X') \subseteq Pol^r_E(X)$ for all $X, X' \subseteq E$,*
and $Y \subseteq Y' \Rightarrow Pol^\ell_F(Y') \subseteq Pol^\ell_F(Y)$ for all $Y, Y' \subseteq F$.

Proof : Let $\perp \subseteq E \times F$ be an arbitrary duality relation with respect to some pole $P \subseteq R$ and function $\ll - \mid - \gg : E \times F \longrightarrow R$.
For (i) : We have $Pol^{\ell}_{F}(\varnothing) \stackrel{def}{=} \{x \in E \mid \forall y \in \varnothing, \ll x \mid y \gg \in P\} = E$.
Similarly, $Pol^{r}_{E}(\varnothing) = F$ (*cf.* also Remark 2.1.4).
For (ii) : Let $x \in X \subseteq E$. Then for all $y \in Pol^{r}_{E}(X)$, $x \perp y$. Therefore $x \in Pol^{\ell}_{F}(Pol^{r}_{E}(X))$. Similarly, $y \in Y \Rightarrow y \in Pol^{r}_{E}\left(Pol^{\ell}_{F}(Y)\right)$ for all $Y \subseteq F$.
For (iii) : Suppose $X \subseteq X'$ with $X, X' \subseteq E$. If $y \in Pol^{r}_{E}(X')$, then for all $x \in X'$, $x \perp y$ (by definition). In particular, $x \perp y$ for all $x \in X$ (by hypothesis). Hence $y \in Pol^{r}_{E}(X)$. Again, similar reasoning shows that $Y \subseteq Y' \Rightarrow Pol^{\ell}_{F}(Y') \subseteq Pol^{\ell}_{F}(Y)$ for all $Y, Y' \subseteq F$. □

Corollary 2.1.6 *For the situation above,* $Pol^{r}_{E}(X) = Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right)$ *for all* $X \subseteq E$ *and* $Pol^{\ell}_{F}(Y) = Pol^{\ell}_{F}\left(Pol^{r}_{E}\left(Pol^{\ell}_{F}(Y)\right)\right)$ *for all* $Y \subseteq F$.

Proof : Using Proposition 2.1.5 (ii), we obviously have that $Pol^{r}_{E}(X) \subseteq Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right)$ for all $X \subseteq E$. By (ii) again, we have $X \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(X))$, and by (iii), this implies that $Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right) \subseteq Pol^{r}_{E}(X)$. Analogous reasoning shows the other equality. □

We can make some immediate observations. In the following, suppose there is a duality between E and F.

Let $\mathscr{E} = (\mathcal{P}(E), \subseteq)$ and $\mathscr{F} = (\mathcal{P}(F), \subseteq)$ be the usual *partially ordered powersets*, considered as categories.

We can define a *functor* from $\mathscr{E}$ to $\mathscr{F}$ such that on objects, it maps $X \subseteq E$ to its polar set $Pol^{r}_{E}(X)$. On arrows, by Proposition 2.1.5 (iii), we know it maps $X \to X'$ to $Pol^{r}_{E}(X') \to Pol^{r}_{E}(X)$, so it is *contravariant*, *i.e.* it is a (covariant) functor from $\mathscr{E}$ to $\mathscr{F}^{op} = (\mathcal{P}(F), \supseteq)$. We can also define an analogous functor from $\mathscr{F}^{op}$ to $\mathscr{E}$ that assigns $Y \in \mathcal{P}(F)$ to $Pol^{\ell}_{F}(Y)$. Such a situation is called a *polarity* in [29] (p.13).

Definition 2.1.7 (Closure operation)

Let $\mathscr{C} = (C, \leqslant)$ be a *preordered set*. A map $cl : C \longrightarrow C$ is called a *closure operation* (see [29], p.12) if it satisfies, for all $a, b \in C$:

(i) $a \leqslant cl(a)$ (we say cl is *extensive*)

(ii) $a \leqslant b \Rightarrow cl(a) \leqslant cl(b)$ (we say cl is *increasing*)

(iii) $cl(cl(a)) \leqslant cl(a)$

Remark 2.1.8

From (i), (ii) and (iii), we see that the map cl is *idempotent*, *i.e.* $cl(cl(a)) = cl(a)$.

Proposition 2.1.9 *The map* $Pol^{\ell}_{F} \circ Pol^{r}_{E} : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ *is a closure operation and similarly for* $Pol^{r}_{E} \circ Pol^{\ell}_{F} : \mathcal{P}(F) \longrightarrow \mathcal{P}(F)$.

Proof: Let $X, X' \subseteq E$. Then $X \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(X))$ and $X \subseteq X' \Rightarrow Pol^{r}_{E}(X') \subseteq Pol^{r}_{E}(X) \Rightarrow Pol^{\ell}_{F}(Pol^{r}_{E}(X)) \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(X'))$ by Proposition 2.1.5 (ii) and (iii), so the map $Pol^{\ell}_{F} \circ Pol^{r}_{E}$ is extensive and increasing. By Corollary 2.1.6, we have $Pol^{r}_{E}(X) = Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right)$. Using Proposition 2.1.5 (iii), we then get $Pol^{\ell}_{F}\left(Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right)\right) \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(X))$. Similar reasoning shows that $Pol^{r}_{E} \circ Pol^{\ell}_{F}$ is a closure operation. □

Proposition 2.1.10 $Pol^{r}_{E} : \mathscr{E} \longrightarrow \mathscr{F}^{op}$ *is* ***left adjoint*** *to* $Pol^{\ell}_{F} : \mathscr{F}^{op} \longrightarrow \mathscr{E}$. *That is:* $\mathscr{F}^{op}(Pol^{r}_{E}(X), Y) \cong \mathscr{E}\left(X, Pol^{\ell}_{F}(Y)\right)$. *This means* $Pol^{r}_{E}(X) \supseteq Y$ *iff* $X \subseteq Pol^{\ell}_{F}(Y)$.

Proof: Suppose we have a morphism $Pol^{r}_{E}(X) \to Y$ in $\mathscr{F}^{op}$ for some $X \in \mathcal{P}(E)$ and $Y \in \mathcal{P}(F)$. Set theoretically, this simply means $Y \subseteq Pol^{r}_{E}(X)$. Hence for all $y \in Y$, we have $y \in Pol^{r}_{E}(X)$, *i.e.* $X \perp y$. Therefore $Pol^{\ell}_{F}(Y) \overset{def}{=} \{x \in E \mid \forall y \in Y, x \perp y\}$ contains X, and we have a morphism $X \to Pol^{\ell}_{F}(Y)$ in $\mathscr{E}$. Conversely, in a similar way, if we have a morphism $X \to Pol^{\ell}_{F}(Y)$ in $\mathscr{E}$, then $X \subseteq Pol^{\ell}_{F}(Y)$, so $X \perp Y$. Clearly $Pol^{r}_{E}(X) \supseteq Y$, and we have an arrow $Pol^{r}_{E}(X) \to Y$ in $\mathscr{F}^{op}$. □

Remark 2.1.11

Saying that Pol^r_E and Pol^ℓ_F are adjoint functors between *poset* categories $\mathscr{E} = (\mathcal{P}(E), \subseteq)$ and $\mathscr{F}^{op} = (\mathcal{P}(F), \supseteq)$ is equivalent to saying that if either $X \subseteq Pol^\ell_F(Y)$ or $Y \subseteq Pol^r_E(X)$, then X and Y are *polar*, *i.e.* $X \perp\!\!\!\!\perp Y$.

Definition 2.1.12 (Fact)

We say that $X \subseteq E$ is a *biorthogonally closed* set, or a *fact*, when

$$X = Pol^\ell_F(Pol^r_E(X))$$

Similarly, $Y \subseteq F$ is a fact when $Y = Pol^r_E\left(Pol^\ell_F(Y)\right)$.

Here are some properties about facts:

Proposition 2.1.13 *Given a duality between sets E and F:*

(i) *polar sets are exactly the facts;*

(ii) *there is a one-to-one correspondence between the set of facts of E and the set of facts of F.*

Proof: For (i) : Let $X \subseteq E$ be a fact. Then X is the polar set of $Y = Pol^r_E(X)$. Conversely, suppose $X \subseteq E$ is a polar set, *i.e.* $X = Pol^\ell_F(Y)$ for some $Y \subseteq F$. By Corollary 2.1.6, we have $Pol^\ell_F(Y) = Pol^\ell_F\left(Pol^r_E\left(Pol^\ell_F(Y)\right)\right)$, hence $X = Pol^\ell_F(Pol^r_E(X))$.

For (ii) : Pol^r_E is an injective map from the facts of E to the facts F. Indeed, suppose $Pol^r_E(X) = Pol^r_E(X')$ for facts $X, X' \subseteq E$. Then $Pol^\ell_F(Pol^r_E(X)) = Pol^\ell_F(Pol^r_E(X'))$, which implies $X = X'$ by definition. Analogously, Pol^ℓ_F is an injection from the facts of F to the facts of E, hence $|E| = |F|$ by Schröder-Bernstein's theorem [23]. $\square$

Now let's consider the (bounded) *lattices* $\mathscr{E} = (\mathcal{P}(E), \subseteq, \cap, \cup)$ and $\mathscr{F} = (\mathcal{P}(F), \subseteq, \cap, \cup)$.

Proposition 2.1.14 *Given a duality between sets E and F, we have, for all $X_i \subseteq E$ and $Y_i \subseteq F$:*

(i) $\bigcap_{i\in I} Pol^r_E(X_i) = Pol^r_E\left(\bigcup_{i\in I} X_i\right)$ *and* $\bigcap_{i\in I} Pol^\ell_F(Y_i) = Pol^\ell_F\left(\bigcup_{i\in I} Y_i\right)$;

(ii) $\bigcup_{i\in I} Pol^r_E(X_i) \subseteq Pol^r_E\left(\bigcap_{i\in I} X_i\right)$ *and* $\bigcup_{i\in I} Pol^\ell_F(Y_i) \subseteq Pol^\ell_F\left(\bigcap_{i\in I} Y_i\right)$.

Proof : Let $\perp$ be an arbitrary duality relation with respect to some pole P.

(i)

$$\begin{aligned}
y \in \bigcap_{i\in I} Pol^r_E(X_i) \quad &\Longleftrightarrow \quad \text{for all } i \in I, y \in Pol^r_E(X_i) \\
&\Longleftrightarrow \quad \text{for all } i \in I \text{ and for all } x \in X_i, \ll x \mid y \gg \in P \\
&\Longleftrightarrow \quad \forall x \in \bigcup_{i\in I} X_i, \ll x \mid y \gg \in P \\
&\Longleftrightarrow \quad y \in Pol^r_E\left(\bigcup_{i\in I} X_i\right)
\end{aligned}$$

Similarly, $x \in \bigcap_{i\in I} Pol^\ell_F(Y_i) \Longleftrightarrow x \in Pol^\ell_F\left(\bigcup_{i\in I} Y_i\right)$ for all $Y_i \subseteq F$.

(ii) Suppose $\bigcap_{i\in I} X_i \neq \varnothing$ (otherwise, it is done).
Let $y \in \bigcup_{i\in I} Pol^r_E(X_i)$. Then $y \in Pol^r_E(X_i)$ for some i, *i.e.* $X_i \perp y$. In particular, for all $x \in \bigcap_{i\in I} X_i$, $x \perp y$. Hence $y \in Pol^r_E\left(\bigcap_{i\in I} X_i\right)$. $\square$

Remark 2.1.15

Proposition 2.1.14 simply tells us that the set of facts is closed under arbitrary intersection, but it may not be closed under union in general.

Proposition 2.1.16 *Given a duality between sets E and F, $Pol^{\ell}_{F}(Pol^{r}_{E}(X))$ is the smallest fact that contains X, for $X \subseteq E$; similarly, $Pol^{r}_{E}\left(Pol^{\ell}_{F}(Y)\right)$ is the smallest fact that contains Y, for $Y \subseteq F$.*

Proof : Let $S = \bigcap_{i \in I} X_i$, where $\{X_i \subseteq E\}$ is the set of all facts containing X.
By Proposition 2.1.13 (i), we can write $X_i = Pol^{\ell}_{F}(Y_i)$ for some $Y_i \subseteq F$, so $S = \bigcap_{i \in I} Pol^{\ell}_{F}(Y_i) = Pol^{r}_{E}\left(\bigcup_{i \in I} Y_i\right)$ by Proposition 2.1.14 (i).
This means S is a polar set, hence a fact (containing X), and it is the smallest. We will show $S = Pol^{\ell}_{F}(Pol^{r}_{E}(X))$:
Clearly $S \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(X))$, since $Pol^{\ell}_{F}(Pol^{r}_{E}(X))$ is a fact containing X.
So by Proposition 2.1.5 (iii), we have

$$\begin{aligned} & X \subseteq S \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(X)) \\ \Rightarrow\ & Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right) \subseteq Pol^{r}_{E}(S) \subseteq Pol^{r}_{E}(X) \end{aligned}$$

But $Pol^{r}_{E}(X) = Pol^{r}_{E}\left(Pol^{\ell}_{F}(Pol^{r}_{E}(X))\right)$ by Corollary 2.1.6, so $Pol^{r}_{E}(X) = Pol^{r}_{E}(S)$, and consequently $Pol^{\ell}_{F}(Pol^{r}_{E}(X)) = Pol^{\ell}_{F}(Pol^{r}_{E}(S)) = S$. □

Now by Remark 2.1.15, we may wonder under what conditions the union of facts is a fact.

Let $X, X' \subseteq E$ be facts. By Proposition 2.1.13 (ii), we know the map Pol^{r}_{E} sets up a bijection between the facts of E and the facts of F, so let $Y = Pol^{r}_{E}(X)$ and $Y' = Pol^{r}_{E}(X')$ be the corresponding facts. Then we have the following :

Proposition 2.1.17 *$X \cup X'$ is a fact iff $Pol^{\ell}_{F}(Y) \cup Pol^{\ell}_{F}(Y') = Pol^{\ell}_{F}(Y \cap Y')$.*

Proof :

$$\begin{aligned} & X \cup X' = Pol^{\ell}_{F}(Pol^{r}_{E}(X \cup X')) \\ \overset{Prop.2.1.14}{\Longleftrightarrow}\ & X \cup X' = Pol^{\ell}_{F}(Pol^{r}_{E}(X) \cap Pol^{r}_{E}(X')) \\ \Longleftrightarrow\ & Pol^{\ell}_{F}(Y) \cup Pol^{\ell}_{F}(Y') = Pol^{\ell}_{F}\left(Pol^{r}_{E}\left(Pol^{\ell}_{F}(Y)\right) \cap Pol^{r}_{E}\left(Pol^{\ell}_{F}(Y')\right)\right) \\ \Longleftrightarrow\ & Pol^{\ell}_{F}(Y) \cup Pol^{\ell}_{F}(Y') = Pol^{\ell}_{F}(Y \cap Y') \end{aligned}$$

□

We may now wonder what are the conditions for *every* subset of E and F to be a fact. We already know that $|E| = |F|$ is a necessary condition by Proposition 2.1.13 (ii), but it is not sufficient.

Here is an adapted version of a criterion which was mentioned in a more general framework by Wright [35]. But first a lemma :

Lemma 2.1.18 *Let $\alpha : \mathcal{P}(E) \longrightarrow \mathcal{P}(F)$ be a homomorphism of the usual Boolean algebras $\mathcal{P}(E)$ and $\mathcal{P}(F)$, i.e. $\alpha(\varnothing) = \varnothing$, $\alpha(E) = F$, $\alpha(X \cap X') = \alpha(X) \cap \alpha(X')$ and $\alpha(X \cup X') = \alpha(X) \cup \alpha(X')$ for all $X, X' \subseteq E$.*
Then, for all $X \subseteq E$,

$$\overline{\alpha(X)} = \alpha(\overline{X})$$

where $\overline{X}$ is the complement of X.

Proof : $\alpha(\varnothing) = \alpha(X \cap \overline{X}) = \alpha(X) \cap \alpha(\overline{X}) = \varnothing$ and $\alpha(E) = \alpha(X \cup \overline{X}) = \alpha(X) \cup \alpha(\overline{X}) = F$ □

Proposition 2.1.19 *Suppose we have a duality between sets E and F. Then every $X \subseteq E$ and $Y \subseteq F$ is a fact if and only if the maps*

$$\begin{array}{rcl} \alpha : \mathcal{P}(E) & \longrightarrow & \mathcal{P}(F) \\ X & \mapsto & \overline{Pol^r_E(X)} \end{array}$$

and

$$\begin{array}{rcl} \beta : \mathcal{P}(F) & \longrightarrow & \mathcal{P}(E) \\ Y & \mapsto & \overline{Pol^\ell_F(Y)} \end{array}$$

are reciprocal isomorphisms of the usual Boolean algebras $\mathcal{P}(E)$ and $\mathcal{P}(F)$.

Proof (Proposition 2.1.19) : For ($\Rightarrow$), we first show α is an isomorphism:

- For all $X, X' \subseteq E$

$$\begin{aligned}
\alpha(X) = \alpha(X') &\Rightarrow \overline{Pol^r_E(X)} = \overline{Pol^r_E(X')} \\
&\Rightarrow Pol^r_E(X) = Pol^r_E(X') \\
&\Rightarrow Pol^\ell_F(Pol^r_E(X)) = Pol^\ell_F(Pol^r_E(X')) \\
&\Rightarrow X = X'
\end{aligned}$$

  hence α is injective.

- for surjectivity: for all $Y \subseteq F$, $\alpha(\underbrace{Pol^\ell_F\left(\overline{Y}\right)}_{\in E}) = \overline{Pol^r_E\left(Pol^\ell_F\left(\overline{Y}\right)\right)} = \overline{\overline{Y}} = Y$

- $\alpha(\varnothing) = \overline{Pol^r_E(\varnothing)} = \overline{F} = \varnothing$

- $\alpha(E) = \overline{Pol^r_E(E)} = \overline{\varnothing} = F$ (for $\varnothing \subseteq F$)

- for all $X, X' \subseteq E$,

$$\begin{aligned}
\alpha(X \cup X') &= \overline{Pol^r_E(X \cup X')} \\
&= \overline{Pol^r_E(X) \cap Pol^r_E(X')} \\
&= \overline{Pol^r_E(X)} \cup \overline{Pol^r_E(X')} \\
&= \alpha(X) \cup \alpha(X')
\end{aligned}$$

- for all $X, X' \subseteq E$,

$$\begin{aligned}
\alpha(X \cap X') &= \overline{Pol^r_E(X \cap X')} \\
&= \overline{Pol^r_E(X) \cup Pol^r_E(X')} \\
&= \overline{Pol^r_E(X)} \cap \overline{Pol^r_E(X')} \\
&= \alpha(X) \cap \alpha(X')
\end{aligned}$$

Now we show $(\beta \circ \alpha)(X) = X$ for all $X \subseteq E$:

$$\begin{aligned} x \in \beta(\alpha(X)) &\Leftrightarrow x \in \beta\left(\overline{Pol^r_E(X)}\right) \\ &\Leftrightarrow x \notin \beta\left(Pol^r_E(X)\right) \text{ by Lemma 2.1.18} \\ &\Leftrightarrow x \notin \overline{Pol^\ell_F\left(Pol^r_E(X)\right)} \\ &\Leftrightarrow x \notin \overline{X} \\ &\Leftrightarrow x \in X \end{aligned}$$

Similarly, $(\alpha \circ \beta)(Y) = Y$ for all $Y \subseteq F$, and we have shown that α and β are reciprocal, *i.e.* $\beta = \alpha^{-1}$. Hence β is also an isomorphism.
For $(\Leftarrow)$: To show that all $X \subseteq E$ are facts, it suffices to show that $Pol^\ell_F\left(Pol^r_E(X)\right) \subseteq X$. Let $x \in Pol^\ell_F\left(Pol^r_E(X)\right) = Pol^\ell_F\left(\overline{\alpha(X)}\right) = \overline{\beta(\overline{\alpha(X)})}$. Then $x \notin \beta(\overline{\alpha(X)})$, hence $x \in \beta(\alpha(X)) = X$. Similarly, $Pol^r_E\left(Pol^\ell_F(Y)\right) \subseteq Y$ for all $Y \subseteq F$. □

In what follows, we will determine another criterion for every subset of E and F to be a fact which is based on the duality relation.

Proposition 2.1.20 *Suppose we have a duality between sets E and F.*
Then

(i) *if $\perp = \varnothing$, the only facts are $\varnothing$, E and F;*

(ii) *if $\perp = E \times F$, the only facts are E and F.*

Proof: For (i): For all $X \subseteq E$, if $X \neq \varnothing$, we have $Pol^r_E(X) = \{y \in F \mid X \perp y\} = \varnothing$, and since $Pol^\ell_F(\varnothing) = E$, this means E is a fact. Similarly, F is a fact.
For (ii): For all $X \subseteq E$, $Pol^r_E(X) = F$, and for all $Y \subseteq F$, $Pol^\ell_F(Y) = E$. □

Definition 2.1.21

Let E, F be non-empty sets such that $|E| = |F|$.
Let $\varphi : E \longrightarrow F$ be a bijection.
Then

$$\perp_\varphi \overset{def}{=} \{(x, y) \in E \times F \mid y \neq \varphi(x)\}$$

will be referred to as a *classical* duality relation.

Remark 2.1.22

Notice that if $|E| = |F| \in \{0,1\}$, then $\perp_\varphi = \varnothing$ and every subset of E and F is a fact. Also, the choice of $\perp_\varphi$ may not always be appropriate to define an R-duality on E: we must ensure that $\varphi^2 = Id$, so that we fulfill the condition that $\perp_\varphi$ is a *symmetric* relation (see Proposition 2.4.2).

Proposition 2.1.23 *Given a duality between sets E and F, suppose every subset of E and every subset of F is a fact. Then the duality relation is classical, i.e. of the form $\perp_\varphi$ for some bijection $\varphi : E \longrightarrow F$.*

Proof : By Proposition 2.1.13 (ii), we know $|E| = |F|$. If $E = F = \varnothing$, then the duality relation is clearly classical. If $|E| = \{a\}$ and $|F| = \{\alpha\}$, the only possible duality relation is $\perp = \varnothing$; also, the only bijection is $\varphi(a) = \alpha$, hence $\perp_\varphi = \perp$. So suppose $|E| = |F| \geqslant 2$. Since every subset is a fact by hypothesis, we must have $Pol^r_E\left(Pol^\ell_F(\varnothing)\right) = \varnothing$ and $Pol^\ell_F(Pol^r_E(\varnothing)) = \varnothing$. Since $Pol^\ell_F(\varnothing) = E$ and $Pol^r_E(\varnothing) = F$, we conclude that $Pol^r_E(E) = \varnothing$ and $Pol^\ell_F(F) = \varnothing$. Therefore, for all $y \in F$, there exists $x \in E$ such that $\ll x \mid y \gg \notin P$, and similarly for all $x \in E$, there exists $y \in F$ such that $\ll x \mid y \gg \notin P$. Now let $x \in E$ and consider $X = E \backslash \{x\}$. Since X is a fact, we cannot have $Pol^r_E(X) = \varnothing$, so there is some $y \in F$ such that $X \perp y$. And since $X \subseteq Pol^\ell_F(\{y\}) \neq E$ ($\{y\}$ is also a fact), we have $Pol^r_E(X) = \{y\}$. Therefore x is the only element in E such that $x \not\perp y$. Moreover, if $\mathcal{R} = \{E\backslash\{x\} \subseteq E \mid x \in E\}$ and $\mathcal{S} = \{\{y\} \subseteq F \mid y \in F\}$, then the following maps are clearly bijective :

$$\begin{array}{rcl} f : E & \longrightarrow & \mathcal{R} \\ x & \mapsto & E\backslash\{x\} \end{array}$$

$$\begin{array}{rcl} Pol^r_E\restriction_{\mathcal{R}} : \mathcal{R} & \longrightarrow & \mathcal{S} \\ E\backslash\{x\} & \mapsto & Pol^r_E((E\backslash\{x\})) = \{y\} \end{array}$$

$$\begin{array}{rcl} g : \mathcal{S} & \longrightarrow & F \\ \{y\} & \mapsto & y \end{array}$$

Therefore the map $\varphi = g \circ Pol^r_E \restriction_{\mathcal{R}} \circ f$ is also a bijection, and it is such that $\varphi(x) = y$ iff $x \not\perp y$. We conclude that $\perp = \perp_\varphi$. $\square$

We can also state the converse :

Proposition 2.1.24 *Let $|E| = |F|$ and suppose we have a duality between E and F with a classical duality relation $\perp_\varphi$, for some bijection $\varphi : E \longrightarrow F$.*
Then every subset of E and F is a fact (with respect to $\perp_\varphi$).

Proof : Obviously, if $|E| \in \{0, 1\}$, then every subset is a fact by Remark 2.1.22.
So let $|E| \geqslant 2$. We already know E and F are facts by Proposition 2.1.5 (i). Now consider any set $X \subseteq E$. It suffices to show that $Pol^\ell_F(Pol^r_E(X)) \subseteq X$:

Since $\perp_\varphi \neq E \times F$ by definition, then $Pol^\ell_F(Pol^r_E(\varnothing)) = Pol^\ell_F(F) = \varnothing$. So suppose $X \neq \varnothing$. By hypothesis, we have

$$Pol^r_E(X) \stackrel{def}{=} \{y \in F \mid X \perp_\varphi y\} = F \backslash \{\varphi(x) \mid x \in X\}$$

Also,

$$Pol^\ell_F(Pol^r_E(X)) \stackrel{def}{=} \{x \in E \mid x \perp_\varphi Pol^r_E(X)\} = E \backslash \{\varphi^{-1}(y) \mid y \in Pol^r_E(X)\}$$

Therefore, if $x \notin X$, then $\varphi(x) \in Pol^r_E(X)$. This means $\varphi^{-1}(\varphi(x)) = x \notin Pol^\ell_F(Pol^r_E(X))$. $\square$

Example 2.1.25 (Subsets as facts)

It might be useful to illustrate Propositions 2.1.23 and 2.1.24.

Consider $E = \{a, b, c\}$ and $F = \{\alpha, \beta, \gamma\}$. Let $\varphi : E \longrightarrow F$ be such that $\varphi(a) = \alpha$,

$\varphi(b) = \beta$ and $\varphi(c) = \gamma$. Consider a duality between E and F such that the duality relation is

$$\perp_\varphi \quad = \quad \{(a,\beta),(a,\gamma),(b,\alpha),(b,\gamma),(c,\alpha),(c,\beta)\}.$$

This ensures every subset of E and F is a fact. Indeed, $Pol_E^r(E) = Pol_F^\ell(F) = \varnothing$ and the following *facts* are in *correspondence* ($X \bowtie Y$ means $Pol_E^r(X) = Y$ and $Pol_F^\ell(Y) = X$):

$$\begin{aligned}
\{a\} &\bowtie \{\beta,\gamma\} \\
\{b\} &\bowtie \{\alpha,\gamma\} \\
\{c\} &\bowtie \{\beta,\gamma\} \\
\{a,b\} &\bowtie \{\gamma\} \\
\{a,c\} &\bowtie \{\beta\} \\
\{b,c\} &\bowtie \{\alpha\}
\end{aligned}$$

2.2 Basic examples of duality between distinct sets E and F

In the last section, while looking at posets $(\mathcal{P}(E), \subseteq)$ and $(\mathcal{P}(F), \subseteq)$ with the extra structure given by polar sets, we recovered the notion of a *Galois correspondence*, which is sometimes referred to as an *antitone* or *order-reversing* Galois connection.

Let's take a look at the original Galois correspondence [3] in the light of our setting:

Example 2.2.1 (Galois groups and subfields as facts)

Let E be a field, and $F = Aut(E)$ be the set of all automorphisms of E, hence a group with respect to composition. Let $R = \{0, 1\}$, and $P = \{1\}$.

Define the map $\ll - \mid - \gg : E \times Aut(E) \longrightarrow R$ such that

$$(x, \sigma) \mapsto \begin{cases} 1 & \text{if } \sigma(x) = x \\ 0 & \text{else} \end{cases}$$

For any $X \subseteq E$, the polar set is given by

$$\begin{aligned} Pol^r_E(X) &= \{\sigma \in Aut(E) \mid X \perp \sigma\} \\ &= \{\sigma \in Aut(E) \mid \forall x \in X\, (\ll x \mid \sigma \gg \in P)\} \\ &= \{\sigma \in Aut(E) \mid \forall x \in X\, (\sigma(x) = x)\} \end{aligned}$$

Dually, for $Y \subseteq Aut(E)$, the polar set is given by

$$Pol^\ell_F(Y) = \{x \in E \mid x \perp Y\}$$

An immediate observation is the following:

Proposition 2.2.2 *If $X \subseteq E$ is a fact, then X is a subfield of E.*

Proof : Suppose $X \subseteq E$ is a fact. Then by Proposition 2.1.13, we can write $X = Pol^\ell_F(Y) = \{x \in E \mid \forall \sigma \in Y, \sigma(x) = x\}$ for some $Y \subseteq Aut(E)$. If $Y = \varnothing$, then $X = E$ and it is done, so suppose Y contains at least one automorphism, say σ. Obviously, 0 and 1 are in X. Suppose $x, x' \in X$. Then $\sigma(x + x') = \sigma(x) + \sigma(x') = x + x'$, so $x + x' \in X$. Similarly, $x \cdot x' \in X$. Also, $\sigma(-x) = \sigma(-1) \cdot \sigma(x) = -1 \cdot x = -x$, hence $-x \in X$, and if $x \neq 0$, then

$$\begin{aligned} \sigma(1) = 1 &\Leftrightarrow \sigma(x^{-1} \cdot x) = 1 \\ &\Leftrightarrow \sigma(x^{-1}) \cdot \sigma(x) = 1 \\ &\Leftrightarrow \sigma(x^{-1}) \cdot x = 1 \\ &\Leftrightarrow \sigma(x^{-1}) = x^{-1} \end{aligned}$$

Thus $x^{-1} \in X$. $\square$

Proposition 2.2.3 *If $Y \subseteq Aut(E)$ is a fact, then Y is a subgroup of $Aut(E)$.*

Proof : Suppose $Y \subseteq Aut(E)$ is a fact, Then $Y = Pol^r_E(X)$ for some $X \subseteq E$ (by Prop. 2.1.13). If X is empty, then $Y = Aut(E)$, so suppose $X \neq \varnothing$.
Obviously, $Pol^r_E(X)$ contains at least the *identity* automorphism, Id, so Y is not empty. If $\sigma, \tau \in Y$, then for all $x \in X$, $\sigma^{-1}(x) = \sigma^{-1}(\sigma(x)) = Id(x) = x$, so $\sigma^{-1} \in Y$, and $\tau(\sigma(x)) = \tau(x) = x$, so $\tau \circ \sigma \in Y$. □

The last two propositions give us necessary conditions for subsets of E and $Aut(E)$ to be facts. The *Fundamental Theorem of Galois Theory* gives us sufficient conditions, as follows:

Assume that E is a *finite*, *separable*, *normal* extension of a field E'.
The subfields K of E that contain E' are facts of E, and the subgroups H of $Pol^r_E(E')$, which are called the *Galois groups of E over E'*, noted $Gal(E/E')$, are facts of $Aut(E)$. The Fundamental Theorem of Galois Theory says they are in a one-to-one correspondence (see [3]), *i.e.*

$$K = Pol^\ell_F(H) \Leftrightarrow H = Pol^r_E(K)$$

In our previous notation, this means $K \bowtie H$. For example, let $E' = \mathbb{Q}$ and $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Then $Pol^r_E(\mathbb{Q})$ is of order 4, with elements σ_i such that

$$\begin{aligned}
\sigma_0(\sqrt{2}) &= \sqrt{2} \quad \text{and} \quad \sigma_0(\sqrt{3}) = \sqrt{3} \\
\sigma_1(\sqrt{2}) &= \sqrt{2} \quad \text{and} \quad \sigma_1(\sqrt{3}) = -\sqrt{3} \\
\sigma_2(\sqrt{2}) &= -\sqrt{2} \quad \text{and} \quad \sigma_2(\sqrt{3}) = \sqrt{3} \\
\sigma_3(\sqrt{2}) &= -\sqrt{2} \quad \text{and} \quad \sigma_3(\sqrt{3}) = -\sqrt{3}
\end{aligned}$$

The facts of $Pol^r_E(\mathbb{Q})$ are $\{\sigma_0\}$, $\{\sigma_0, \sigma_1\}$, $\{\sigma_0, \sigma_2\}$, $\{\sigma_0, \sigma_3\}$ and $Pol^r_E(\mathbb{Q})$ itself, corresponding respectively to facts $\mathbb{Q}(\sqrt{2}, \sqrt{3})$, $\mathbb{Q}(\sqrt{2}\}$, $\mathbb{Q}(\sqrt{3})$, $\mathbb{Q}(\sqrt{6})$ and $\mathbb{Q}$, *i.e.* $\{\sigma_0\} \bowtie \mathbb{Q}(\sqrt{2}, \sqrt{3})$, etc.

Since we are looking at situations where E and F may be distinct sets, many mathematical examples may arise. The last example had some *historical* value; the next two examples may be considered as *pedagogical* for the present work:

Example 2.2.4 (Lines and points as facts)

This one is really a *toy example* inspired by a metaphor used in [16] (p.8).

Let E be the set of *points* in the plane and F the set of (straight) *lines*.
For the moment, let $E = \mathbb{R}^2$ and let an element ℓ of F be a set

$$\ell = \{(x, y) \in \mathbb{R}^2 \mid Ax + By + C = 0, \text{ for } A, B, C \in \mathbb{R}\}$$

Let $R = \{0, 1\}$ and $P = \{1\}$.

Define the map $\ll - \mid - \gg : E \times F \longrightarrow R$ such that

$$\ll (x, y) \mid \ell \gg = \begin{cases} 1 & \text{if } (x, y) \in \ell \\ 0 & \text{else} \end{cases}$$

In words, we have stated that a point and a line are in duality if and only if "the point *belongs* to the line". Now $Pol^r_E(\{(x, y)\})$ is the set of lines that "intersect at (x, y)", while $Pol^r_E(\{(x_1, y_1), (x_2, y_2)\})$, with $(x_1, y_1) \neq (x_2, y_2)$, is the singleton that contains the "only line that passes through (x_1, y_1) and (x_2, y_2)". Dually, $Pol^\ell_F(\{\ell\})$ is the set of all points that belong to ℓ, *i.e.* $Pol^\ell_F(\{\ell\}) = \ell$.
We observe that $Pol^r_E(X) = \varnothing$ if the points in X are not aligned, and similarly $Pol^\ell_F(Y) = \varnothing$ if the lines in Y do not intersect at a common point.
Therefore the only facts are E, F, $\varnothing$, singletons and their polar set.

This said, we can *abstract* this polarity and see "points" and "lines" as *primitive* undefined terms. Our familiar definition of a *line* ℓ as a set of *points* can be recovered by $Pol^\ell_F(\ell)$; but dually, we have a definition of a *point* p as a set of *lines* (such that p is their *intersection*) given by $Pol^r_E(p)$.

Example 2.2.5 (Convex sets as facts)

This one was pointed out in [29] (p.13). Let E be the set of *points* of a plane, say $E = \mathbb{R}^2$, and F the set of *half planes*, where $H \in F$ is given by $H = \{(x, y) \in \mathbb{R}^2 \mid Ax + By + C \geqslant 0, \text{ for } A, B, C \in \mathbb{R}\}$.
Let $R = \{0, 1\}$ and $P = \{1\}$.

Define the map $\ll - \mid - \gg : E \times F \longrightarrow R$ such that

$$\ll(x, y) \mid H\gg = \begin{cases} 1 & \text{if } (x, y) \in H \\ 0 & \text{else} \end{cases}$$

Hence $(x, y) \perp H$ if and only if the point (x, y) *belongs* to the half-plane H.
Then for $X \subseteq E$, $Pol^{\ell}_{F}\,(Pol^{r}_{E}\,(X))$ is the intersection of all half-planes containing X, *i.e.* the *convex hull* of X. Hence a *convex set* X is a fact, and we can write $X = Pol^{\ell}_{F}\,(Y)$ for some set of half-planes $Y \subseteq F$ such that $Y = Pol^{r}_{E}\,(X)$.
Now we can abstract our notion of a *convex set* and describe it in two equivalent ways : as the usual set of points $Pol^{\ell}_{F}\,(Y)$, or dually by the half-planes containing it, *i.e.* $Pol^{r}_{E}\,(X)$.

Remark 2.2.6 In our setting of polarity, we also recognize the idea of a *game* in the sense of Lafont and Streicher [26]. By definition, a game over a set R (whose elements are called *scalars*) is a pair of sets E, F, whose elements are respectively called the *vectors* and the *forms*, equipped with a map $\ll - \mid - \gg : E \times F \longrightarrow R$. This is a generalization of the original description given by Von Neumann and Morgenstern, where E and F represented *strategies* for each player, and the map $E \times F \longrightarrow \mathbb{R}$ was the *payoff* function as mentioned in [26]. As Lafont and Streicher point out, a game over R is simply a special case of *Chu space* over R.

2.3 Metamathematical examples

Example 2.3.1 (Propositional logic)

Let $E = PROP$, *i.e.* the usual set of well-formed formulas of propositional logic. Take F to be the set of valuations Val, *i.e.* the set of maps $[\![-]\!] : PROP \longrightarrow \{0,1\}$ which are valuations. (Refer to [34] for the terminology.) Let $R = \{0,1\}$ and $P = \{1\}$. We define the map $\ll - \mid - \gg \, : PROP \times Val \longrightarrow R$ such that $\ll A \mid [\![-]\!] \gg \, = [\![A]\!]$. Traditionally, $[\![A]\!] = 1$ means that A is a *valid* (or *true*) proposition, while $[\![A]\!] = 0$ means it is not.

Now we can express some metamathematical theorems in terms of polar sets. Let $\Gamma \subseteq PROP$ be a set of formulas. Given a *formal system*, $\Gamma \vdash \varphi$ means that the formula $\varphi \in PROP$ is *derivable* from Γ (using the *deduction rules* of the system). We say that Γ is *deductively closed* if for any $\varphi \in PROP$, $\Gamma \vdash \varphi$ implies that $\varphi \in \Gamma$.

We also say that a formal system is *sound* (or *adequate*, *correct*) with respect to Val if for any $\varphi \in PROP$,

$$\vdash \varphi \Rightarrow \models \varphi$$

where $\models \varphi$ means that for all valuations $[\![-]\!] \in Val$, we have $[\![\varphi]\!] = 1$.
Let $\Theta = \{\varphi \in PROP \mid \vdash \varphi\}$. Θ is the set of all *provable* formulas in the system, *i.e.* the *theorems*. Obviously, the set of theorems is deductively closed.

Proposition 2.3.2 *For a given formal system, the following are equivalent:*

(i) *The system is sound (with respect to Val);*

(ii) $\Theta \subseteq Pol^{\ell}_{F}(Val)$*;*

(iii) $Val = Pol^{r}_{E}(\Theta)$.

Proof : For (i)$\Rightarrow$(ii): Suppose *soundness*. If $\theta \in \Theta$, then $\vdash \theta$, which implies $\models \theta$ by hypothesis. This means for all valuations $[\![-]\!] \in Val$, $[\![\theta]\!] = 1$, hence $\theta \in Pol^{\ell}_{F}(Val)$.

For (ii)⇒(iii): By Proposition 2.1.5, we know Val is a polar set, *i.e.* a fact (Prop. 2.1.13), so

$$\begin{aligned}\Theta \subseteq Pol^{\ell}_{F}(Val) &\Rightarrow Pol^{r}_{E}\left(Pol^{\ell}_{F}(Val)\right) \subseteq Pol^{r}_{E}(\Theta)\\ &\Rightarrow Val \subseteq Pol^{r}_{E}(\Theta)\end{aligned}$$

For (iii)⇒(i): Suppose $Val \subseteq Pol^{r}_{E}(\Theta)$. Take any $\varphi \in PROP$ and suppose $\vdash \varphi$. Then $\varphi \in \Theta$, and since $Pol^{r}_{E}(\Theta)$ contains every valuation by hypothesis, we conclude by definition that $\varphi \perp Val$, hence $\models \varphi$. □

We know that if

$$\Theta_c \stackrel{def}{=} \{\varphi \in PROP \mid \vdash_c \varphi\}$$

where $\vdash_c \varphi$ means that φ is derivable from *classical* rules, and if

$$\Theta_i \stackrel{def}{=} \{\varphi \in PROP \mid \vdash_i \varphi\}$$

where $\vdash_i \varphi$ means that φ is derivable from *intuitionistic* rules, then

$$\Theta_i \subseteq \Theta_c$$

since the deductive rules of intuitionistic logic are *included* in those of classical logic (indeed, intuitionistic logic is simply classical logic without the *reductio ad absurdum* rule).

Using Proposition 2.1.5, we get

$$Pol^{r}_{E}(\Theta_c) \subseteq Pol^{r}_{E}(\Theta_i)$$

Hence *soundness* of classical logic implies *soundness* of intuitionistic logic (with respect to Val) by Proposition 2.3.2.

Now we say that a formal system is *complete* with respect to Val if for any $\varphi \in PROP$,

$$\models \varphi \Rightarrow \vdash \varphi$$

Proposition 2.3.3 *For a given formal system, the following are equivalent:*

(i) *The system is complete (with respect to Val);*

(ii) $Pol^{\ell}_F(Val) \subseteq \Theta$.

Proof: For (i)⇒(ii): If $\varphi \in Pol^{\ell}_F(Val)$, then by definition $\varphi \perp Val$, hence $\models \varphi$. By *completeness*, $\vdash \varphi$, so $\varphi \in \Theta$. For (ii)⇒(i): Clear by definition. □

It is well-known that classical logic is *complete* with respect to Val, which can be expressed by $Pol^{\ell}_F(Val) \subseteq \Theta_c$. And since it is also *sound*, we can see that Θ_c is a fact corresponding to Val, *i.e.* $\Theta_c \bowtie Val$.

We can generalize the last example to (first order) *predicate logic*, which was first pointed out by Lawvere in 1969 (reprinted in [30]):

Example 2.3.4 (Predicate logic)

Consider a *language* $\mathscr{L}$ and let E be the set of all well-formed $\mathscr{L}$-sentences.
Let F be the set of $\mathscr{L}$-structures, *i.e.* the structures that can give a *meaning* to the language $\mathscr{L}$. Let $R = \{0, 1\}$ and $P = \{1\}$.
We define the map $\ll - \mid - \gg : E \times F \longrightarrow R$ such that:

$$\ll\sigma \mid \mathcal{M}\gg = \begin{cases} 1 & \text{if } \mathcal{M} \models \sigma \quad (\mathcal{M} \text{ is a } \textit{model} \text{ for } \sigma) \\ 0 & \text{otherwise} \end{cases}$$

In words, we have simply defined a duality between *syntax* and *semantics*.
We recall that a *theory* $\Theta \subseteq E$ is *closed under derivability*, *i.e.* $\Theta \vdash \sigma$ implies $\sigma \in \Theta$ ([34], p.104).
Also, a set $\Pi \subseteq E$ such that $\Theta = \{\sigma \mid \Pi \vdash \sigma\}$ is called an *axiom set* of the theory Θ. We call such a theory Θ a *deductive theory*.

Moreover, we can define the *theory of (a structure)* $\mathcal{M} \in F$, noted $Th(\mathcal{M})$ (see

[32], p.82), by $Th(\mathcal{M}) \stackrel{def}{=} Pol^{\ell}_{F}(\{\mathcal{M}\})$.
Notice that in a sense, $Th(\mathcal{M})$ is *induced* from a structure $\mathcal{M}$, so we may refer to it as an *induced theory*.

Let's take a concrete example. Let $\mathscr{L}$ be the usual language of *Peano arithmetic*. Let $PA \subseteq E$ be a set of *axioms* [32]. We expect that $Pol^{r}_{E}(PA)$ contains $\mathbb{N}$, the *standard* model. Now using Proposition 2.1.9, we know $PA \subseteq Pol^{\ell}_{F}(Pol^{r}_{E}(PA)) \subseteq Pol^{\ell}_{F}(\{\mathbb{N}\}) = Th(\mathbb{N})$. And we know from Gödel that if we consider the theory $\widehat{PA} = \{\sigma \in E \mid PA \vdash \sigma\}$, then $\widehat{PA} \subsetneq Th(\mathbb{N})$.

Proposition 2.3.5 *If $\widehat{PA}$ is a fact, then $\mathbb{N}$ is not the only model for $\widehat{PA}$.*

Proof :

$$\begin{aligned} \widehat{PA} \text{ is a fact} \quad & \Leftrightarrow \quad \widehat{PA} = Pol^{\ell}_{F}\left(Pol^{r}_{E}\left(\widehat{PA}\right)\right) \\ & \Rightarrow \quad Pol^{r}_{E}\left(\widehat{PA}\right) \neq \{\mathbb{N}\} \text{ since } Pol^{\ell}_{F}(\{\mathbb{N}\}) = Th(\mathbb{N}) \\ & \Rightarrow \quad \{\mathbb{N}\} \subsetneq Pol^{r}_{E}\left(\widehat{PA}\right) \end{aligned}$$

□

Remark 2.3.6

We know there are *non-standard* models for arithmetic, but do we know if there is such a set of models in correspondence with $\widehat{PA}$ so that $\widehat{PA}$ is a fact?

Let's try to understand Popper's philosophy in the light of *polarity*:

Example 2.3.7 (Popper's *refutationism*)

Popper's *logic* of scientific discovery is based on the idea of *conjectures and refutations*. In order to *refute* a *conjecture*, we put it to the *test* : if it holds, we say it has not been refuted (yet); if it fails, we simply reject it (see [31]).

Let E be a set of $\mathscr{L}$-sentences with respect to a language $\mathscr{L}$. Let F be a set of *tests* for E. Let $R = \{0,1\}$ and $P = \{1\}$. We define the map $\ll - \mid - \gg : E \times F \longrightarrow R$ such that

$$\gamma \perp \mathcal{T} \Leftrightarrow \ll\gamma \mid \mathcal{T}\gg = 1 \Leftrightarrow \text{ “conjecture } \gamma \text{ is not refuted by the test } \mathcal{T}\text{”}$$

We define a *theory* as a subset $\Gamma \subseteq E$ of conjectures. Notice that we do not require a theory to be *deductive*.

We say that a theory $\Gamma \subseteq E$ is *contingent*, *i.e.* has *not been refuted yet*, when $Pol^r_E(\Gamma) = F$. We might say that it is “valid” in some sense (indeed, in some fields like medicine, propositions are tested by experimental means, hence *truth* is given by “so far so good”). Notice that it looks like a generalization of *soundness*, so we may then say that Γ is *sound* with respect to the tests in F.
In the same line of thought, we may say that Γ is *complete* (with respect to F) when $Pol^\ell_F(F) \subseteq \Gamma$. This simply means that the theory can explain everything that has been tested successfully.

Consider some cases:

- If $F = \varnothing$, then no test is associated to what can be *said* in E (with respect to a language $\mathscr{L}$). Then a *theory* $\Gamma \subseteq E$ is not *testable*. We can say it is *sound*, but so it is for any theory! Moreover, anything can be induced from F, even contradictory statements! Therefore E is the only fact (of E) and it corresponds to F, *i.e.* $E \bowtie F$. In words, E is the largest valid theory, so everything is *true*.
  We retrieve here Popper' criterion of *testability* (or *refutability*) for a theory to be called “scientific”: it has to be exposed to a non-empty set of tests. Otherwise, it is called “metaphysics” (see [31]).

- Since a theory $\Gamma \subseteq E$ is *true* when $Pol^r_E(\Gamma) = F$, *absolute truth* would mean

that $Pol_E^r(\Gamma)$ contains *every* possible test. Obviously, the question is : how can we tell F is sufficiently large and would Γ be *testable* in practice?
So instead of talking of *absolute truth* (which either does not exist or is not achievable), we will say a theory is less *false* the more F contains (*severe*) tests that "do not invalidate" the theory.

- The leading example in science is Newton's laws of physics, say $N \subseteq E$, where the elements of E are expressed in the language of Euclidian geometry. This theory is *sound* (*true*) with respect to some set F of tests, *i.e.* $Pol_E^r(N) = F$. It may be convenient here to think of tests as *observations* in some sense. Now N contains a prediction that "light travels in a straight line" which has been tested with success in F. But if we consider $F' = F \cup \{\mathcal{T}\}$, where $\mathcal{T}$ is a test at a *cosmic scale*, then the prediction fails. Einstein's relativity is a theory that is less false in this sense, since its polar set contains F'.

- The *logic* of scientific discovery from Popper's viewpoint lies in the search of an increasing sequence of testing sets $F \subseteq F' \subseteq F'' \subseteq ...$ until a theory $\Gamma \subseteq E$ fails; we have then made progress towards (absolute) *truth*, since at least we have discovered that some conjecture $\gamma \in \Gamma$ was not *true*!

The main critique offered by Girard ([13],[15]) to Popper's philosophy is that it gives the questionable status of "science" to some fields like medicine (where *theories* tend to reduce the more they are tested), while it ignores the *scientific* value of Gödels' incompleteness theorem since it cannot be "tested" in Popper's sense.

Example 2.3.8 (Lakatos' *Proofs and Refutations*)

Lakatos [27] had an analogous viewpoint for the *logic* of mathematical discoveries. A theory $\Gamma \subseteq E$ for Lakatos is a set of *conjectures* (which are *tentatively* or *informally* "proven", *i.e.* that were not deduced from a set of axioms) which may be sound for a set of mathematical structures F, until we consider more structures that falsify an element of Γ. For example, the conjecture "Euler's formula $V - E + F = 2$ applies

to any polyhedron" may be *true* when tested with the set of "convex polyhedra", but fails otherwise.

2.4 R-Duality on a set E

We recall that when we have a duality between a set E and itself, we call it an *R-duality on E* and ask that $Pol^r_E = Pol^\ell_E$.

Definition 2.4.1 (Cyclic pole)

Given a map $\ll - \mid - \gg : E^2 \longrightarrow R$, a pole $P \subseteq R$ is *cyclic* if for all $x, y \in E$, $\ll x \mid y \gg \in P \Rightarrow \ll y \mid x \gg \in P$.

Equivalently, we can say that $\perp \subseteq E^2$ is a *symmetric* relation, *i.e.* $x \perp y \Rightarrow y \perp x$.

What follows is a necessary and sufficient condition to define an R-duality on E:

Proposition 2.4.2 *We have an R-duality on E iff the pole P is cyclic.*

Proof : For *necessity*:
Obviously, if the pole is empty, then it is cyclic, so suppose $P \neq \varnothing$. Let $a, b \in E$ and suppose $\ll a \mid b \gg \in P$, *i.e.* $a \perp b$. Consider $Pol^r_E(\{a\}) \stackrel{def}{=} \{y \in E \mid a \perp y\}$. Then $b \in Pol^r_E(\{a\})$. And since $Pol^r_E(\{a\}) = Pol^\ell_E(\{a\})$ by hypothesis, we have $b \in Pol^\ell_E(\{a\}) \stackrel{def}{=} \{x \in E \mid x \perp a\}$, which means $b \perp a$. Hence $\ll b \mid a \gg \in P$. For *sufficiency*: clear. $\square$

We introduce the notation $A^\perp$ for $Pol^r_E(A)$ (and $Pol^\ell_E(A)$). Now it might be convenient to recall some properties of Section 2.1:

Proposition 2.4.3 *Given an R-duality on E, we have, for all $A, A_i, B \subseteq E$,*

(i) $A \subseteq A^{\perp\perp}$;
(ii) $A \subseteq B \Rightarrow B^\perp \subseteq A^\perp$;

(iii) $A^{\perp} = A^{\perp\perp\perp}$;
(iv) $\bigcap_{i\in I} A_i^{\perp} = \left(\bigcup_{i\in I} A_i\right)^{\perp}$;
(iv) $\bigcup_{i\in I} A_i^{\perp} \subseteq \left(\bigcap_{i\in I} A_i\right)^{\perp}$.

Proof : See Propositions 2.1.5, 2.1.14 and Corollary 2.1.6. □

Remark 2.4.4

It is interesting to observe that if we think of $A^{\perp}$ as the *intuitionistic negation* "$\neg A := A \to \bot$", intersection, union and inclusion respectively as "$\wedge, \vee, \vdash$", then the *formulas* given by Proposition 2.4.3 are provable intuitionistically. In fact, when the inclusion is not given in Proposition 2.4.3, the associated sequent is not provable. For example, $\neg(A \wedge B) \vdash \neg A \vee \neg B$ is not provable, while $\neg A \vee \neg B \vdash \neg(A \wedge B)$ is:

$$\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \qquad \bot \vdash \bot}{A \to \bot, A \vdash \bot}}{A \to \bot, A \wedge B \vdash \bot}}{A \to \bot \vdash (A \wedge B) \to \bot} \qquad \dfrac{\dfrac{\dfrac{B \vdash B \qquad \bot \vdash \bot}{B \to \bot, B \vdash \bot}}{B \to \bot, A \wedge B \vdash \bot}}{B \to \bot \vdash (A \wedge B) \to \bot}}{(A \to \bot) \vee (B \to \bot) \vdash (A \wedge B) \to \bot}$$

Besides being a good *mnemonic* way to remember which *De Morgan* law is provable and which one is not in intuitionistic calculus, maybe we can hope that this interpretation of negation of A is an alternative for the traditional "A implies *absurdity*".

Now let's look at some algebraic properties of $\left(\mathcal{P}(E), \cup, \cap, (-)^{\perp}\right)$ with the ordering given by "$\subseteq$". Set theoretically, we obviously know that for any $A, B, C \in \mathcal{P}(E)$:

$$A \cup B = B \cup A$$

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$A \cup A = A$$

$$A \cap (A \cup B) = A$$

These also hold *dually* by replacing "$\cup$" by "$\cap$" (and vice versa). We also have neutral elements $\varnothing$ and E; so we may ask what other properties we get with respect to $(-)^{\perp}$, for any pole P.

By Proposition 2.4.3, we automatically have

$$A^{\perp} \cap B^{\perp} = (A \cup B)^{\perp}$$

If we consider a classical relation $\perp := \perp_{\varphi}$ for some bijection $\varphi : E \longrightarrow E$ (see Definition 2.1.21), then every subset is a fact, so moreover

$$A^{\perp} \cup B^{\perp} = (A \cap B)^{\perp}$$

by Proposition 2.1.17.

We may wonder if $(\mathcal{P}(E), \cup, \cap, (-)^{\perp}, \varnothing, E)$ is a Boolean algebra (where $\perp$ is classical). Not quite: $A \cup A^{\perp} = E$ and $A \cap A^{\perp} = \varnothing$ fail in general. We then say it is simply a *De Morgan algebra* ([25]).
In fact, it is Boolean for exactly one duality relation:

Proposition 2.4.5 $(\mathcal{P}(E), \cup, \cap, (-)^{\perp}, \varnothing, E)$ *is a Boolean algebra iff the duality relation is classical of the form* $\perp_{Id}$*, where* $Id : E \longrightarrow E$ *is the identity map.*

Proof : ($\Rightarrow$): Take $\{a\}$ for some $a \in E$. By hypothesis, $\{a\} \cap \{a\}^{\perp} = \varnothing$, so $a \notin \{a\}^{\perp}$. But also $\{a\} \cup \{a\}^{\perp} = E$, thus $\{a\}^{\perp} = E \backslash \{a\}$. This means $a \perp x$ for all $x \in E$, except for $x = a$. But since a was arbitrary, $\perp = \{(x, y) \in E^2 \mid x \neq y\}$, which is simply a classical duality relation $\perp_{\varphi}$ where $\varphi(x) = x$.
($\Leftarrow$): We already know that "De Morgan laws" hold, so all we need to verify is that $A \cup A^{\perp_{Id}} = E$ and $A \cap A^{\perp_{Id}} = \varnothing$ for all $A \subseteq E$. But this is obvious since $A^{\perp_{Id}} = E \backslash A$.

$\square$

Remark 2.4.6

Notice that we retrieved the notion of the *complement* of $A \subseteq E$ as a particular case of a classical duality relation. Indeed, $A^{\perp_{Id}} = E \backslash A$.

Now consider $\mathcal{H} \subseteq \mathcal{P}(E)$ such that $(\mathcal{H}, \subseteq, \cap, \cup, N, \top, \Rightarrow)$ is a Heyting algebra, where N and $\top$ are respectively the least and the greatest elements, and for all $A, B \in \mathcal{H}$ the *Heyting implication* is defined as:

$$A \Rightarrow B \stackrel{def}{=} \bigcup \{X \in \mathcal{H} | A \cap X \subseteq B\}$$

i.e. $A \Rightarrow B$ is the largest set (in $\mathcal{H}$) such that its intersection with A is contained in B (in other words, $X \cap A \subseteq B$ iff $X \subseteq A \Rightarrow B$).

The *pseudo-complement* of $A \in \mathcal{H}$ is given by :

$$\neg_N A \stackrel{def}{=} A \Rightarrow N$$

We may wonder how the pseudo-complement is related to our notion of polar set.

Proposition 2.4.7 *Let $\mathcal{H} \subseteq \mathcal{P}(E)$ be a filter, where E is a non-empty set. Given an R-duality on $\top$, we have $\neg_N A = A^{\perp}$ for all $A \in \mathcal{H}$ if and only if $\perp = \perp_{Id} \cup N^2$.*

Proof : For all $A \in \mathcal{H}$, we can easily see that $X = (\top \backslash A) \cup N$ is the biggest set in the filter such that $A \cap X$ is in N. Therefore $\neg_N A = X$. Suppose $\neg_N A = A^{\perp}$. We have previously observed that when the duality relation is of the form $\perp_{Id}$, $A^{\perp_{Id}} = \top \backslash A$. So all we have to add to $\perp_{Id}$ are the couples of the form (x, x) such that $x \in N$, since $N \subseteq A$ for any A. Conversely, if $\perp = \perp_{Id} \cup N^2$, then $A^{\perp} := \{x \in \top \mid A \perp x\} = (\top \backslash A) \cup N \stackrel{def}{=} (A \Rightarrow N)$. $\square$

Proposition 2.4.8 *Let $\mathcal{H} \subseteq \mathcal{P}(E)$ be a total order, i.e. a chain, where E is a non-empty set. Given an R-duality on $\top$, we have $\neg_N A = A^{\perp}$ for all $A \in \mathcal{H}$ if and only if $\perp = (N \times \top) \cup (\top \times N)$.*

Proof: ($\Rightarrow$): By definition,

$$\neg_N A := (A \Rightarrow N) = \begin{cases} \top & \text{if } A = N \\ N & \text{else} \end{cases}$$

Since by hypothesis $\neg_N A = A^{\perp}$, we consider two cases: if $A = N$, then $N^{\perp} := \{x \in E \mid N \perp x\} = \top$, which means $N \perp \top$ (and $\top \perp N$ since $\perp$ is symmetric). And if $A \neq N$, $A^{\perp} := \{x \in E \mid A \perp x\} = N$; since every $A \in \mathcal{H}$ is contained or equal to $\top$, we have $\top \perp N$ (and $N \perp \top$).
($\Leftarrow$): For any $\mathcal{H} \ni A \neq N$, $A^{\perp} = N$; and clearly $N^{\perp} = \top$. Therefore $A^{\perp} = \neg_N A$ as given above. $\square$

Examples 2.4.9 (Polarity in filters and chains)

Let's illustrate this:

- When $\varnothing \in \mathcal{H}$, we have $N = \varnothing$. If $\mathcal{H}$ is a filter, then it is *improper* and we retrieve the notion of complement. If $\mathcal{H}$ is a chain, then $\perp = \varnothing$.

- Consider $E = \{a, b, c, d\}$ and take the following sublattice $\mathcal{H}$ of $\mathcal{P}(E)$:

$$\begin{tikzcd} & E & \\ \{a,b,c\} \arrow[ur] & & \{a,b,d\} \arrow[ul] \\ & \{a,b\} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

  Using Proposition 2.4.7, if the pole is of the form

$$\begin{aligned} \perp \;=\; & \{(a,a),(a,b),(b,a),(a,c),(c,a),(a,d),(d,a),(b,b),(b,c),(c,b), \\ & (b,d),(d,b),(c,d),(d,c)\} \end{aligned}$$

  then the pseudo-complement and the polar set of A coincide.

- Now consider the chain $\mathcal{C}$

$$\begin{tikzcd} E \\ \{a,b,c\} \arrow[u] \\ \{a,b\} \arrow[u] \\ \{a\} \arrow[u] \end{tikzcd}$$

Using Proposition 2.4.8, if the duality relation is of the form

$$\perp = \{(a,a),(a,b),(b,a),(a,c),(c,a),(a,d),(d,a)\}$$

then for all $A \in \mathcal{C}$, $\neg_{\{a\}}A = A^{\perp}$.

Now let E be a non-empty set and take any R-duality on E, with duality relation $\perp$. Define

$$N := \{x \in E \mid x \perp E\}$$

and

$$I := \{x \in E \mid x \perp x\}$$

In general, $N \subseteq I$, and the following properties hold:

Proposition 2.4.10 *For all $A \subseteq E$, we have*

(i) $A \subseteq N \Rightarrow A^{\perp} = E$;

(ii) $E^{\perp} = N$;

(iii) $N \subseteq A^{\perp}$;

(iv) $A \cap A^{\perp} \subseteq I$;

(v) *If A is a fact, then* $\left(A \cup A^{\perp}\right)^{\perp\perp} = E \Leftrightarrow A \cap A^{\perp} = N$.

Proof : (i) to (iv) are obvious by definition. For (v), we know by Proposition 2.4.3 that $\left(A \cup A^{\perp}\right)^{\perp\perp} = \left(A^{\perp} \cap A^{\perp\perp}\right)^{\perp}$, which is equal to $\left(A^{\perp} \cap A\right)^{\perp}$ since A is a fact. So $\left(A \cup A^{\perp}\right)^{\perp\perp} = E \Rightarrow \left(A \cup A^{\perp}\right)^{\perp\perp\perp} = E^{\perp} \Rightarrow \left(A^{\perp} \cap A\right)^{\perp\perp} = E^{\perp}$, hence $A^{\perp} \cap A = N$ by (ii). The converse is obvious since $N^{\perp} = E$ by (i). □

Here are some examples of R-dualities on a set.

Example 2.4.11 (Vector spaces as facts)

This example motivates the notation used by Girard ([15],[17]):

Let $(E, +, \cdot)$ be a vector space over $\mathbb{R}$. Consider an $\mathbb{R}$-duality on E with $P = \{0\}$ as the pole, and define a symmetric bilinear map $\ll - \mid - \gg\, : E^2 \longrightarrow \mathbb{R}$. In particular, when $\ll - \mid - \gg$ is *positive definite*, *i.e.* $\ll v \mid v \gg\ \in [0, \infty+[$ and $\ll v \mid v \gg = 0$ if and only if $v = 0$ for all $v \in E$, we retrieve the notion of *scalar product* ($\ll v \mid v \gg = 0$ iff $v = 0$ simply means $N = I = \{0\}$). Notice that the pole is cyclic and for any $A \subseteq E$, we say $A^{\perp}$ is the set of vectors *orthogonal* to the ones in A [22]. Here are some other interesting observations:

Proposition 2.4.12 *In vector spaces E, every fact is a subspace of E.*

Proof : By Proposition 2.1.13 (i), facts are polar sets, so let $A^{\perp}$ be a polar set for some $A \subseteq E$. Clearly $0 \in A^{\perp}$, since we have $< a \mid 0 >= 0$ for all $a \in A$ (by bilinearity), so $A^{\perp} \neq \varnothing$. Also, let $v, w \in A^{\perp}$. Then for any $\lambda, \mu \in \mathbb{R}$, $\lambda v + \mu w \in A^{\perp}$: for all $a \in A$, $\ll a \mid \lambda v + \mu w \gg = \ll a \mid \lambda v \gg + \ll a \mid \mu w \gg = \lambda \ll a \mid v \gg + \mu \ll a \mid w \gg = \lambda \cdot 0 + \mu \cdot 0 = 0$.

□

Definition 2.4.13

Let E be a vector space and $A, B \subseteq E$ be two subspaces. The *sum* of A and B is given by

$$A + B \stackrel{def}{=} \{v \in E \mid \exists a \in A, \exists b \in B \text{ such that } v = a + b\}$$

The following is well-known (see [22]):

Proposition 2.4.14 *For all subspaces A, B of E, $A + B$ is a subspace of E.*

We also know from [22] (p.325):

Proposition 2.4.15 *Let E be of finite dimension. Then for any subspace A of E,*

(i) $\dim E = \dim A + \dim A^{\perp} - \dim(A \cap N)$;

(ii) $A^{\perp\perp} = A + N$.

In particular, we have

Proposition 2.4.16 *If E is a vector space of finite dimension and $N = \{0\}$, then every subspace A of E is a fact, i.e. $A^{\perp\perp} = A$.*

Therefore by Propositions 2.4.12 and 2.4.16, we know that for any vector space of finite dimension, facts are exactly the subspaces. Moreover,

Proposition 2.4.17 *Let E be of finite dimension. Then for all subspaces A, B of E, $A + B = (A \cup B)^{\perp\perp}$.*

Proof : Clearly $A \cup B \subseteq A + B$, and since $(A \cup B)^{\perp\perp}$ is the smallest fact containing $A \cup B$ by Proposition 2.1.16, it is the smallest subspace containing it (Proposition 2.4.12 and 2.4.16). All we need to show is that $A + B$, which is a subspace by Proposition 2.4.14, is contained in $(A \cup B)^{\perp\perp}$. Suppose $v \in A + B$, *i.e.* $v = a + b$ for $a \in A$, $b \in B$. Take any $w \in (A \cup B)^{\perp}$: $w \perp v'$ for all $v' \in A \cup B$, and in particular $w \perp a$ and $w \perp b$. Therefore $\ll w \mid v \gg = \ll w \mid a+b \gg = \ll w \mid a \gg + \ll w \mid b \gg = 0$. Since w is arbitrary, this means $w \perp v$ for all $w \in (A \cup B)^{\perp}$, hence $v \in (A \cup B)^{\perp\perp}$. $\square$

Definition 2.4.18

Let E be a vector space and $A, B \subseteq E$ be two subspaces. The *direct sum* of A and B is given by

$$A \oplus B \stackrel{def}{=} \{v \in E \mid \exists! a \in A, \exists! b \in B \text{ such that } v = a + b\}$$

The following is well-known (see [22]):

Proposition 2.4.19 *For all subspaces A, B of E, $v \in A \oplus B$ if and only if $v \in A + B$ and $A \cap B = \{0\}$.*

Proposition 2.4.20 *Let A be a subspace of a vector space E of finite dimension. Then $A \oplus A^{\perp\!\!\!\!\!\!\cup} = E \Leftrightarrow A \cap A^{\perp\!\!\!\!\!\!\cup} = \{0\} = N$.*

Proof :

$$\begin{aligned} A \oplus A^{\perp\!\!\!\!\!\!\cup} = E \quad &\Leftrightarrow \quad A + A^{\perp\!\!\!\!\!\!\cup} = E \text{ and } A \cap A^{\perp\!\!\!\!\!\!\cup} = \{0\} \text{ by Proposition 2.4.19} \\ &\Leftrightarrow \quad \left(A \cup A^{\perp\!\!\!\!\!\!\cup}\right)^{\perp\!\!\!\!\!\!\cup\perp\!\!\!\!\!\!\cup} = E \text{ and } A \cap A^{\perp\!\!\!\!\!\!\cup} = \{0\} \text{ by Proposition 2.4.17} \\ &\Leftrightarrow \quad A \cap A^{\perp\!\!\!\!\!\!\cup} = N \text{ and } A \cap A^{\perp\!\!\!\!\!\!\cup} = \{0\} \text{ by Proposition 2.4.10 (v)} \end{aligned}$$

$\square$

Remark 2.4.21

Notice that if E is a *complex* vector space (of finite dimension), we have analogous results when $\ll - \mid - \gg : E^2 \longrightarrow \mathbb{C}$ is an *hermitian form*, *i.e.* it is a *sesquilinear form* :

- $\ll x + x' \mid y \gg = \ll x \mid y \gg + \ll x' \mid y \gg$ and $\ll x \mid y + y' \gg = \ll x \mid y \gg + \ll x \mid y' \gg$;
- $\ll x \mid \lambda y \gg = \lambda \ll x \mid y \gg$ and $\ll \lambda x \mid y \gg = \overline{\lambda} \ll x \mid y \gg$.

and

$$\ll x \mid y \gg = \overline{\ll y \mid x \gg}$$

Indeed, if the pole is $\{0\}$, then it is cyclic : $\ll x \mid y \gg = 0 \Leftrightarrow \overline{\ll y \mid x \gg} = 0 \Leftrightarrow$ $\ll y \mid x \gg = 0$ since $0 = \overline{0}$.

Example 2.4.22 (Coherence spaces as facts)

Let X be a set and consider $\mathcal{E} = \mathcal{P}(X)$.

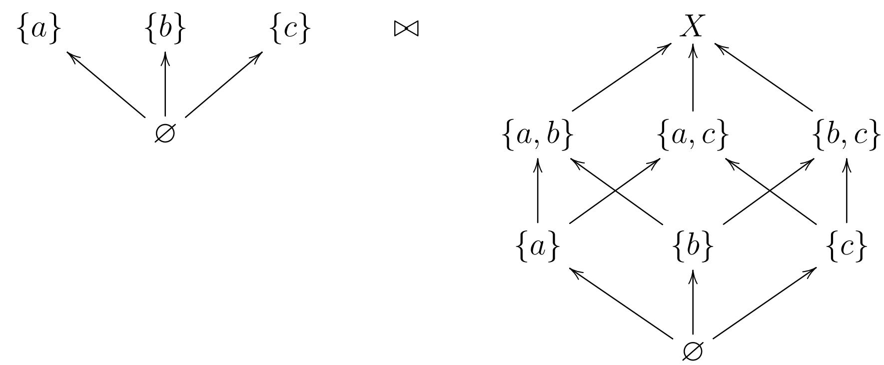
Define an $\mathbb{N}$-duality on $\mathcal{E}$ as follows: take the map

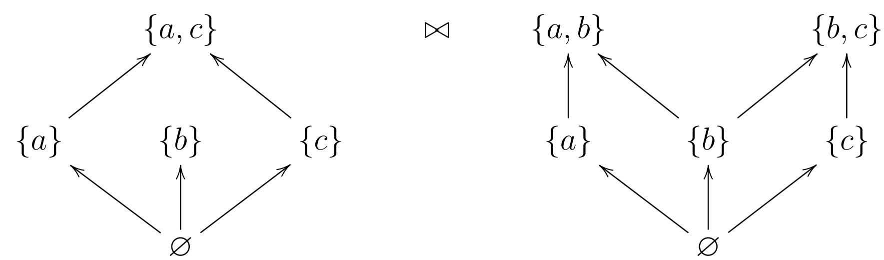
$$\begin{array}{rcl} \ll - \mid - \gg : \mathcal{E} \times \mathcal{E} & \to & \mathbb{N} \\ (A, B) & \mapsto & |A \cap B| \end{array}$$

and let the pole be $P = \{0, 1\}$.

Here are some examples:

- Let $X = \varnothing$, and $\mathcal{E} = \{\varnothing\}$. Then $\perp\!\!\!\!\!\!\smile = \{(\varnothing, \varnothing)\}$, and by Proposition 2.1.19, the only fact is $\mathcal{E}$.
- Let $X = \{*\}$, and $\mathcal{E} = \{\varnothing, \{*\}\}$. Then again $\perp\!\!\!\!\!\!\smile = \mathcal{E} \times \mathcal{E}$, and by Proposition 2.1.19, the only fact is $\mathcal{E}$.
- Let $X = \{t, f\}$. Then $\perp\!\!\!\!\!\!\smile = (\mathcal{E} \times \mathcal{E}) \backslash \{(X, X)\}$. The only facts are $\mathcal{E}$ and $\{\varnothing, \{t\}, \{f\}\}$.
- Let $X = \{a, b, c\}$. Here are some *corresponding* facts:





Definition 2.4.23 A *coherence space* [10] is a subset $\mathcal{A} \subseteq \mathcal{P}(X)$, where X is a set, which satisfies:
(i) Down-closure: if $A \in \mathcal{A}$ and $A' \subseteq A$, then $A' \in \mathcal{A}$.
(ii) Binary completeness: if $\mathcal{M} \subseteq \mathcal{A}$ and if for all $A_1, A_2 \in \mathcal{M}$, $A_1 \cup A_2 \in \mathcal{A}$, then $\bigcup \mathcal{M} \in \mathcal{A}$. In particular (if $\mathcal{M} = \varnothing$), we have $\varnothing \in \mathcal{A}$.

It will be convenient to write $|\mathcal{A}|$ for $\{a \in X \mid \{a\} \in \mathcal{A}\}$. The elements of $|\mathcal{A}|$ are called the *tokens* by Girard [10].
Moreover, we can define a *coherence relation modulo* $\mathcal{A}$ between tokens by

$$a \supset b \mod \mathcal{A} \Leftrightarrow \{a, b\} \in \mathcal{A}$$

$|\mathcal{A}|$ equipped with $\supset$ is a *graph*, called the *web* of $\mathcal{A}$: the *vertices* are the tokens, and there is an *edge* between a and b iff $a \supset_{\mathcal{A}} b$, where $a \supset_{\mathcal{A}} b$ means $a \supset b \mod \mathcal{A}$. The *cliques* are exactly the elements of $\mathcal{A}$ (see [10]).
In what follows, given a coherence space $\mathcal{A}$, we define an $\mathbb{N}$-duality on $\mathcal{P}(|\mathcal{A}|)$ as

discussed previously:

$$\begin{aligned} \ll - \mid - \gg : \mathcal{P}(|\mathcal{A}|) \times \mathcal{P}(|\mathcal{A}|) &\rightarrow \mathbb{N} \\ (A, B) &\mapsto |A \cap B| \end{aligned}$$

with $P = \{0, 1\}$.

Proposition 2.4.24 *Let $\mathcal{A}$ be a coherence space. Then $\{a, b\} \in \mathcal{A} \Leftrightarrow \{a, b\} \notin \mathcal{A}^{\perp}$ for $a, b \in |\mathcal{A}|$, $a \neq b$.*

Proof : For $(\Rightarrow)$: obvious by definition of the duality.
For $(\Leftarrow)$: Since $\{a, b\} \notin \mathcal{A}^{\perp}$, then there exists $B \in \mathcal{A}$ such that $|\{a, b\} \cap B| > 1$. Since $\{a, b\} \subseteq B$, we conclude $\{a, b\} \in \mathcal{A}$ by (i) of Definition 2.4.23. $\square$

Remark 2.4.25

Proposition 2.4.24 tells us that the web of $\mathcal{A}$ is the *complement* of the web of $\mathcal{A}^{\perp}$ (if we ignore the *loops*). Also, if $a \frown_{\mathcal{A}} b$, then $a \asymp_{\mathcal{A}^{\perp}} b$, which means a and b are *incoherent* in $\mathcal{A}^{\perp}$, *i.e.* $\{a, b\} \notin \mathcal{A}^{\perp}$ or $a = b$. In other words, the *coherent* subsets of $\mathcal{A}$ correspond to the *incoherent* subsets of $\mathcal{A}^{\perp}$, and *vice versa*.

The following equivalence can be found in [15] (p.194):

Proposition 2.4.26 *$\mathcal{A}$ is a coherence space if and only if $\mathcal{A}^{\perp\perp} = \mathcal{A}$.*

Proof : For $(\Rightarrow)$: it suffices to show $\mathcal{A}^{\perp\perp} \subseteq \mathcal{A}$. Let $A \subseteq |\mathcal{A}|$ and suppose $A \notin \mathcal{A}$. Since A is neither $\varnothing$ nor a singleton, we have $|A| > 1$. Consider two cases :
If $A \in \mathcal{A}^{\perp}$, then clearly $A \notin \mathcal{A}^{\perp\perp}$ by definition;
if $A \notin \mathcal{A}^{\perp}$, then $|A| > 2$ by Proposition 2.4.24. Consider $\mathcal{M} = \{\{x\} \mid x \in A\}$. Clearly $\bigcup \mathcal{M} = A \notin \mathcal{A}$. Hence by Definition 2.4.23 (ii), since $\mathcal{M} \subseteq \mathcal{A}$, there exists $A_1, A_2 \in \mathcal{M}$ such that $A_1 \cup A_2 \notin \mathcal{A}$. Said differently, this means there exists $\{x, y\} \subseteq A$ such that $\{x, y\} \notin \mathcal{A}$. Therefore $\{x, y\} \in \mathcal{A}^{\perp}$, and $A \notin \mathcal{A}^{\perp\perp}$ by definition of the duality.

Now for $(\Leftarrow)$: Suppose $A \in \mathcal{A}$ and $A' \subseteq A$. We know $A \in \mathcal{A}^{\perp\perp}$, so *a fortiori* $A' \in \mathcal{A}^{\perp\perp} = \mathcal{A}$, which satisfies condition (i) of Definition 2.4.23. For (ii), suppose $\mathcal{M} \subseteq \mathcal{A}$ and $\forall A, B \in \mathcal{M}$, $A \cup B \in \mathcal{A}$. Consider $\bigcup \mathcal{M}$: we have to show it belongs to $\mathcal{A}$.
If $\mathcal{M} = \{\varnothing\}$ or $\mathcal{M} = \{\varnothing, \{a\}\}$ (for any $a \in |\mathcal{A}|$), then clearly $\bigcup \mathcal{M} \in \mathcal{A}$. Otherwise, $\mathcal{M}$ contains two distinct non-empty subsets, say A, B, and since by hypothesis $A \cup B \in \mathcal{A}$, then $\bigcup \mathcal{M} \notin \mathcal{A}^{\perp}$ (indeed, $A \cup B \subseteq \bigcup \mathcal{M}$ and $|A \cup B| > 1$).
Let's proceed by contradiction : suppose $\bigcup \mathcal{M} \notin \mathcal{A}^{\perp\perp}$. Then there exists $B' \in \mathcal{A}^{\perp}$ such that $|B' \cap \bigcup \mathcal{M}| > 1$, *i.e.* there are at least two elements a, b in $B' \cap \bigcup \mathcal{M}$. But a, b belong to some sets in $\mathcal{M}$, say $a \in A$ and $b \in B$ (A, B may not be distinct). So by hypothesis, $A \cup B \in \mathcal{A}$. Therefore $B' \notin \mathcal{A}^{\perp}$, contradiction. $\square$

Before moving to the next example, consider the following:
Let X be a *finite* set and define an $\mathbb{N}$-duality on $\mathcal{P}(X)$ as in the previous example, *i.e.* $A \perp B \Leftrightarrow \ll A \mid B \gg \in \{0,1\} \Leftrightarrow |A \cap B| \in \{0,1\}$. Also, define an $\mathbb{N}$-duality on $\{0,1\}^X$, the set of maps from X to $\{0,1\}$, such that $\lll f \mid g \ggg := \sum_{x \in X} f(x)g(x)$, with the duality relation given by $f \approx g \Leftrightarrow \lll f \mid g \ggg \in \{0,1\}$.
Now consider the bijection

$$\begin{array}{rcl} \varphi : \mathcal{P}(X) & \longrightarrow & \{0,1\}^X \\ A & \mapsto & \left(X \xrightarrow{\chi_A} \{0,1\}\right) \end{array}$$

such that χ_A is the *characteristic function* associated to A :

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \textit{else} \end{cases}$$

We can show the following:

Proposition 2.4.27 $A \perp B \Leftrightarrow \varphi(A) \approx \varphi(B)$

Proof: Indeed, $A \perp B \Leftrightarrow A \cap B = \varnothing$ or $A \cap B = \{a\}$ for some $a \in A, B$. The first case is equivalent to saying $\lll\varphi(A) \mid \varphi(B)\ggg = 0$ since $\varphi(A)(x) \neq \varphi(B)(x)$ for all $x \in X$. The second case is equivalent to $\lll\varphi(A) \mid \varphi(B)\ggg = 1$ since

$$\varphi(A)(x)\varphi(B)(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{else} \end{cases}$$

In both cases, $\varphi(A) \underset{\approx}{\perp} \varphi(B)$. □

This suggests the notion of some kind of "*duality morphism*":

Definition 2.4.28 (duality morphism)

Given an R-duality on E and an R'-duality on E', with duality relations respectively $\perp$ and $\underset{\approx}{\perp}$, a map $\varphi : E \longrightarrow E'$ is a *morphism* of dualities if $x \perp y \Leftrightarrow \varphi(x) \underset{\approx}{\perp} \varphi(y)$. We say it is a *monomorphism* when φ is injective, and an *isomorphism* when the map is bijective.

In our example, since φ is a bijection by Proposition 2.4.27, this means the facts in $\{0,1\}^X$ correspond exactly to the facts of $\mathcal{P}(X)$, *i.e.* the coherence spaces (Proposition 2.4.26). Therefore we have yet another alternative definition for coherence spaces (for finite X), which will be generalized in the following example.

Example 2.4.29 (Probabilistic coherence spaces as facts)

Let X be a finite set and define an $\mathbb{R}_+$-duality on $\mathbb{R}_+^X$, the set of functions from X to the positive real numbers, such that $\ll f \mid g \gg = \sum_{x \in X} f(x)g(x)$ with $P = [0,1]$, *i.e.*

$$f \perp g \Leftrightarrow \ll f \mid g \gg \in [0,1]$$

In this duality, facts are called *probabilistic coherence spaces* and are characterized by the following theorem proved by Girard in [17] (p.410):

Theorem 2.4.30 *A subset $\mathcal{A} \subseteq \mathbb{R}_+^X$, where X is a finite set, is a probabilistic coherence space, i.e. a fact, if and only if it satisfies:*

(i) $\mathcal{A} \neq \varnothing$ *(in particular, it contains the "null" function* $0(x) = 0$ *for all* $x \in X$*).*

(ii) $\mathcal{A}$ *is a closed convex.*

(iii) *If* $f \leqslant g \in \mathcal{A}$*, i.e.* $f(x) \leqslant g(x)$ *for all* $x \in X$*, then* $f \in \mathcal{A}$*.*

This idea can be generalized by the following example of Ehrhard [6]:
Let X be an *arbitrary* at most countable set, and consider an R-duality on $\mathbb{R}^X$, the set of functions from X to $\mathbb{R}$, by letting

$$\ll f \mid g \gg = \sum_{x \in X} |f(x)g(x)|$$

Here R is the *rig* $\mathbb{R} \cup \{\infty\}$ and $P = \mathbb{R}$.

In other words, $f \perp g \Leftrightarrow \ll f \mid g \gg < \infty$, *i.e.* iff the sum *converges.*
As pointed out by Ehrhard, facts are called (real) *Köthe spaces* (of *web* X).

Notice that in the previous example, if $X = \{x_1, x_2, ..., x_n\}$, we can represent $f \in \mathbb{R}_+^X$ by the following *diagonal matrix*:

$$M(f) = \begin{pmatrix} f(x_1) & 0 & \dots & 0 \\ 0 & f(x_2) & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \dots & 0 & f(x_n) \end{pmatrix}$$

so that given another $g \in \mathbb{R}_+^X$, we have $\ll f \mid g \gg = Tr\,(M(f)M(g))$. This remark leads *naturally* to *quantum coherence spaces.*

Example 2.4.31 (Quantum coherence spaces as facts)

In this example [17], we consider X with the structure of a (complex) Hilbert space of finite dimension. Let E be the space of *hermitian* operators over X, *i.e.* such that $< h(x) \mid y >_X = < x \mid h(y) >_X$ for all $h \in E$, where $< - \mid - >_X$ is the *(hermitian) scalar product.*

E is an euclidian space, *i.e.* a real hermitian space (or a real prehilbertian space of finite dimension), with $\ll h \mid k \gg \stackrel{def}{=} Tr(hk)$ for all $h, k \in E$.
Now we define a $\mathbb{C}$-duality on E by

$$h \perp k \Leftrightarrow \ll h \mid k \gg \in [0,1]$$

Here, facts are called *quantum coherence spaces* and are characterized by the following theorem proved by Girard in [17] (p.414):

Theorem 2.4.32 *A subset $\mathcal{A} \subseteq E$, with E as above, is a quantum coherence space, i.e. a fact, iff it satisfies:*

(i) $0 \in \mathcal{A}$.

(ii) *$\mathcal{A}$ is a closed convex.*

(iii) *If $nf \in \mathcal{A}$ for all $n \in \mathbb{N}$, then $-f \in \mathcal{A}$.*

(iv) *If $f, g \in \mathcal{A}$, $\lambda, \mu \geqslant 0$ and $\lambda f + \mu g \in \mathcal{A}$, then $\lambda f \in \mathcal{A}$.*

2.5 Duality on a set E

We recall that by a *duality on E*, we mean an E-duality on E.
In other words, the map $\ll - \mid - \gg : E^2 \longrightarrow E$ plays the role of an *internal product*.
Notice that the map $\ll - \mid - \gg$ may be defined using the structure on E if there is any. For example, if $(E, \cdot, 1)$ is a *monoid*, then we may let $\ll a \mid b \gg \stackrel{def}{=} a \cdot b$. But then we must be careful when choosing the pole $P \subseteq E$. Indeed, for the *free* monoid, say Σ^*, if the *word* $abc \in P$, then we must also have $bca \in P$ and $cab \in P$, since the pole has to be cyclic by Proposition 2.4.2.
Obviously, $P \in \{\varnothing, E\}$ are always possible poles, and if E is a *commutative* monoid, then any subset of E can be a pole.

The next lemmas and propositions will show how a duality on E can serve to characterize additional structure on E. Let $(E, \cdot, 1)$ be a monoid. We define $\ll a \mid b \gg \stackrel{def}{=} a \cdot b$.

Notation: In the following propositions referring to group theory, in order to avoid confusion with the notion of cyclic group, we write "cyclic" (in quotes) to denote cyclicity of the pole in a polarity.

Lemma 2.5.1 *Let $(G, \cdot, 1)$ be a group with $g \in G$ a given element. Then the pole $P = \{g\}$ is "cyclic" iff g is in the center of G.*

Proof: $(\Rightarrow)$: Suppose P is "cyclic" and take $a \in G$. Since $\ll a \mid a^{-1} \cdot g \gg$ and $\ll g \cdot a^{-1} \mid a \gg$ are in P, then $\ll a^{-1} \cdot g \mid a \gg$ and $\ll a \mid g \cdot a^{-1} \gg$ are also in P (by hypothesis). Therefore $a \cdot a^{-1} \cdot g = a \cdot g \cdot a^{-1}$, which means $g \cdot a = a \cdot g$.
$(\Leftarrow)$: Suppose for all $a \in G$, we have $a \cdot g = g \cdot a$. Then take $a, b \in G$ and suppose $\ll a \mid b \gg \in P$, *i.e.* $a \cdot b = g$. Then $b = a^{-1} \cdot g \stackrel{hyp}{\Rightarrow} b = g \cdot a^{-1} \Rightarrow b \cdot a = g$. Thus $\ll b \mid a \gg \in P$. $\square$

Proposition 2.5.2 *Let $(G, \cdot, 1)$ be a group, $|G| \geqslant 2$ and $g \in G$. Given a duality on G such that $P = \{g\}$, the only facts are $\varnothing$, G and every singleton.*

Proof: Since $G^{\perp} = \{b \in G \mid \forall a \in G, a \cdot b = b \cdot a = g\} = \varnothing$, then $G^{\perp\perp} = G$. Now take any non-empty set A different than G and consider $A^{\perp} = \{b \in G \mid \forall a \in A, a \cdot b = b \cdot a = g\}$. If $|A| \geqslant 2$, then $A^{\perp} = \varnothing$, so A is not a fact. If A is a singleton, say $\{a\}$, then $A^{\perp} = \{b \in G \mid a \cdot b = b \cdot a = g\}$. But $A^{\perp}$ is not empty iff $a^{-1} \cdot g = g \cdot a^{-1}$, which is the case by Lemma 2.5.1. Hence $A^{\perp\perp} = A$. $\square$

Lemma 2.5.3 *Let $(G, \cdot, 1)$ be a group. Then the pole $P = G \backslash \{g\}$, for some $g \in G$, is "cyclic" iff g is in $Z(G)$, the center of G.*

Proof : ($\Rightarrow$)

$$\begin{aligned} g \notin P &\Rightarrow g \cdot h \cdot h^{-1} \notin P \text{ for all } h \in G \\ &\Rightarrow h^{-1} \cdot g \cdot h \notin P \text{ for all } h \in G \text{ (since } P \text{ is ``cyclic'')} \\ &\Rightarrow g = h^{-1} \cdot g \cdot h \text{ for all } h \in G \text{ (since there is only one element not in } P) \\ &\Rightarrow h \cdot g = g \cdot h \text{ for all } h \in G, \textit{ i.e. } g \in Z(G). \end{aligned}$$

Conversely, let $P = G\backslash\{g\}$ with $g \in Z(G)$. Suppose $\ll a \mid b \gg \in P$, *i.e.* $a \cdot b \neq g$. Now suppose $\ll b \mid a \gg \notin P$. This means

$$\begin{aligned} b \cdot a = g &\Rightarrow b = g \cdot a^{-1} \\ &\Rightarrow b = a^{-1} \cdot g \text{ (since } g \in Z(G)) \\ &\Rightarrow a \cdot b = g \end{aligned}$$

which is a contradiction. □

Proposition 2.5.4 *Let* $(G, \cdot, 1)$ *be a group and suppose there is a duality on* G. *Then the pole is of the form* $P = G\backslash\{g\}$ *for some* $g \in G$ *iff for all* $A \subseteq G$, A *is a fact with respect to* P.

Proof : Notice that if $G = \{1\}$, then it's clear since $\varnothing$ and G are the only subsets of G. If $|G| = 2$, *i.e.* $G = \{1, \sigma\}$, with $\sigma^2 = 1$, then for $P = \varnothing$, singletons are not facts; for $P \in \{\{1\}, \{\sigma\}\}$, every subset is a fact.

So consider when $|G| > 2$.

($\Rightarrow$) Clearly $\varnothing$ and G are facts, since $\varnothing^{\perp} = G$ and $G^{\perp} = \varnothing$ (indeed, for any $h \in G$, $(g \cdot h^{-1}) \cdot h \notin P$).
Now consider any non-empty set $A \subseteq G$. By Proposition 2.4.3, we know $A \subseteq A^{\perp\perp}$.
Let's proceed by contraposition:
Take $h \in G$ and suppose $h \notin A$. Since $A^{\perp} = \{b \in G \mid \forall a \in A, a \cdot b \in P\}$,

then $h^{-1} \cdot g \in A^{\perp}$ (inverses are unique). Hence $g^{-1} \cdot (h \cdot g) \notin A^{\perp\perp}$ since $A^{\perp\perp} = \{a \in G \mid \forall b \in A^{\perp}, b \cdot a \in P\}$. But $g^{-1} \cdot (h \cdot g) = g^{-1} \cdot (g \cdot h) = (g^{-1} \cdot g) \cdot h = h$ since $g \in Z(G)$ by Lemma 2.5.3.
Therefore $h \notin A^{\perp\perp}$, and $A^{\perp\perp} \subseteq A$.

($\Leftarrow$) Suppose for all $A \subseteq G$, we have $A = A^{\perp\perp}$.
By Proposition 2.1.23, we know the duality relation is classical of the form $\perp_{\varphi}$, where $\varphi : G \longrightarrow G$ is a bijection.
Said differently, this means for all $a \in G$, $\ll a \mid b \gg = a \cdot b \in P$ for all but one $b \in G$. The Cayley table of $(G, \cdot, 1)$ tells us that for any given a, every element of G is in P except for some g_a (depending on a). And for every other distinct a', there is also only one $g_{a'} \notin P$. Hence g_a must be equal to $g_{a'}$. Therefore the pole contains every element of G except for one. $\square$

Proposition 2.5.5 *Let $(M, \cdot, 1)$ be a monoid. Suppose we have a duality on M given by $P = M \backslash \{1\}$.*
Then $(M, \cdot, 1)$ is a group iff $A^{\perp\perp} = A$ for all $A \subseteq M$.

Proof : ($\Rightarrow$) : Clear by Proposition 2.5.4.
($\Leftarrow$) : Suppose every subset of M is a fact. By Proposition 2.1.23, the duality relation is classical, hence any $a \in M$ is *polar* to any other element except for some $m_a \in M$, where $\ll a \mid m_a \gg = a \cdot m_a = 1$. And since the pole is "cyclic", we also have $m_a \cdot a = 1$, which means m_a is the *inverse* of a. $\square$

2.6 Alternative notations

In Girard's original paper on linear logic [8], given a *phase space* $(M, \perp)$, *i.e.* a commutative monoid M and $\perp \subseteq M$, the *dual* of $X \subseteq M$ was defined as follows:

$$X^{\perp} \stackrel{def}{=} \{y \in M \mid \forall x(x \in X \rightarrow yx \in \perp)\}$$

Linear implication was then defined by

$$X \multimap Y \overset{def}{=} \left(XY^{\perp}\right)^{\perp}$$

where $XY = \{xy \mid x \in X \text{ and } y \in Y\}$.

This led to the alternative definition

$$X \multimap Y \overset{def}{=} \{y \in M \mid \forall x \in X, yx \in Y\}$$

The notation was followed later by [24].

Later in [15], Girard introduced $\underset{\sim}{\downarrow}$ for $\perp$ and wrote $\sim X$ for $X^{\perp}$, called the *negation* of X. He also introduced $x \underset{\sim}{\downarrow} y$ for $xy \in \underset{\sim}{\downarrow}$ and $X \underset{\sim}{\downarrow} Y$ for $x \underset{\sim}{\downarrow} y$ $(x \in X, y \in Y)$. It is interesting to notice that Girard then defined $\sim X$ as a *particular* case of linear implication, *i.e.* $\sim X \overset{def}{=} X \multimap \underset{\sim}{\downarrow}$.

That idea was generalized in [15]. The subset $\perp \subseteq M$ could be any *pole* $P \subseteq R$, R being some set called the *scalar space* (see [18]). The *product* yx could be any binary map $\langle x \mid y\rangle : E \times F \longrightarrow R$, and the *polar set* of $X \subseteq E$ (and $Y \subseteq F$) would be given by:

$$X^P \overset{def}{=} \{y \in F \mid \forall x \in X, \langle x \mid y\rangle \in P\}$$

and

$$Y^P \overset{def}{=} \{x \in E \mid \forall y \in Y, \langle x \mid y\rangle \in P\}$$

Notice that Girard's notation of $x \underset{\sim}{\downarrow} y$ suggests that $\underset{\sim}{\downarrow}$ is a binary relation $\underset{\sim}{\downarrow} \subseteq M \times M$, but clearly it is not since $\underset{\sim}{\downarrow} \subseteq M$. Hence the definitions given in Section 2.1 of the present work seem less confusing. The introduction of the notation "$A^{\underset{\sim}{\downarrow}}$" for *polar sets* also tries to unify previous notations used by Girard and its followers, namely "$A^{\perp}$" and "$\sim A$".

Chapter 3

Examples in logic

3.1 *Tarskian* semantics

3.1.1 Classical Logic revisited

In the light of Section 2, we can explore further the ideas mentioned in the introduction. Given an R-duality on E, we can see the map $\ll - \mid - \gg : E^2 \longrightarrow R$ as the *testing process* and the pole $P \subseteq R$ as the "positive results", *i.e.* $\mu \perp \nu$ means "μ passed test ν" (and *vice versa* by symmetry), so that $A^{\perp}$ are the *tests* for A. Therefore, since a proposition is the set A of all elements that passed the tests for A, we have

$$\begin{array}{rcl} A & := & \{\mu \in E \mid \mu \perp A^{\perp}\} \\ & = & A^{\perp\perp} \end{array}$$

We can now give a meaning to the traditional connectives. Suppose $A, B \subseteq E$ are sets defining two propositions, *i.e.* A, B are facts. Set theoretically, we can imagine

the set representing the proposition "A *and* B" as given by

$$
\begin{aligned}
\text{“}A \text{ and } B\text{”} &= \{\mu \in E \mid \mu \perp A^{\perp} \text{ and } \mu \perp B^{\perp}\} \\
&= \{\mu \in E \mid \mu \perp A^{\perp}\} \cap \{\mu \in E \mid \mu \perp B^{\perp}\} \\
&= A^{\perp\perp} \cap B^{\perp\perp} \\
&= A \cap B
\end{aligned}
$$

By De Morgan's law, we can give a meaning to "A *or* B":

$$
\begin{aligned}
\text{“}A \text{ or } B\text{”} &:= \text{“ not (not } A \text{ and not } B\text{) ”} \\
&= \left(A^{\perp} \cap B^{\perp}\right)^{\perp} \\
&= (A \cup B)^{\perp\perp} \text{ by Proposition 2.4.3}
\end{aligned}
$$

We need to check that the previous meanings of *conjunction* and *disjunction* fit our definition by testing:

$$
\begin{aligned}
\text{“}A \text{ or } B\text{”} &:= \{\mu \in E \mid \mu \perp (\text{“}A \text{ or } B\text{”})^{\perp}\} \\
&= \{\mu \in E \mid \mu \perp (A \cup B)^{\perp\perp\perp}\} \\
&= (A \cup B)^{\perp\perp\perp\perp} \\
&= (A \cup B)^{\perp\perp}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{“}A \text{ and } B\text{”} &:= \{\mu \in E \mid \mu \perp (\text{“}A \text{ and } B\text{”})^{\perp}\} \\
&= \{\mu \in E \mid \mu \perp (A \cap B)^{\perp}\} \\
&= (A \cap B)^{\perp\perp} \\
&= A \cap B \text{ since the intersection of facts is a fact (see Remark 2.1.15).}
\end{aligned}
$$

Heuristically, if we think of A as a set of *observations* or *models* associated to some proposition, then we expect that $A^{\perp}$ may contain observations of the *negation* or *counter-models*. But it may also happen that an element belongs to both a set and

its polar. This may look odd at first sight, but it occurs in many common situations. For example, if the proposition "this house is inhabited" is interpreted by A, it may contain irrelevant observations, *e.g.* $x \in A$ that says "when we look through the window, we can't see anyone". But then $A^{\perp\!\!\!\perp}$, corresponding to the assertion that "this house is not inhabited", will also contain x.

Now to give a *semantical* interpretation for classical propositional logic, $[\![-]\!] : PROP_{CL} \longrightarrow \mathcal{P}(E)$, we need a notion of *truth* or *validity*. Since classical logic is based on the principle that a proposition is *true* if and only if its negation is *false*, we consider a distinguished element $\boldsymbol{v} \in E$ and say a proposition $A \in PROP_{CL}$ is *true* when $\boldsymbol{v} \in [\![A]\!]$. Then we define an R-duality on E such that given the map $\ll - \mid - \gg : E^2 \longrightarrow R$, the pole $P \subseteq R$ is chosen in a way that $\ll \boldsymbol{v} \mid x \gg \in P$ for all $x \neq \boldsymbol{v}$. In other words, we have

$$(\boldsymbol{v}, x) \in \perp\!\!\!\perp \Leftrightarrow \boldsymbol{v} \neq x$$

for all $x \in E$. We can see $\boldsymbol{v}$ as a *critical observation* that is not in contradiction with any other observation of E except for itself.

Here is how the interpretation goes for $PROP_{CL}$: atomic propositions are mapped to facts of E, with the usual constants interpreted as follows:

$$\begin{array}{lcl} \top & \mapsto & E \\ \bot & \mapsto & E^{\perp\!\!\!\perp} \end{array}$$

Other formulas are defined inductively:

$$\begin{array}{rcl} \neg A & \mapsto & [\![A]\!]^{\perp\!\!\!\perp} \\ A \wedge B & \mapsto & [\![A]\!] \cap [\![B]\!] \\ A \vee B := \neg(\neg A \wedge \neg B) & \mapsto & \left([\![A]\!]^{\perp\!\!\!\perp} \cap [\![B]\!]^{\perp\!\!\!\perp} \right)^{\perp\!\!\!\perp} = ([\![A]\!] \cup [\![B]\!])^{\perp\!\!\!\perp\perp\!\!\!\perp} \\ A \to B := \neg A \vee B & \mapsto & \left([\![A]\!]^{\perp\!\!\!\perp\perp\!\!\!\perp} \cap [\![B]\!]^{\perp\!\!\!\perp} \right)^{\perp\!\!\!\perp} = \left([\![A]\!]^{\perp\!\!\!\perp} \cup [\![B]\!] \right)^{\perp\!\!\!\perp\perp\!\!\!\perp} \end{array}$$

Proposition 3.1.1 *All formulas of $PROP_{CL}$ are mapped to facts under the above interpretation.*

Proof : Clear, since E is a fact by Proposition 2.1.5, polar sets are facts by Proposition 2.1.13, and the intersection of polar sets is a polar set by Proposition 2.4.3. $\square$

Remark 3.1.2

Notice that using Proposition 2.4.3, $A \vee B$ is mapped to $(\llbracket A \rrbracket \cup \llbracket B \rrbracket)^{\perp\perp}$, which is the smallest fact that contains $\llbracket A \rrbracket \cup \llbracket B \rrbracket$ by Proposition 2.1.16. Also, $A \to \bot$ is mapped to $\left(\llbracket A \rrbracket^{\perp} \cup E^{\perp}\right)^{\perp\perp}$, and by Proposition 2.4.3, $\varnothing \subseteq \llbracket A \rrbracket^{\perp} \Rightarrow \llbracket A \rrbracket^{\perp\perp} \subseteq \varnothing^{\perp} \Rightarrow \varnothing^{\perp\perp} \subseteq \llbracket A \rrbracket^{\perp\perp\perp}$, hence $E^{\perp} \subseteq \llbracket A \rrbracket^{\perp}$. Therefore $\left(\llbracket A \rrbracket^{\perp} \cup E^{\perp}\right)^{\perp\perp} = \left(\llbracket A \rrbracket^{\perp}\right)^{\perp\perp} = \llbracket A \rrbracket^{\perp}$, which is the interpretation of $\neg A$. Hence we retrieve the traditional equivalence of propositions : $\neg A = A \to \bot$.

We will show that (any) formal system used to derive classical formulas is both *sound* and *complete* with respect to this interpretation.
We say that an *inference rule* is valid (in the formal system) if whenever the (interpreted) premises contain $\boldsymbol{v}$, then so does the conclusion.
We also say that A is a *tautology* ($\models A$) if and only if $\boldsymbol{v} \in \llbracket A \rrbracket$ for *any* assignment $\llbracket - \rrbracket : PROP_{CL} \longrightarrow \mathcal{P}(E)$.

Lemma 3.1.3 *Given an R-duality on E as above, we have $\boldsymbol{v} \in A \Leftrightarrow \boldsymbol{v} \notin A^{\perp}$ for any $A \subseteq E$.*

Proof : Clear by the definition of the duality relation. $\square$

Obviously, it follows that

Corollary 3.1.4 $\boldsymbol{v} \in A \cup A^{\perp}$

Lemma 3.1.5 *For any fact $A, B, C \subseteq E$, we have*

(i) $\boldsymbol{v} \in \Big([\![A]\!]^{\perp} \cup \big([\![B]\!]^{\perp} \cup [\![A]\!] \big)^{\perp\perp} \Big)^{\perp\perp}$

(ii)

$$\boldsymbol{v} \in \Bigg(\bigg([\![A]\!]^{\perp} \cup \big([\![B]\!]^{\perp} \cup [\![C]\!] \big)^{\perp\perp} \bigg)^{\perp\perp\perp} \cup \bigg(\big([\![A]\!]^{\perp} \cup [\![B]\!] \big)^{\perp\perp\perp} \cup \big([\![A]\!]^{\perp} \cup [\![C]\!] \big)^{\perp\perp} \bigg)^{\perp\perp} \Bigg)^{\perp\perp}$$

(iii)

$$\boldsymbol{v} \in \bigg(\Big([\![B]\!]^{\perp\perp} \cup [\![A]\!]^{\perp} \Big)^{\perp\perp\perp} \cup \Big([\![A]\!]^{\perp} \cup [\![B]\!] \Big)^{\perp\perp} \bigg)^{\perp\perp}$$

Proof : For (i): clear by Lemma 3.1.3 and Corollary 3.1.4.
For (ii): If $\boldsymbol{v} \in [\![A]\!]^{\perp}$ or $\boldsymbol{v} \in [\![C]\!]$, this is done, so suppose it is not the case. Then suppose $\boldsymbol{v} \notin [\![B]\!]$. Thus $\boldsymbol{v} \notin [\![A]\!]^{\perp} \cup [\![B]\!]$, which means $\boldsymbol{v} \in \Big([\![A]\!]^{\perp} \cup [\![B]\!] \Big)^{\perp\perp\perp}$ by Lemma 3.1.3. And if $\boldsymbol{v} \in [\![B]\!]$, then $\boldsymbol{v} \notin [\![B]\!]^{\perp}$, hence $\boldsymbol{v} \notin [\![A]\!]^{\perp} \cup \Big([\![B]\!]^{\perp} \cup [\![C]\!] \Big)^{\perp\perp}$ and consequently $\boldsymbol{v} \in \Bigg([\![A]\!]^{\perp} \cup \Big([\![B]\!]^{\perp} \cup [\![C]\!] \Big)^{\perp\perp} \Bigg)^{\perp\perp\perp}$, which concludes the proof.
For (iii): If $\boldsymbol{v} \in [\![A]\!]^{\perp}$ or $\boldsymbol{v} \in [\![B]\!]$, then $\boldsymbol{v} \in \Big([\![A]\!]^{\perp} \cup [\![B]\!] \Big)^{\perp\perp}$ and it's done, so suppose it is not the case. Then $\boldsymbol{v} \in \Big([\![B]\!]^{\perp\perp} \cup [\![A]\!]^{\perp} \Big)^{\perp\perp\perp}$. □

Theorem 3.1.6 (*Soundness*) *Given an R-duality on $E \neq \varnothing$ as described above, if $\vdash_c A$ for $A \in PROP_{CL}$ then for all assignments $[\![-]\!] : PROP_{CL} \longrightarrow \mathcal{P}(E)$, we have $\boldsymbol{v} \in [\![A]\!]$.*

Proof : We need to check these Hilbert system's axioms and inference rule *Modus ponens* :

- For $A \to (B \to A)$, we have $[\![A \to (B \to A)]\!] = [\![\neg A \vee (\neg B \vee A)]\!] \ni \boldsymbol{v}$ by Lemma 3.1.5 (i).
- For $(A \to (B \to C)) \to ((A \to B) \to (A \to C))$, we have $\boldsymbol{v} \in [\![\neg(\neg A \vee (\neg B \vee C)) \vee (\neg(\neg A \vee B) \vee (\neg A \vee C))]\!]$ by Lemma 3.1.5 (ii).

- For $(\neg B \to \neg A) \to (A \to B)$, we have $\boldsymbol{v} \in [\![\neg(\neg\neg B \vee \neg A) \vee (\neg A \vee B)]\!]$ by Lemma 3.1.5 (iii).
- For the inference rule $\dfrac{A \qquad A \to B}{B}$ (*Modus ponens*):
  If $\boldsymbol{v} \in [\![A]\!]$ and $\boldsymbol{v} \in [\![A \to B]\!] = \left([\![A]\!]^{\perp} \cup [\![B]\!] \right)^{\perp\perp}$, then $\boldsymbol{v} \notin [\![A]\!]^{\perp}$ and $\boldsymbol{v} \in [\![B]\!]$ by Lemma 3.1.3.

□

Theorem 3.1.7 (*Completeness*) *Given an R-duality on $E \neq \varnothing$ as described above and $A \in PROP_{CL}$, if $\models A$, then $\vdash_c A$.*

Proof : Indeed, suppose $\models A$. This means for any assignment $[\![-]\!] : PROP_{CL} \longrightarrow \mathcal{P}(E)$, we have $\boldsymbol{v} \in [\![A]\!]$. We know A is equivalent to a conjunctive normal form $D_1 \wedge ... \wedge D_n$ where $D_i = l_1 \vee ... \vee l_k$ (l_j are literals, *i.e.* atoms or negation of atoms). Since $\boldsymbol{v} \in [\![A]\!]$, then $\boldsymbol{v} \in [\![D_1 \wedge ... \wedge D_n)]\!] = [\![D_1]\!] \cap ... \cap [\![D_n]\!]$. So $\boldsymbol{v} \in [\![D_i]\!]$ for every i. But since $[\![D_i]\!] = ([\![l_1]\!] \cup ... \cup [\![l_k]\!])^{\perp\perp}$, we have $\boldsymbol{v} \in [\![l_1]\!] \cup ... \cup [\![l_k]\!]$ by Lemma 3.1.3. This means every $[\![D_i]\!]$ is of the form $\left([\![l_1]\!] \cup ... \cup [\![p]\!] \cup ... \cup [\![p]\!]^{\perp} \cup ... \cup [\![l_k]\!] \right)^{\perp\perp}$ for some atomic proposition p, since it is the only way to have $\boldsymbol{v} \in [\![D_i]\!]$ for any interpretation, guaranteed by Corollary 3.1.4 (indeed, if it were not of this form, we could have an interpretation such that for every atomic proposition q, $\boldsymbol{v} \notin [\![q]\!]$, and consequently $\boldsymbol{v} \notin [\![D_i]\!]$).
Since we know $p \vee \neg p$ is provable classically, it follows that every D_i is provable, and consequently $D_1 \wedge ... \wedge D_n$. □

Remark 3.1.8

We observe that choosing a *classical* duality relation that is *not* reflexive is a sufficient condition to interpret classical logic. Moreover, the interpretation of "$A \vee B$" becomes simply "$[\![A]\!] \cup [\![B]\!]$". To make things even simpler, we see that if E is a singleton, then any formula A is assigned to one of the only two facts, E or $\varnothing$. We then retrieve a semantics *isomorphic* to the traditional *true*/*false* semantics. However,

this last interpretation has the structure of a Boolean algebra, while we see that in general, our interpretations are De Morgan algebras (with classical duality relations, see Section 2.4).

Example 3.1.9 (*Classical* duality on a group)

Here's a very concrete particular case of the general setting described above. Let $(G, \cdot, 1)$ be a group. Define a duality on G by letting $\ll a \mid b \gg = a \cdot b$ and taking the pole $P = G \backslash \{1\}$. Since every subset of G is a fact by Proposition 2.5.4, we can have the following interpretation (where A is valid if $1 \in [\![A]\!]$):

$$\begin{array}{rcl}
\text{atomic proposition } p & \mapsto & \text{some subset of } E \\
\top & \mapsto & E \\
\bot & \mapsto & \varnothing \\
\neg A & \mapsto & [\![A]\!]^{\perp} \\
A \wedge B & \mapsto & [\![A]\!] \cap [\![B]\!] \\
A \vee B & \mapsto & [\![A]\!] \cup [\![B]\!] \\
A \to B & \mapsto & [\![A]\!]^{\perp} \cup [\![B]\!]
\end{array}$$

Observe that since we never used *associativity* of " $\cdot$ ", our interpretation holds for any magma $(G, \cdot, 1)$ with unity where every element has an *inverse*.

Remark 3.1.10

Notice that when $[\![A \to B]\!] = E$, we have $[\![A]\!]^{-1} \subseteq [\![B]\!]$, where $[\![A]\!]^{-1} := \{a^{-1} \mid a \in [\![A]\!]\}$. Indeed, for all $x \in E$, if $x \in [\![A]\!]^{\perp} \cup [\![B]\!]$, then $x \in [\![A]\!]^{\perp}$ or $x \in [\![B]\!]$. Since $[\![A]\!]^{\perp} = E \backslash [\![A]\!]^{-1}$, this means either $x \notin [\![A]\!]^{-1}$ or $x \in [\![B]\!]$. Classically speaking, this is equivalent to saying if $x \in [\![A]\!]^{-1}$, then $x \in [\![B]\!]$.

In the next interpretation, things will be made so that the *validity* of a proposition A will be given when $[\![A]\!] = E$.

Let E be a non-empty set and take any R-duality on E, with duality relation $\perp$. As in Section 2.4, let $N := \{x \in E \mid x \perp E\}$ and $I := \{x \in E \mid x \perp x\}$.

Lemma 3.1.11 *If $N = I$, then for all $A, B \subseteq E$ we have*

(i) $\left(A \cap \left(B \cap A^{\perp}\right)^{\perp\perp}\right)^{\perp} = E$;

(ii) $\left(\left(B^{\perp} \cap A^{\perp\perp}\right)^{\perp} \cap \left(A \cap B^{\perp}\right)^{\perp\perp}\right)^{\perp} = E$.

Proof: For (i), we have $\left(B \cap A^{\perp}\right) \subseteq A^{\perp} \Rightarrow A^{\perp\perp} \subseteq \left(B \cap A^{\perp}\right)^{\perp} \Rightarrow \left(B \cap A^{\perp}\right)^{\perp\perp} \subseteq A^{\perp\perp\perp} = A^{\perp}$. Since $A \cap A^{\perp} \subseteq N$ by Proposition 2.4.10 (iv), *a fortiori* $A \cap \left(B \cap A^{\perp}\right)^{\perp\perp} \subseteq N$. Thus the result follows by Proposition 2.4.10 (i).
For (ii), since $A \subseteq A^{\perp\perp}$, we have $A \cap B^{\perp} \subseteq A^{\perp\perp} \cap B^{\perp} \Rightarrow \left(A^{\perp\perp} \cap B^{\perp}\right)^{\perp} \subseteq \left(A \cap B^{\perp}\right)^{\perp} \Rightarrow \left(A \cap B^{\perp}\right)^{\perp\perp} \subseteq \left(A^{\perp\perp} \cap B^{\perp}\right)^{\perp\perp} = A^{\perp\perp} \cap B^{\perp}$ since $A^{\perp\perp} \cap B^{\perp}$ is a fact. The result follows by Proposition 2.4.10 (iv) and (i). $\square$

Lemma 3.1.12 *Suppose we have an R-duality on E such that $N = I$ and for all facts $X, Y, Z \subseteq E$, if $x \in X \cap Y$ and $X \neq Y$, then $x \in Z$ or $x \in Z^{\perp}$. Then for all facts $A, B, C \subseteq E$, we have*

$$\left(\left(A \cap \left(B \cap C^{\perp}\right)^{\perp\perp}\right)^{\perp} \cap \left(\left(A \cap B^{\perp}\right)^{\perp} \cap \left(A \cap C^{\perp}\right)^{\perp\perp}\right)^{\perp\perp}\right)^{\perp} = E$$

Proof: Let $x \in A \cap C^{\perp}$, $x \notin N$. If $x \in B$, then $x \notin \left(A \cap \left(B \cap C^{\perp}\right)^{\perp\perp}\right)^{\perp}$ or else we have $x \in B^{\perp}$, which implies $x \notin \left(A \cap B^{\perp}\right)^{\perp}$. The result follows using Proposition 2.4.10 (i). $\square$

The interpretation of classical logic is given as follows: we have an R-duality on E such that

$$\begin{array}{l}\text{(i) } N = I \text{ and} \\ \text{(ii) for all facts } X, Y, Z \subseteq E \text{ if } x \in X \cap Y \text{ and } X \neq Y \text{ ,} \\ \text{then } x \in Z \text{ or } x \in Z^{\perp}.\end{array} \qquad (*)$$

As before, we assign atomic propositions to facts, and

$$\begin{array}{rcl} \top & \mapsto & E \\ \bot & \mapsto & E^{\perp} \\ \neg A & \mapsto & [\![A]\!]^{\perp} \\ A \wedge B & \mapsto & [\![A]\!] \cap [\![B]\!] \\ A \vee B := \neg(\neg A \wedge \neg B) & \mapsto & \left([\![A]\!]^{\perp} \cap [\![B]\!]^{\perp} \right)^{\perp} = ([\![A]\!] \cup [\![B]\!])^{\perp\perp} \\ A \rightarrow B := \neg A \vee B & \mapsto & \left([\![A]\!] \cap [\![B]\!]^{\perp} \right)^{\perp} \end{array}$$

By Proposition 3.1.1, every proposition is assigned to a fact, so we can prove *soundness*:

Theorem 3.1.13 (*Soundness*) *Given an R-duality on $E \neq \varnothing$ as described by $(*)$, if $\vdash_c A$ for $A \in PROP_{CL}$ then for all assignments $[\![-]\!] : PROP_{CL} \longrightarrow \mathcal{P}(E)$, we have $[\![A]\!] = E$.*

Proof : We check these axioms as before :

- $[\![A \rightarrow (B \rightarrow A)]\!] = E$ by Lemma 3.1.11 (i).
- $[\![(A \rightarrow (B \rightarrow C)) \rightarrow ((A \rightarrow B) \rightarrow (A \rightarrow C))]\!] = E$ by Lemma 3.1.12.
- $[\![(\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B)]\!] = E$ by Lemma 3.1.12 (ii).

For *modus ponens*, we clearly see that if $[\![A]\!] = E$ and $[\![A \rightarrow B]\!] = \left([\![A]\!]^{\perp} \cup [\![B]\!] \right)^{\perp\perp} = E$, then $[\![B]\!] = E$. Indeed, $\left([\![A]\!]^{\perp} \cup [\![B]\!] \right)^{\perp\perp} = E$ is equivalent to saying $\left([\![A]\!]^{\perp\perp} \cap [\![B]\!]^{\perp} \right)^{\perp} = E$, which means $[\![A]\!] \cap [\![B]\!]^{\perp} = N$. Hence $[\![B]\!]^{\perp} = N$, and $[\![B]\!] = E$. $\square$

Remark 3.1.14

The previous conditions on the duality given by $(*)$ seem too *strong* to ensure the *soundness* theorem for classical logic. Indeed, in example 3.1.21, the condition (i) $N = I$ holds, but not (ii). Therefore examples 3.1.16 and 3.1.17 of interpretations of classical propositional calculus illustrate Theorem 3.1.13 but are rather "specialized".

Example 3.1.15 (Subsets as propositions)

Consider an R-duality on $E \neq \varnothing$ with classical duality relation $\perp_{Id}$. We have $N = I = \varnothing$ and every $x \in E$ is either in $X \subseteq E$ or in its *complement*. We retrieve the traditional Boolean algebra $\mathcal{P}(E)$ interpretation where *conjunction*, *disjunction* and *negation* are simply the set theoretic *intersection*, *union* and *complement*.

Example 3.1.16 (Subspaces as propositions)

The $\mathbb{R}$-duality on $\mathbb{R}^2$ with the scalar product $< \vec{v} \mid \vec{w} >= v_1w_1 + v_2w_2$, where $\vec{v} = (v_1, v_2)$, $\vec{w} = (w_1, w_2)$ and $v \perp w \Leftrightarrow < v \mid w >= 0$, also fulfills the requirements for our interpretation: $N = I = \{\vec{0}\}$ and for any facts V, W, *i.e.* subspaces (see Example 2.4.11), if $V \neq W$, we have $V \cap W = \{\vec{0}\}$, and obviously $\vec{0} = (0, 0)$ is in any subspace. Using Proposition 2.4.17, we get the following interpretation for the disjunction:

$$[\![A \vee B]\!] := (\ [\![A]\!] \cup\ [\![B]\!])^{\perp\perp} =\ [\![A]\!] +\ [\![B]\!]$$

Example 3.1.17 (Coherence spaces as propositions)

Let $X = \{a, b, c\}$ be a set and consider an $\mathbb{N}$-duality on $\mathcal{P}(X)$ such that facts are coherence spaces (see Section 2.4). Here N consists of all singletons and the empty set, and clearly $N = I$. Suppose $A \in \mathcal{A} \cap \mathcal{B}$ for distinct facts $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. Notice that $A \neq X$, otherwise $\mathcal{A} = \mathcal{B} = \mathcal{P}(X)$. If $A \in N$, then it belongs to every fact. If $|A| = 2$, then $A \in \mathcal{C}$ or $A \in \mathcal{C}^{\perp}$ for all $\mathcal{C} \subseteq \mathcal{P}(X)$ by Proposition 2.4.24. Let's mention that this also works *a fortiori* for coherence spaces representing the *Booleans*, *i.e.* where $X = \{t, f\}$ (see Example 2.4.22).

3.1.2 Intuitionistic Logic

Let E be a non-empty set. Consider $\mathcal{H} \subseteq \mathcal{P}(E)$ such that $(\mathcal{H}, \subseteq, \cap, \cup, N, \top, \Rightarrow)$ is a Heyting algebra (see section 2.4). We interpret the formulas of propositional intuitionistic calculus, noted $PROP_{IL}$, in $\mathcal{H}$ as follows:

Define $[\![-]\!] : PROP_{IL} \longrightarrow \mathcal{H}$ such that atomic propositions are mapped to any subset of $\mathcal{H}$ and other formulas are assigned inductively:

$$\begin{array}{rcl} \perp & \mapsto & N \\ A \wedge B & \mapsto & [\![A]\!] \cap [\![B]\!] \\ A \vee B & \mapsto & [\![A]\!] \cup [\![B]\!] \\ A \rightarrow B & \mapsto & \bigcup\{X \in \mathcal{H} | \ [\![A]\!] \cap X \subseteq [\![B]\!]\} \\ \neg A := A \rightarrow \perp & \mapsto & \bigcup\{X \in \mathcal{H} | \ [\![A]\!] \cap X \subseteq N\} \end{array}$$

In this interpretation, the validity of A is given when $[\![A]\!] = \top$. Heuristically, if we think of E as a set of *explanations*, we can see $\top$ as a *sufficient* set of elements that can explain or justify the *truth* of a proposition.

In Section 2.4, we have seen that in some cases we can give a meaning to $\neg A$ in terms of a polar set $[\![A]\!]^{\perp}$ by defining an *appropriate* R-duality on $\top$ (see Propositions 2.4.7 and 2.4.8). An immediate observation is that in our examples of Heyting algebras, the least element N coincides exactly with our previous definition $N := \{x \in E \mid x \perp E\}$. Notice also that when the interpretation of $PROP_{IL}$ is given on $\mathcal{H} = \mathcal{P}(E)$, $[\![A \rightarrow B]\!] = (E \setminus [\![A]\!]) \cup [\![B]\!]$, so the interpretation of $[\![\neg A]\!]$ is the *complement* of $[\![A]\!]$, and $A \vee \neg A$ is always valid. This is the usual Boolean algebra interpretation (since $\mathcal{H} = \mathcal{P}(E)$ is then a Boolean algebra).

Remark 3.1.18

In our previous interpretation for intuitionistic propositional logic, formulas are not necessarily mapped to *facts*, *e.g.* when the Heyting algebra is an arbitrary filter

or chain. Intuitively, this suggests that in the framework of polarity, intuitionistic propositions may be represented by sets that were not "properly tested", *i.e.* that are not biorthogonally closed (see also Remark 2.4.4).

3.1.3 Linear Logic

It is a good thing to recall Girard's *phase semantics* (see [15]) as an historical example of the use of polar sets to express the idea of *negation* in logic.

Basically, the idea is to define a duality on a monoid $(M, \cdot, 1)$ by letting

$$\begin{array}{rcl} \ll - \mid - \gg : M \times M & \longrightarrow & M \\ (m, n) & \mapsto & m \cdot n \end{array}$$

and choose a cyclic pole $P \subseteq M$. Also, for all $A, B \subseteq M$, we define

$$A \cdot B \overset{def}{=} \{a \cdot b \mid a \in A \text{ and } b \in B\}$$

Such a structure is called a *phase space* by Girard [8] (when the monoid is commutative).

Originally, Girard's definition of a polar set (see Section 2.6) didn't imply that the pole was cyclic in general, so the monoid had to be commutative to ensure this condition. Without a cyclic pole, we would have lost associativity of $\otimes$, but the drawback of having a commutative monoid is that we restricted phase semantics to the interpretation of commutative linear logic only.
Later, even with a generalized definition of polar sets (for $E \neq F$), Girard stuck to the original definition and used a commutative monoid for phase semantics. He simply mentioned that if we wanted to have associativity of $\otimes$ without commutativity (and retrieve things like Lambek's syntactic calculus [28]), we could take any non-commutative monoid but choose a cyclic pole (see [15]). Notice that from our definition and the discussion in Section 2.4, an R-duality on E forces the pole to be

cyclic (or equivalently that $\perp$ be a symmetric relation).

We define the interpretation $[\![-]\!] : PROP_{LL} \longrightarrow \mathcal{P}(M)$ such that constants (*i.e.* neutral elements respectively for &, $\oplus$, $\otimes$ and $⅋$) are given by

$$\begin{array}{rcl} \top & \mapsto & M \\ \mathbf{0} & \mapsto & M^{\perp} \\ \mathbf{1} & \mapsto & P^{\perp} \\ \perp & \mapsto & P \end{array}$$

Remark 3.1.19

It is interesting to notice that $P^{\perp}$ is a submonoid of M: clearly $1 \in P^{\perp} := \{m \in M \mid \forall p \in P, m \cdot p, p \cdot m \in P\}$. And if $m, n \in P^{\perp}$, *i.e.* $m \cdot p, p \cdot m, n \cdot p, p \cdot n \in P$ for all $p \in P$, then $(m \cdot n) \cdot p = m \cdot \underbrace{(n \cdot p)}_{\in P} \in P$ and similarly $p \cdot (m \cdot n) \in P$, which means that $m \cdot n \in P^{\perp}$. Also, M is a fact by Proposition 2.1.5 (i), and so is P: Indeed, suppose $m \in P^{\perp\perp}$. By definition, this means $m \perp P^{\perp}$, and in particular, $m \perp 1$. Therefore $m \cdot 1 = m \in P$.

Now since we want negation to be involutive, we assign every atomic proposition to facts, *i.e.* subsets $X \subseteq M$ such that $X^{\perp\perp} = X$ (with respect to the choice of P), and define other formulas of $PROP_{LL}$ inductively:

$$\begin{array}{rcl} \sim A & \mapsto & [\![A]\!]^{\perp} \\ A \,\&\, B & \mapsto & [\![A]\!] \cap [\![B]\!] \\ A \oplus B & \mapsto & ([\![A]\!] \cup [\![B]\!])^{\perp\perp} \\ A \otimes B & \mapsto & ([\![A]\!] \cdot [\![B]\!])^{\perp\perp} \\ A ⅋ B & \mapsto & \left([\![A]\!]^{\perp} \cdot [\![B]\!]^{\perp}\right)^{\perp} \\ !A & \mapsto & ([\![A]\!] \cap \mathcal{I})^{\perp\perp} \\ ?A & \mapsto & \left([\![A]\!]^{\perp} \cap \mathcal{I}\right)^{\perp} \end{array}$$

where $\mathcal{I}$ is the set of *idempotent* elements of M that are in $P^{\perp}$.
Define $A \multimap B :=\sim A \parr B$. In particular, we have $A \multimap P = \sim A$.

We observe the use of *double negation* in order to ensure closure of facts.
Also, we can show $\otimes$ is associative (using the property $[\![A]\!]^{\perp\perp} \cdot [\![B]\!]^{\perp\perp} \subseteq ([\![A]\!] \cdot [\![B]\!])^{\perp\perp}$, see [8]).

Defining A to be *valid* when $1 \in [\![A]\!]$, we can show *soundness* and *completeness* in the sense that A is provable in linear logic if and only if $1 \in [\![A]\!]$ for any interpretation in any phase space (see [8]). The proof of *completeness* is instructive and here is the general idea : if we suppose a proposition A is valid in any phase space, then in particular consider the *free* commutative monoid $(M, \cdot, \varnothing)$ of *contexts* (*multisets* of formulas) with " $\cdot$ " defined by *concatenation.* We choose the pole $P = \{\Gamma \mid \vdash \Gamma \text{ is provable}\}$. The interpretation goes as follows:

$$A \quad \mapsto \quad [\![A]\!] = \{\Gamma \mid \vdash \Gamma, A \text{ is provable}\}$$

It can be shown that $[\![A]\!]$ is indeed a fact, and since $\varnothing \in [\![A]\!]$ by hypothesis, then we have $\vdash A$.

Remark 3.1.20

Notice that if we define a duality on the monoid $(\mathcal{P}(X), \cup, \varnothing)$ with the pole $P = \{X\}$, where X is a non-empty set, the two negations coincide and $!A = A$, so we retrieve an interpretation of classical logic where facts are *filters.* This comment will be made more explicit in the following example.

Example 3.1.21 (Classical logic in a phase semantics)

Let X be a non-empty set and define a phase space as follows: consider the monoid $(\mathcal{P}(X), \cup, \varnothing)$ and define a duality on $\mathcal{P}(X)$ such that

$$A \perp B \Leftrightarrow A \cup B = X$$

Notice that the set of idempotent elements of the monoid $(\mathcal{P}(X), \cup, \varnothing)$ is $\mathcal{I} = \mathcal{P}(X)$, hence $[\![!A]\!] = [\![?A]\!] = [\![A]\!]$ in the above interpretation of linear logic.

Definition 3.1.22 Let $A \in \mathcal{P}(X)$ for some non-empty set X. The set $\Uparrow(A) = \{A' \in \mathcal{P}(X) \mid A \subseteq A'\}$ is called the *principal filter* generated by $A \subseteq X$ (in the poset $(\mathcal{P}(X), \subseteq)$).

Proposition 3.1.23 *With respect to the above duality on $\mathcal{P}(X)$, the facts are of the form $\Uparrow(A)$ for $A \subseteq X$. Moreover, for any $A \subseteq X$, $\Uparrow(A)^{\perp} = \Uparrow(\overline{A})$.*

Proof : Clear by definition. □

We define the *locative product* (see [17]) of $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ by

$$\mathcal{A} \uplus \mathcal{B} \stackrel{def}{=} \begin{cases} \{\Gamma \cup \Delta \mid \Gamma \in \mathcal{A}, \Delta \in \mathcal{B}\} & \text{if } \mathcal{A}, \mathcal{B} \neq \varnothing \\ \varnothing & \text{else} \end{cases}$$

Proposition 3.1.24 *Let $A, B \in \mathcal{P}(X)$ and let $\mathcal{A} = \Uparrow(A)$ and $\mathcal{B} = \Uparrow(B)$. Then $\mathcal{A} \uplus \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.*

Proof : Clearly $\mathcal{A} \uplus \mathcal{B} = \Uparrow(A \cup B)$, and $\mathcal{A} \cap \mathcal{B}$ contains every element of $\mathcal{P}(X)$ that contains both A and B, *i.e.* $\mathcal{A} \cap \mathcal{B} = \{C \in \mathcal{P}(X) \mid A \cup B \subseteq C\}$. □

If we interpret linear logic as above, atomic propositions are *filters*, and using Proposition 3.1.24 we easily see that :

$$[\![\mathbf{1}]\!] = \{X\}^{\perp} = \Uparrow(\varnothing) = \mathcal{P}(X) = [\![\top]\!]$$

$$[\![\mathbf{0}]\!] = \mathcal{P}(X)^{\perp} = \Uparrow(\varnothing)^{\perp} = \Uparrow(X) = \{X\} = [\![\bot]\!]$$

$$[\![A \otimes B]\!] = \Big([\![A]\!] \uplus [\![B]\!] \Big)^{\perp\perp} = ([\![A]\!] \cap [\![B]\!])^{\perp\perp} = [\![A]\!] \cap [\![B]\!] = [\![A \,\&\, B]\!]$$

$$[\![A \parr B]\!] = \Big([\![A]\!]^{\perp} \uplus [\![B]\!]^{\perp} \Big)^{\perp} = \Big([\![A]\!]^{\perp} \cap [\![B]\!]^{\perp} \Big)^{\perp} = ([\![A]\!] \cup [\![B]\!])^{\perp\perp} = [\![A \oplus B]\!]$$

Also, $[\![A \Rightarrow B]\!] = [\![A \multimap B]\!] = \Big([\![A]\!] \cap [\![B]\!]^{\perp} \Big)^{\perp}$, and we can prove the *soundness* theorem:

Theorem 3.1.25 (*Soundness*) *Given a duality on $E = \mathcal{P}(X)$ as above, where X is a non-empty set, if $\varphi \in PROP_{CL}$ is provable (classically) then for all assignments $[\![-]\!] : PROP_{CL} \longrightarrow \mathcal{P}(E)$ as defined above, we have $[\![\varphi]\!] = E$.*

Proof : Since $N = I = \{X\}$, we can use Lemma 3.1.11 and show $[\![a \to (b \to a)]\!] = E$ and $[\![(\neg b \to \neg a) \to (a \to b)]\!] = E$.
Modus ponens is clear and for $\varphi = (a \to (b \to c)) \to ((a \to b) \to (a \to c))$, we can see that $[\![\varphi]\!] = E$. Indeed, suppose $[\![a]\!] = \Uparrow(A)$, $[\![b]\!] = \Uparrow(B)$ and $[\![c]\!] = \Uparrow(C)$ for some $A, B, C \subseteq X$. Using Proposition 3.1.23, we can show $[\![a \to (b \to c)]\!] = \Big(\Uparrow(A) \cap \big(\Uparrow(B) \cap \Uparrow(C)^{\perp}\big)^{\perp\perp}\Big)^{\perp} = \Uparrow\big(\overline{A} \cap \overline{B} \cap C\big)$ and $[\![(a \to b) \to (a \to c)]\!] = \Uparrow\big(\big(A \cup \overline{B}\big) \cap \overline{A} \cap C\big)$. It follows that

$$
\begin{aligned}
[\![\varphi]\!] &= \Big(\ [\![a \to (b \to c)]\!] \cap\ [\![(a \to b) \to (a \to c)]\!]^{\perp}\Big)^{\perp} \\
&= \big(\Uparrow\big(\overline{A} \cap \overline{B} \cap C\big) \cap \Uparrow\big(\big(\overline{A} \cap B\big) \cup A \cup \overline{C}\big)\big)^{\perp} \\
&= \Uparrow\big(\big(\overline{A} \cap \overline{B} \cap C\big) \cup \big(\overline{A} \cap B\big) \cup A \cup \overline{C}\big)^{\perp} \\
&= \Uparrow(X)^{\perp} \\
&= \Uparrow(\varnothing) = E
\end{aligned}
$$

□

Remark 3.1.26

The interpretations presented in this Section were meant to illustrate the idea that propositions in logic may be defined in terms of sets of elements with a notion of *validity*, *i.e. provability*, and a meaning of negation as the result of some testing. However, none of these examples seems appropriate to define a proposition as a set of its *formal proofs*.

3.2 *Heyting* interpretation

This section illustrates how the notion of polarity as negation arose in Girard's work in *categorical* logic, where proofs are *morphisms* (in a *Cartesian closed category*).

But before, we present a generalized version for the interpretation of the classical connectives "and", "or" as discussed at the beginning of Section 3.1.1.

3.2.1 Interpretation of the connectives by *testing*

In the proposition "A and B", the universe and testing process for which A has been defined may be different from B. Hence "A and B" comes from some other testing that should *preserve* the previous testings in a sense. Let's try to make it clear. Consider an R-duality on E, with testing map $\ll - \mid - \gg_E : E^2 \longrightarrow R$, pole $P \subseteq R$ and duality relation $\overset{\perp}{\sim} \subseteq E^2$, and an R'-duality on E' with testing map $\ll - \mid - \gg_{E'} : E'^2 \longrightarrow R'$, pole $P' \subseteq R'$ and duality relation $\overset{\perp}{\approx} \subseteq E'^2$. Next we consider an R''-duality on E'', with $\ll - \mid - \gg_{E''} : E''^2 \longrightarrow R''$, $P'' \subseteq R''$ and $\overset{\perp}{\underset{\approx}{\sim}} \subseteq E''^2$, and two duality *monomorphisms* $\iota_E : E \longrightarrow E''$ and $\iota_{E'} : E' \longrightarrow E''$. The interpretation of the connectives is then given by

$$
\begin{aligned}
\text{“}A \text{ and } B\text{”} &:= \left\{\mu \in E'' \mid \mu \overset{\perp}{\underset{\approx}{\sim}} \iota_E\left(A^{\overset{\perp}{\sim}}\right) \text{ and } \mu \overset{\perp}{\underset{\approx}{\sim}} \iota_{E'}\left(B^{\overset{\perp}{\approx}}\right)\right\} \\
&= \left\{\mu \in E'' \mid \mu \overset{\perp}{\underset{\approx}{\sim}} \iota_E\left(A^{\overset{\perp}{\sim}}\right)\right\} \cap \left\{\mu \in E'' \mid \mu \overset{\perp}{\underset{\approx}{\sim}} \iota_{E'}\left(B^{\overset{\perp}{\approx}}\right)\right\} \\
&= \iota_E\left(A^{\overset{\perp}{\sim}}\right)^{\overset{\perp}{\underset{\approx}{\sim}}} \cap \iota_{E'}\left(B^{\overset{\perp}{\approx}}\right)^{\overset{\perp}{\underset{\approx}{\sim}}}
\end{aligned}
$$

and

$$
\begin{aligned}
\text{“}A \text{ or } B\text{”} &:= \text{“ not (not } A \text{ and not } B\text{)”} \\
&= \left(\iota_E\left(A^{\overset{\perp}{\sim}\overset{\perp}{\sim}}\right)^{\overset{\perp}{\underset{\approx}{\sim}}} \cap \iota_{E'}\left(B^{\overset{\perp}{\approx}\,\overset{\perp}{\approx}}\right)^{\overset{\perp}{\underset{\approx}{\sim}}}\right)^{\overset{\perp}{\underset{\approx}{\sim}}} \\
&= \left(\iota_E(A) \cup \iota_{E'}(B)\right)^{\overset{\perp}{\underset{\approx}{\sim}}\,\overset{\perp}{\underset{\approx}{\sim}}}
\end{aligned}
$$

where $\iota_E(A) := \{\iota_E(a) \mid a \in A\}$.

3.2.2 A new look at coherence spaces

Initially motivated by a will to find a categorical interpretation of intuitionistic logic (inspired by Scott's domains), the following example led to the discovery of linear logic.

Example 3.2.1 (Category of coherence spaces)

Consider a category where objects are *coherence spaces* and morphisms are *linear stable maps* in the following sense:

Definition 3.2.2 A *linear stable map* f from $\mathcal{A}$ to $\mathcal{B}$, where $\mathcal{A}, \mathcal{B}$ are coherence spaces, satisfies the following properties :

(i) If $A \in \mathcal{A}$, then $f(A) \in \mathcal{B}$, where A is any element of $\mathcal{A}$. In words, it simply says that a *clique* is mapped to another *clique*.
(ii) If $A \subseteq A' \in \mathcal{A}$, then $f(A) \subseteq f(A')$.
(iii) $f\left(\bigcup_{i\in I} A_i\right) = \bigcup_{i\in I} f(A_i)$, *i.e.* f preserves union.
(iv) If $A \cup A' \in \mathcal{A}$, then $f(A \cap A') = f(A) \cap f(A')$ (*stability*).

Negation plays a central role in this interpretation and is defined as in section 2.4: given a coherence space $\mathcal{A}$, the $\mathbb{N}$-duality is defined on $\mathcal{P}(|\mathcal{A}|)$, with duality relation $\perp\!\!\!\perp \subseteq \mathcal{P}(|\mathcal{A}|)^2$, by

$$A \perp\!\!\!\perp A' \Leftrightarrow |A \cap A'| \leqslant 1$$

and the polar set of $\mathcal{A}$ given by

$$\mathcal{A}^{\perp\!\!\!\perp} = \{A' \subseteq |\mathcal{A}| \mid \forall A \in \mathcal{A}, A \perp\!\!\!\perp A'\}.$$

Two *conjunctions* coexist in this interpretation. The first one, noted "&" (*with*), which is related to the usual *intuitionistic* conjunction "$\wedge$", is given as follows, where $\mathcal{A}, \mathcal{B}$ are coherence spaces:

- $|\mathcal{A} \,\&\, \mathcal{B}| \stackrel{def}{=} |\mathcal{A}| \times \{1\} \cup |\mathcal{B}| \times \{2\}$, *i.e.* the *tokens* of $\mathcal{A} \,\&\, \mathcal{B}$ are in the *disjoint* union of tokens of $\mathcal{A}$ and $\mathcal{B}$. In other words, the tokens of $\mathcal{A} \,\&\, \mathcal{B}$ are the tokens of both $\mathcal{A}$ *and* $\mathcal{B}$, and we have a way to distinguish them.

- The coherence is given by

$$(a, 1) \frown_{\mathcal{A}\&\mathcal{B}} (b, 2) \text{ for all } a \in |\mathcal{A}|, b \in |\mathcal{B}|$$

and

$$
\begin{aligned}
(a,1) \smile_{\mathcal{A}\&\mathcal{B}} (a',1) \quad &\Leftrightarrow \quad a \smile_{\mathcal{A}} a' \\
(b,2) \smile_{\mathcal{A}\&\mathcal{B}} (b',2) \quad &\Leftrightarrow \quad b \smile_{\mathcal{B}} b'
\end{aligned}
$$

In terms of polarity, we can give an alternative definition of the coherence space $\mathcal{A}\&\mathcal{B}$. Indeed, consider an $\mathbb{N}$-duality on $E = \mathcal{P}(|\mathcal{A}|)$, an $\mathbb{N}$-duality on $E' = \mathcal{P}(|\mathcal{B}|)$ and an $\mathbb{N}$-duality on $E'' = \mathcal{P}(|\mathcal{A} \,\&\, \mathcal{B}|)$, with duality relations $\perp$, $\underset{\approx}{\perp}$ and $\underset{\approx\atop\approx}{\perp}$ respectively (defined as above). We then consider the two duality *embeddings*

$$
\begin{aligned}
\iota_E : E \quad &\longrightarrow \quad E'' \\
A \quad &\mapsto \quad A \times \{1\} \\
\iota_{E'} : E' \quad &\longrightarrow \quad E'' \\
B \quad &\mapsto \quad B \times \{2\}
\end{aligned}
$$

Indeed, if $A \perp A'$, then $A \cap A'$ contains at most one element; and consequently, $A \times \{1\} \cap A' \times \{1\}$ contains also at most one element (and similarly for $B \underset{\approx}{\perp} B'$). We can show

Proposition 3.2.3 $\mathcal{A} \,\&\, \mathcal{B} = \iota_E\left(\mathcal{A}^{\perp}\right)^{\underset{\approx\atop\approx}{\perp}} \cap \iota_{E'}\left(\mathcal{B}^{\underset{\approx}{\perp}}\right)^{\underset{\approx\atop\approx}{\perp}}$

Proof: The key idea is to use Proposition 2.4.24: $\{a,a'\} \in \mathcal{A} \Leftrightarrow \{a,a'\} \notin \mathcal{A}^{\perp}$, for $a \neq a'$, which simply means $a \smile_{\mathcal{A}} a'$ iff $a \not\smile_{\mathcal{A}^{\perp}} a'$ (for $a \neq a'$). This means the only two-element sets that are contained in $\iota_E\left(\mathcal{A}^{\perp}\right)$ are of the form $\{(a,1),(a',1)\}$, where $a \smile_{\mathcal{A}^{\perp}} a'$, *i.e.* $a \not\smile_{\mathcal{A}} a'$. Hence $\iota_E\left(\mathcal{A}^{\perp}\right)^{\underset{\approx\atop\approx}{\perp}}$ contains every two-element set (in E'') except for those of the form $\{(a,1),(a',1)\}$, where $a \not\smile_{\mathcal{A}} a'$. A similar reasoning shows that $\iota_{E'}\left(\mathcal{B}^{\underset{\approx}{\perp}}\right)^{\underset{\approx\atop\approx}{\perp}}$ contains every two-element set (in E'') except for those of the form $\{(b,2),(b',2)\}$, where $b \not\smile_{\mathcal{B}} b'$. Therefore $\iota_E\left(\mathcal{A}^{\perp}\right)^{\underset{\approx\atop\approx}{\perp}} \cap \iota_{E'}\left(\mathcal{B}^{\underset{\approx}{\perp}}\right)^{\underset{\approx\atop\approx}{\perp}}$ contains all two-element sets in E'' of the form

$$\{(a,1),(b,2)\} \text{ for all } a \in |\mathcal{A}|,\ b \in |\mathcal{B}|$$

$$\{(a,1),(a',1)\} \text{ for } a \smile_{\mathcal{A}} a'$$

$$\{(b,2),(b',2)\} \text{ for } b \frown_{\mathcal{B}} b'$$

which gives an equivalent definition for $\mathcal{A} \,\&\, \mathcal{B}$, since $a \frown_{\mathcal{A}} a' \overset{def}{\Leftrightarrow} \{a,a'\} \in \mathcal{A}$. $\square$

The second conjunction, noted "$\otimes$" (*times*), is closer in spirit to the set theoretic cartesian product:

- $|\mathcal{A} \otimes \mathcal{B}| \overset{def}{=} |\mathcal{A}| \times |\mathcal{B}|$.
- The coherence is given by

$$(a,b) \frown_{\mathcal{A}\otimes\mathcal{B}} (a',b') \quad \Leftrightarrow \quad a \frown_{\mathcal{A}} a' \text{ and } b \frown_{\mathcal{B}} b'$$

Alternatively, given an $\mathbb{N}$-duality on $\mathcal{P}(|\mathcal{A}| \times |\mathcal{B}|)$ such that $C \,\underline{\underline{\perp}}\, C' \Leftrightarrow |C \cap C'| \leqslant 1$, we have

Proposition 3.2.4 $\mathcal{A} \otimes \mathcal{B} = \{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}^{\underline{\underline{\perp}}\,\underline{\underline{\perp}}}$

Proof: It suffices to show that $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}^{\underline{\underline{\perp}}\,\underline{\underline{\perp}}}$ is the smallest fact, *i.e.* the smallest coherence space, that contains exactly the two-element sets of the form $\{(a,b),(a',b')\}$, where $\{a,a'\} \in \mathcal{A}$ and $\{b,b'\} \in \mathcal{B}$. For all two-element sets $\{a,a'\} \in \mathcal{A}$, $\{b,b'\} \in \mathcal{B}$, the set $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ contains sets of the form $\{(a,b),(a,b'),(a',b),(a',b')\}$. Therefore $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}^{\underline{\underline{\perp}}}$ doesn't contain the two-element sets that are contained in $\{(a,b),(a,b'),(a',b),(a',b')\}$. By Proposition 2.4.24, this means they are contained in $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}^{\underline{\underline{\perp}}\,\underline{\underline{\perp}}}$. Also, by construction $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ doesn't contain a set $A \times B$ that has pairs $(a,b),(a',b')$ where $\{a,a'\} \notin \mathcal{A}$ or $\{b,b'\} \notin \mathcal{B}$. Therefore the two-element sets of the form $\{(a,b),(a',b')\}$, where $\{a,a'\} \notin \mathcal{A}$ or $\{b,b'\} \notin \mathcal{B}$, are in the polar set, and consequently they are not in $\{A \times B \mid A \in \mathcal{A}, B \in \mathcal{B}\}^{\underline{\underline{\perp}}\,\underline{\underline{\perp}}}$ by Proposition 2.4.24. $\square$

According to our two conjunctions, we can then define two *disjunctions* "$\oplus$" and "$\parr$" by De Morgan's property:

$$
\begin{aligned}
\mathcal{A} \oplus \mathcal{B} &\overset{def}{=} \left(\mathcal{A}^{\sim\perp} \,\&\, \mathcal{B}^{\underset{\approx}{\perp}}\right)^{\underset{\approxeq}{\perp}} = \left(\iota_E\left(\mathcal{A}\right) \cup \iota_{E'}\left(\mathcal{B}\right)\right)^{\underset{\approxeq}{\perp}\ \underset{\approxeq}{\perp}} \\
\mathcal{A} \parr \mathcal{B} &\overset{def}{=} \left(\mathcal{A}^{\sim\perp} \otimes \mathcal{B}^{\underset{\approx}{\perp}}\right)^{\underset{\cong}{\perp}}
\end{aligned}
$$

And a *linear* implication "$\multimap$":

$$
\mathcal{A} \multimap \mathcal{B} \overset{def}{=} \mathcal{A}^{\sim\perp} \parr \mathcal{B}
$$

We can retrieve the usual *intuitionistic* implication in terms of linear implication:

$$
\mathcal{A} \to \mathcal{B} \overset{def}{=} !\mathcal{A} \multimap \mathcal{B} = \left((!\mathcal{A})^{\sim\perp\sim\perp} \otimes \mathcal{B}^{\underset{\approx}{\perp}}\right)^{\underset{\cong}{\perp}}
$$

where the coherence space $!\mathcal{A}$ is given by

- $|!\mathcal{A}| \overset{def}{=} \mathcal{A}_{fin}$.
- $A \frown_{!\mathcal{A}} A' \Leftrightarrow A \cup A' \in \mathcal{A}$

with $\mathcal{A}_{fin} := \{A \in \mathcal{A} \mid A \text{ is finite}\}$

Remark 3.2.5

It is important to observe that at the categorical level, coherence spaces (or probabilistic or quantum coherence spaces) were originally meant to describe *types*. For example, for $X = \{t, f\}$, the coherence space $\{\varnothing, \{t\}, \{f\}\}$ represents the *Booleans* [10] (p.55). Again, it is not clear whether these objects are appropriate candidates to define propositions as a set of *proofs*.

Chapter 4

Girard's *hell*

In [15], Girard introduced three "*enfers*"[1] $(-1, -2, -3)$ where logics can be studied. *Tarskian* semantics (see Section 3.1) exemplified the first underground (-1), where we can at most tell the *provability* of a proposition, *i.e.* if it is provable or not. According to Girard [20], if we can say that the proposition "A or B" is provable in the first underground, we should not be able to tell which one of A *or* B is. Said differently, if an element is meant to represent the interpretation of "A or B", then we have lost the information whether it comes from the set A or B. Therefore in this underground it is reasonable to think of the set of elements identified with a proposition as a set of *models* or *observations*, but not as *formal proofs*.

Heyting interpretations characterize the second underground (-2). At this level, we can "dig" further towards a more accurate description of logic by interpreting the *structure* of proofs, rather than provability, which allows to distinguish different proofs of a same *formula*. Categories offer examples that live in this underground, but we must not think categorical logic is exclusive to this level. Indeed, although we may view the Heyting level as given by a category where *arrows* are proofs, the *poset reflection* of this category lives at the Tarski level, *i.e.* the first underground.

[1] *Undergrounds* in [15], or *infernos* in [21]

The third underground is characterized by a change of paradigm in the study of logic where, according to Girard [14], instead of "studying the rules of logic, we study the logic of the rules". This level focuses on the *interaction* of proofs (*e.g.* cut-elimination). In [20] and [21], Girard split this third underground in two - one renamed the "third" underground (-3) and the other one the "fourth" (-4). For example, in this *new* third underground, our interpretation of logic can be seen as some sort of game "proofs of A *vs* proofs of *not* A" and we are interested in *evaluating* the result of the interaction, *i.e.* in knowing which one is a *winner* (a *correct* proof). In the fourth underground, we focus on *how* the rules are established. At this level, *transcendental syntax*[2] refers to an *idealized language* that can explain the *structure* of logic. *Proofs* are elements of a universe E, called *epistates*. Epistates correspond to some *strategies* that *interact* with each other in a "game" without rule or winner. Now some of these strategies are not really trying to *prove* anything, but only to *forbid* others, so the notion of *proof* is somewhat *liberalized* (we can call them simply *tests* or *paraproofs*). Therefore epistates play a more general *normative* (or *deontic* [21]) role, *i.e.* they establish which strategies are allowed and which ones are not, based on a *norm* or *standard* given by the definition of the testing process. This said, given an R-duality on E, we call *dichology* ("*dichologie*" in French) a set $\mathbf{A}$ of epistates that is a *fact*. The dichology $\mathbf{A}^{\perp}$ plays the role of some *forbidden space* for $\mathbf{A}$, and *vice versa*. In other words, we can say that $\mathbf{A}^{\perp}$ and $\mathbf{A}$ define what strategies their *dual* is allowed to play. Intuitively, a dichology $\mathbf{A}$ corresponds to a *proposition*, while $\mathbf{A}^{\perp}$ is its negation. Table 1 summarizes the terminology used by Girard.

[2] "To find the - modestly some - hypotheses making logic possible, this is *transcendental syntax*." ([11], p. 7)

	Intuitive meaning	In Ludics	In GoI
Epistates	Judge, test, trial, *épreuve*, paraproof, strategy	Design	Project
Dichology	Set of epistates equal to its *biorthogonal* (*i.e.* a *fact*)	Behaviour	Conduct

Table 1: Terminology of Girard's transcendental syntax

4.1 Towards *existentialism*

What follows is inspired by the philosophical considerations of Girard [20].

In philosophy, *essentialism* suggests that there is a pre-existing *truth* that can judge the validity of our statements or reasoning (this is sometimes called *pure essence*). This notion is omnipresent in most situations of duality (refer to the previous examples in Section 2.3):

- For propositional logics, theories are judged by *truth* assigments (valuations);
- for *scientific* theories (in the sense of Popper), *observations* or *experiments/tests* act as judges;
- for mathematics, theories are judged by mathematical objects, *i.e.* *models* along with Tarski's definition of truth.

Even if Popper offered a severe critique to *essentialism* in metaphysics (by imposing that the *laws* of a theory had to be tested), there is still a predominance of *truth* (given by the *tests*) over *theories*. But as Girard points out, what if we put the *quality* of the test in perspective ([15], p.31), *i.e.* its *validity*, then who's judging the judge? In logic, this suggests seeking a way to define a testing process that is not one-way only, where semantics (essence) tests and syntax (existence) is tested. According to Girard, we need a *symmetric* process, where the "judge is judged in return".

Examples presented in this work showed us that such a viewpoint is possible in the framework of *polarity*, if we abstract to a testing process $\ll x \mid y \gg$ where we can't distinguish which one of x or y should be predominant, *i.e.* where syntax and semantics live side by side according to Girard. This said, we could ask for more: that all tests live in the same *universe*, *i.e.* that the R-duality is defined on some set E. Girard [12] calls such a duality between homogeneous objects a *monist duality*. Now we could have that everything lives in semantics (see Section 3.1), but following Girard, logic is about syntax [20], hence we should think of our universe E as a set of *proofs*. So instead of having the traditional paradigm where a proof of A is tested by a *model* for $\neg A$ (and *vice versa*), we aim for a monist duality where a *proof* of A is tested by a *proof* of $\neg A$. Hence proofs are not tested with a notion of pre-existing semantics, *e.g.* truth values, but with other (*para*-) proofs (those of the negation).

We see that this last approach leads us, philosophically speaking, to *existentialism*. By analogy, an existentialist viewpoint sees the human being as being defined by interaction with his environment, as opposed to a predetermined meaning (the essentialist viewpoint). Therefore we want a notion of proof which is defined by some *internalized* interactions, and this can be achieved *technically* using polarity.

In [15] and [17], Girard mentioned that some definitions can be *desessentialized*, in the sense, it seems, that they can be expressed equivalently by a *fact* $A \subseteq E$, *i.e.* $A^{\perp\perp} = A$, given by an R-duality on E. Let's call "*desessentialized objects*" such objects that can be expressed as facts in Girard's sense.

4.1.1 Examples of *desessentialized objects*

Here are some leading examples given by Girard:

- Coherence spaces, probabilistic coherence spaces and quantum coherence spaces (see Examples 2.4.22, 2.4.29 and 2.4.31).

- This one is mentioned in [17] (p. 413): Given the real vector space $\mathbb{R}^X$ with scalar product $< f \mid g >= \sum_{x \in X} f(x)g(x)$, the *positive* maps $\mathbb{R}^X_+$ can be *desessentialized* by the following R-duality on $\mathbb{R}^X$:

$$f \perp g \Leftrightarrow < f \mid g > \in [0, +\infty]$$

  $\mathbb{R}^X_+$ is a *fact*; moreover, it is *self-polar*, *i.e.* $\left(\mathbb{R}^X_+\right)^{\perp} = \mathbb{R}^X_+$.

Moreover, according to Girard ([17], p. 304), the following definition cannot be *desessentialized*:

Definition 4.1.1 (Atomic coherence)

Let X be an enumerable set. The *atomic coherence* $\mathcal{A}$ (of a *hypercoherence* in [5], or ω*-coherence* in [1]) is a subset of $\mathcal{P}^*_{fin}(X)$ such that $\{x\} \in \mathcal{A}$ for all $x \in X$, where $\mathcal{P}^*_{fin}(X)$ contains every finite subset of X except for $\varnothing$.

The *linear negation* is defined by $\sim \mathcal{A} := (\mathcal{P}^*_{fin}(X) \backslash \mathcal{A}) \cup \{\{x\} \mid x \in X\}$.

But atomic coherences can be expressed as *facts* given an appropriate R-duality on $\mathcal{P}^*_{fin}(X)$:

Proposition 4.1.2 *Given a* $\{0,1\}$*-duality on* $\mathcal{P}^*_{fin}(X)$ *with testing map*

$$\ll A \mid B \gg := \begin{cases} 1 & \text{if } |A \cap B| = 1 \\ 1 & \text{if } |A \cap B| < |A \cup B| \\ 0 & \text{else} \end{cases}$$

and pole $P = \{1\}$*, then the facts are exactly the atomic coherences.*

Proof: Notice that given any set $\mathcal{S} \subseteq \mathcal{P}^*_{fin}(X)$, the polar set $\mathcal{S}^{\perp}$ contains every singleton, since for any $Y \in \mathcal{S}$ and $x \in X$, either $|Y \cap \{x\}| = 1$ or $|Y \cap \{x\}| = 0 < |Y \cup \{x\}|$. Therefore if $\mathcal{A} \subseteq \mathcal{P}^*_{fin}(X)$ is a fact, it is also an atomic coherence by definition. Conversely, if $\mathcal{A}$ is an atomic coherence, then clearly it can be expressed by the following polar set : $\left(\mathcal{P}^*_{fin}(X) \backslash \mathcal{A}\right)^{\perp}$. $\square$

This said, it seems that facts can be used to express more definitions than simply those given by Girard. Let's call "*objects by testing*" such objects that are exactly the facts $A \subseteq E$ given an R-duality on E.

4.1.2 Examples of *objects by testing*

- Let E be a set. Then a set X is a *subset* of E if and only if X is a fact given an R-duality on E with any *classical* duality relation $\perp_{\varphi}$.
- In Example 2.4.11, given a vector space of finite dimension, *subspaces* are exactly the facts when $N = \{0\}$.
- For a finite cyclic group, *subgroups* are exactly the facts with the following R-duality on the group:

Let $(G, \cdot, 1)$ be a *finite cyclic* group of order n. Define an R-duality on G such that

$$g \perp h \Leftrightarrow n \equiv 0 \mod (o(g)\, o(h))$$

where $o(g)$ is the *order* of $g \in G$. Notice that the duality relation is symmetric, since $o(g)\, o(h) = o(h)\, o(g)$. We recall that $o(g)$ is simply $|\langle g \rangle|$, the cardinal of the subgroup generated by g (see [3]). Also, the following results are well-known [3]:

Lemma 4.1.3 (Lagrange's Theorem) *The order of any subgroup of a finite group G is a divisor of the order of G.*

Lemma 4.1.4 (Fundamental Theorem of Finite Cyclic Groups) *Let G be a cyclic group of finite order n.*

(i) *Every subgroup of G is cyclic;*

(ii) *For each positive divisor d of n, G has exactly one subgroup of order d.*

Proposition 4.1.5 *Given the duality above, the facts are exactly the subgroups of G.*

Proof : Suppose $H' \subseteq G$ is a fact. Then $H' = H^{\perp\!\!\!\!\!\smile}$ for some subset $H \subseteq G$. If $H = \varnothing$, then $H^{\perp\!\!\!\!\!\smile} = G$, so H' is a subgroup. Otherwise, we have to check the following:

- $H' \neq \varnothing$, since $1 \in H'$. Indeed, $1 \perp\!\!\!\!\!\smile g$ for all $g \in G$: clearly $o(1) = |\langle 1 \rangle| = 1$, and since $\langle g \rangle$ is a subgroup, it follows from Lagrange's theorem that $|\langle g \rangle| = o(g)$ divides n; therefore $n \equiv 0 \mod (o(1)\, o(g))$.

- Suppose $g \in H'$. Then for all $h \in H$, $g \perp\!\!\!\!\!\smile h$, *i.e.* $n \equiv 0 \mod (o(g)\, o(h))$. But $o(g) = o(g^{-1})$: indeed, $g^{o(g)} = 1$ is equivalent to $1 = (g^{-1})^{o(g)}$.

- Suppose $g, g' \in H'$. Then for all $h \in H$, $g \perp\!\!\!\!\!\smile h$ and $g' \perp\!\!\!\!\!\smile h$, *i.e.* $o(g)\, o(h)$ and $o(g')\, o(h)$ both divide n. Suppose that m is the *least common multiple* of the orders of every element of H. Then if $m \nmid n$, cleary $o(x) = 1$ for all $x \in H'$, which means $x = 1$ and H' is the trivial group. But if $m \mid n$, say $n = km$, then $o(x) \mid k$ for all $x \in G$. Moreover, we know from the previous lemma there exists a (unique) cyclic subgroup of order k, hence generated by some element $g'' \in G$. Therefore $g'' \in H'$, and since $o(g) \mid k$ and $o(g') \mid k$, it follows that $\langle g \rangle$ and $\langle g' \rangle$ are cyclic subgroups of $\langle g'' \rangle$. Hence $g \cdot g' \in \langle g'' \rangle$, which means $|\langle g \cdot g' \rangle|$ divides k. We conclude that $o(g \cdot g')\, o(h)$ divides n for all $h \in H$, and consequently that $g \cdot g' \in H'$.

For the converse, suppose $H \subseteq G$ is a cyclic subgroup of G such that $|H| = k$. Then $H^{\perp\!\!\!\!\!\smile}$ is also a cyclic subgroup (since it is a fact), and it is such that $|H^{\perp\!\!\!\!\!\smile}||H| = |G|$. Now $H^{\perp\!\!\!\!\!\smile\perp\!\!\!\!\!\smile}$ is again a cyclic subgroup and $|H^{\perp\!\!\!\!\!\smile\perp\!\!\!\!\!\smile}||H^{\perp\!\!\!\!\!\smile}| = |G|$, which means $|H^{\perp\!\!\!\!\!\smile\perp\!\!\!\!\!\smile}| = |H| = k$. And since G has a unique subgroup of order k by our lemma, we conclude that $H^{\perp\!\!\!\!\!\smile\perp\!\!\!\!\!\smile} = H$. $\square$

We may wonder to what extent we can define a duality relation so that the facts are exactly the objects we would like to describe. Here is an example that shows it is not always possible:

Proposition 4.1.6 *Let $(G, \cdot, 1)$ be the dihedral group of order 8. There exists no R-duality on G such that the facts are exactly the subgroups of G.*

Proof : We recall the *Cayley table* for G (where ab means $a \cdot b$):

$\cdot$	1	b	a	a^2	a^3	ab	a^2b	a^3b
1	1	b	a	a^2	a^3	ab	a^2b	a^3b
b	b	1	a^3b	a^2b	ab	a^3	a^2	a
a	a	ab	a^2	a^3	1	a^2b	a^3b	b
a^2	a^2	a^2b	a^3	1	a	a^3b	b	ab
a^3	a^3	a^3b	1	a	a^2	b	ab	a^2b
ab	ab	a	b	a^3b	a^2b	1	a^3	a^2
a^2b	a^2b	a^2	ab	b	a^3b	a	1	a^3
a^3b	a^3b	a^3	a^2b	ab	b	a^2	a	1

We proceed by contradiction: suppose there is a duality relation $\perp$ such that facts are exactly the subgroups of G. Here is the lattice of subgroups ordered by inclusion:

$$\begin{tikzcd}
& & G & & \\
\{1,a^2,b,a^2b\} \arrow[urr] & & \{1,a,a^2,a^3\} \arrow[u] & & \{1,a^2,ab,a^3b\} \arrow[ull] \\
\{1,b\} \arrow[u] & \{1,a^2b\} \arrow[ul] & \{1,a^2\} \arrow[ull] \arrow[u] \arrow[urr] & \{1,ab\} \arrow[ur] & \{1,a^3b\} \arrow[u] \\
& & \{1\} \arrow[ull] \arrow[ul] \arrow[u] \arrow[ur] \arrow[urr] & &
\end{tikzcd}$$

Since $\{1\}$ is contained in every subgroup, then $\{1\}^{\perp}$ must contain every subgroup by Proposition 2.4.3 (ii). Hence $\{1\} \bowtie G$. Now consider any subgroup of order 2. None can be self-dual. Indeed, without loss of generality, suppose $\{1,b\}^{\perp} = \{1,b\}$. Then since $\{1,b\} \subseteq \{1,a^2,b,a^2b\}$, we must have $\{1,a^2,b,a^2b\}^{\perp} \subseteq \{1,b\}^{\perp} = \{1,b\}$ by Proposition 2.4.3 again. This forces $\{1,a^2,b,a^2b\}^{\perp} = \{1\}$, hence $\{1,a^2,b,a^2b\} \bowtie \{1\}$, which contradicts what we said earlier. An analogous reasoning shows that the subgroups of order 4 are not self-dual. Moreover, two distinct subgroups of order 2 cannot be in correspondence, since they are each included in a subgroup of order 4, and the polar set of such group should then be included in a two-element set, which

leads to obvious contradictions. Therefore the five subgroups of order 2 should be in correspondence with the three subgroups of order 4, which is not possible. □

4.2 Examples of *Transcendental syntaxes*

4.2.1 Ludics

Here *epistates* are called *designs* ("*desseins*") and *dichologies* are *behaviours* ("*comportements*"). We may think of a design as the *locative* structure of a proof presented in sequent calculus ([14], p.6), where *locative* means that each formula is given a distinct "*address*". Said differently, a design can be understood as a "skeleton of a sequent calculus derivation, where we do not manipulate formulas, but their locations (the addresses where the formulas are stored)" ([7], p.713). Thus the syntax is some abstracted form of sequent calculus that is closer to Böhm trees [7].
Let E be the set of designs. Define an R-duality on E by letting

$$\ll - \mid - \gg : E \times E \longrightarrow R = \{\mathfrak{Dai}, \mathfrak{Fid}\}$$

such that $\ll \mathfrak{D} \mid \mathfrak{E} \gg$ is the "result" of a *normalisation* procedure, *i.e.* some kind of *cut-elimination* for ludics, given by the interaction between the rules (or *actions*) of $\mathfrak{D}$ and $\mathfrak{E}$. Designs are like *programs* that *test* other behaviours' designs, but can be tested in return [13]. By analogy, we may view this *reciprocal testing* as some kind of *game* [7]. When $\ll \mathfrak{D} \mid \mathfrak{E} \gg = \mathfrak{Dai}$, this means the last rule *played* (by either $\mathfrak{D}$ or $\mathfrak{E}$) is the *daimon*:

$$\frac{}{\vdash \Lambda} \maltese$$

This rule corresponds to "immediate terminaison" [14], which means the normalisation *converges*. When the normalisation process *diverges*, *i.e.* never ends (like an

infinite loop), we have $\ll\mathfrak{D} \mid \mathfrak{E}\gg = \mathfrak{Fid}$ which means we admit the rule "Ω"[3]:

$$\dfrac{}{\vdash \Lambda}\,\Omega$$

Let $P = \{\mathfrak{Dai}\}$ be the pole, *i.e.* $\mathfrak{D} \perp \mathfrak{E} \Leftrightarrow \ll\mathfrak{D} \mid \mathfrak{E}\gg = \mathfrak{Dai}$. This said, when we get a situation such that for some $\mathfrak{D} \in \mathbf{A}$ and $\mathfrak{E} \in \mathbf{A}^{\perp}$, $\ll\mathfrak{D} \mid \mathfrak{E}\gg = \mathfrak{Fid}$, this means that we "don't know" if $\mathbf{A}$ (or $\mathbf{A}^{\perp}$) contains a *winning* strategy. Also, when $\ll\mathfrak{D} \mid \mathfrak{E}\gg = \mathfrak{Dai}$, we know at least one of $\mathbf{A}$ or $\mathbf{A}^{\perp}$ gave up in the testing process. Therefore a fact $\mathbf{A}$ and its polar $\mathbf{A}^{\perp}$ are behaviours that interact in *consensus*, *i.e.* that contain designs that *object to* the possibility of an infinite dispute with their polar set's designs. We see that $\mathbf{A}$ defines the "*rule* for $\mathbf{A}^{\perp}$" and *vice versa* ([14], p.33).

As mentioned before, some *epistates* (designs) do not really prove anything: they only play a *normative* role. We define a "proof" as a *winning* design $\mathfrak{D}$, *i.e.* such that for all $\mathfrak{E} \in \mathbf{A}^{\perp}$, $\ll\mathfrak{D} \mid \mathfrak{E}\gg = \mathfrak{Dai}$, and the ✠-rule is played by $\mathfrak{E}$. Also, we say $\mathbf{A}$ is *true* if it contains such a winning design.

The interpretation of "&" is given by the intersection of *negative* and *disjoint* behaviours (in the sense of Definition 54, p. 335, and Proposition 26 in [17], p. 344), while "$\oplus$" was the *biorthogonal* of the union of *positive disjoint* behaviours. It is worth mentioning that this latter definition was *generalized* by Girard (see [17], p. 344):

$$\text{“}\mathbf{A} \oplus \mathbf{B}\text{”} \quad := \quad (\varphi(\mathbf{A}) \cup \psi(\mathbf{B}))^{\perp\perp}$$

where φ, ψ are *delocations* of *loci*, *i.e.* some kind of injective maps (see [14], p. 34). This idea seems to have been originally conceived by Girard to express a *spiritual*

[3]The notation comes from λ-calculus ([17], p.315). Indeed consider the λ-term $\lambda x.xx$. When applied to itself, and by using β-reduction, we get Ω:

$$\begin{aligned} \Omega \quad &:= \quad (\lambda x.xx)\,(\lambda x.xx) \\ &\overset{\beta}{=} \quad (xx)\,[(\lambda x.xx)/\,x] \\ &= \quad (\lambda x.xx)\,(\lambda x.xx) = \Omega \end{aligned}$$

interpretation of the connectives, *i.e.* a construction that is closer to the (traditional) categorical interpretation. But it is also close in a sense to our general interpretation of Section 3.2.1; the interpretation in *Ludics* may then be seen as a particular case when φ and ψ are the *identity* map.

4.2.2 Towards Geometry of Interaction

Example 4.2.1 (Partitions as epistates)

Let I be a (non-empty) finite set.
Consider $E \subseteq \mathcal{P}(\mathcal{P}(I))$ the set of (non-empty) partitions of I.
We define an R-duality on E such that, for all $\mathcal{P}, \mathcal{Q} \in E$, we have

$$\mathcal{P} \perp \mathcal{Q} \Leftrightarrow G\left[\mathcal{P}, \mathcal{Q}\right] \text{ is a tree}$$

where $G\left[\mathcal{P}, \mathcal{Q}\right]$ is the *bipartite graph* such that *vertices* are elements of $\mathcal{P}$ and $\mathcal{Q}$, and an *edge* between $X \in \mathcal{P}$ and $Y \in \mathcal{Q}$ is given by an element of $X \cap Y$.

The duality defined above can be used to interpret *proof-nets* for linear logic. Indeed, suppose every literal of a formula is assigned to a distinct *location* $i \in I = \{1, 2, 3, ..., n\}$. Moreover, different instances of the same literal are assigned to different locations, so *resources* are taken into account.

The basic idea is to assign a partition of I to the *proof* of a formula. Following Danos and Regnier [2], a derivation of the shape

$$\frac{\vdash A_1, A_2, ..., A_n \qquad \vdash B_m, B_{m+1}, ..., B_{m+k}}{\vdash D\left(A_1, A_2, ..., A_n, B_m, B_{m+1}, ..., B_{m+k}\right)}$$

corresponds to the partition $\{\{1, 2, ..., n\}, \{m, m+1, ..., m+k\}\}$, while the derivation

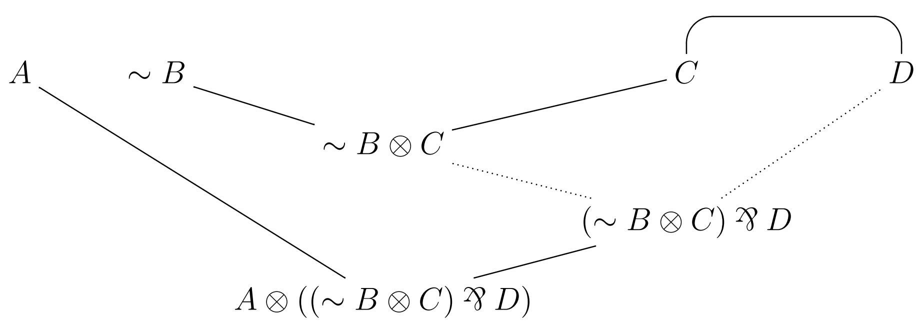
$$\frac{\vdash A_1, A_2, ..., A_n}{\vdash D'\left(A_1, A_2, ..., A_n\right)}$$

corresponds to $\{\{1, 2, ..., n\}\}$. An element of the partition corresponds to the *linked* literals in the proof-net. Let's take an example to illustrate the discussion that follows:

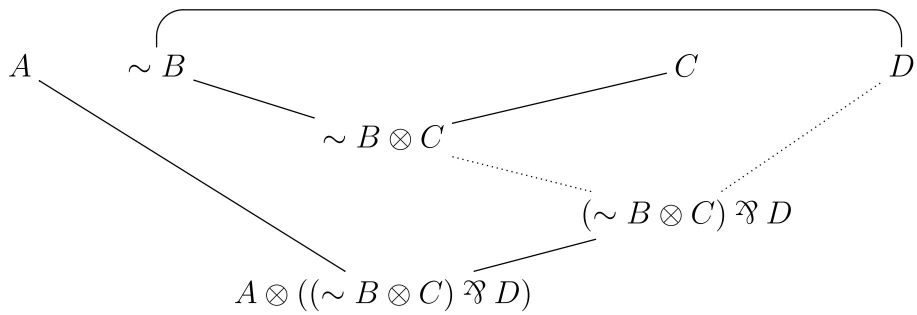
Let $\varphi := A \otimes ((\sim B \otimes C) \parr D)$.
There are two sequent calculus *proofs*:

$$\dfrac{\vdash A \qquad \dfrac{\dfrac{\vdash\sim B \qquad \vdash C, D}{\vdash\sim B \otimes C, D}}{\vdash (\sim B \otimes C) \parr D}}{\vdash A \otimes ((\sim B \otimes C) \parr D)} \quad \text{and} \quad \dfrac{\vdash A \qquad \dfrac{\dfrac{\vdash\sim B, D \qquad \vdash C}{\vdash\sim B \otimes C, D}}{\vdash (\sim B \otimes C) \parr D}}{\vdash A \otimes ((\sim B \otimes C) \parr D)}$$

Given that $1, 2, 3, 4$ are the respective locations for A, $\sim B$, C, D, the proofs above correspond respectively to the partitions $\mathcal{P} = \{\{1\}, \{2\}, \{3, 4\}\}$ and $\mathcal{P}' = \{\{1\}, \{2, 4\}, \{3\}\}$.
The proof-net for the first derivation is



and for the second one



where *dot* lines represent the choices for the *switches* ("*interrupteurs*").

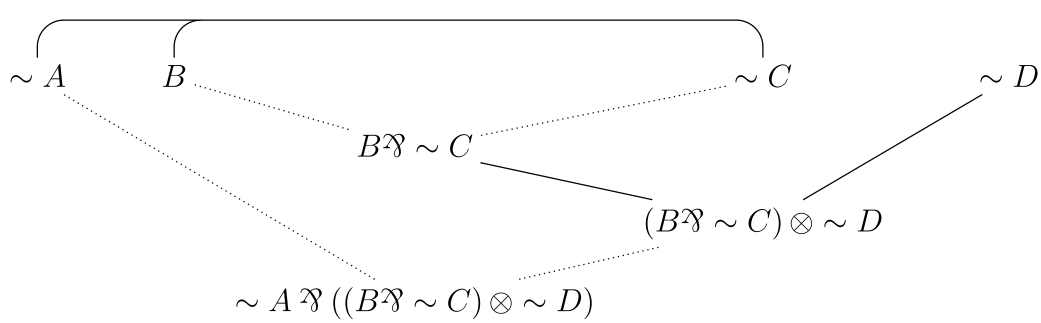
Now the interesting feature of this interpretation comes when we look at the *negation*:

Let $\sim \varphi := \sim A \parr ((B \parr \sim C) \otimes \sim D)$.

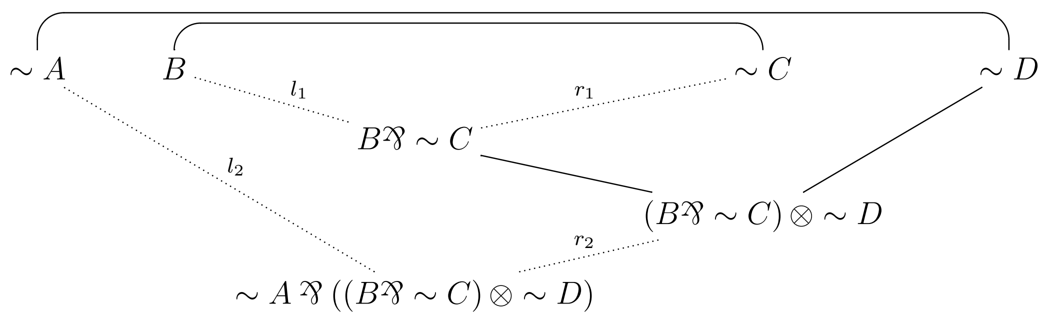
The proofs are

$$\dfrac{\dfrac{\dfrac{\vdash \sim A, B, \sim C}{\vdash \sim A, B \parr \sim C} \qquad \vdash \sim D}{\vdash \sim A, (B \parr \sim C) \otimes \sim D}}{\vdash \sim A \parr ((B \parr \sim C) \otimes \sim D)} \qquad \text{and} \qquad \dfrac{\dfrac{\vdash \sim A, \sim D \qquad \dfrac{\vdash B, \sim C}{\vdash B \parr \sim C}}{\vdash \sim A, (B \parr \sim C) \otimes \sim D}}{\vdash \sim A \parr ((B \parr \sim C) \otimes \sim D)}$$

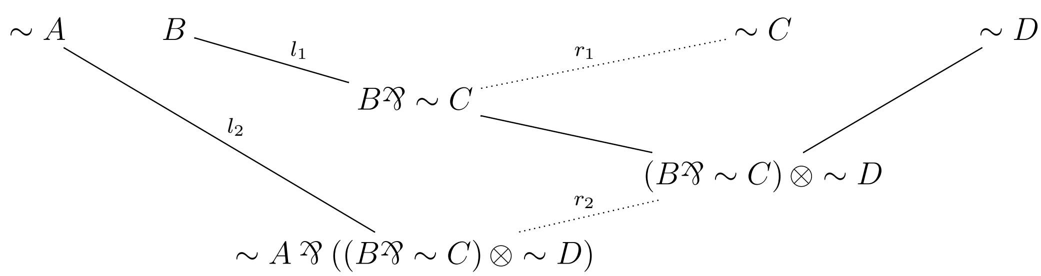
with partitions $\mathcal{Q} = \{\{1,2,3\},\{4\}\}$ and $\mathcal{Q}' = \{\{1,4\},\{2,3\}\}$ which correspond to the proof-nets



and



If we look at the possible choices of *switches* in the last proof-net, we get $\{l_1, l_2\}$, $\{l_1, r_2\}$, $\{r_1, l_2\}$ and $\{r_1, r_2\}$. Forgetting the links between literals, we see that each choice *divides* $\sim A, B, \sim C, \sim D$ in some *connected components*. For example, given $\{l_1, l_2\}$, we get the partition $\{\sim A\}$, $\{\sim C\}$ and $\{B, \sim D\}$ which means B and $\sim D$ are connected:



In terms of locations, this corresponds to the partition $\mathcal{P}'$ above, which was a *proof* of φ. In general, we observe that a choice of *switch* in the proof-net of a formula corresponds to a *proof* of its negation [17].

Therefore, with the duality defined above, we have

$$\{\mathcal{P}, \mathcal{P}'\}^{\perp} = \{\mathcal{Q}, \mathcal{Q}'\} \quad \text{and} \quad \{\mathcal{P}, \mathcal{P}'\} = \{\mathcal{Q}, \mathcal{Q}'\}^{\perp}$$

which means the set of proofs of φ is a fact, corresponding to the proofs of $\sim \varphi$.

Now we can show the following "bipolar" theorem due to Girard in [17] (p.445):

Theorem 4.2.2 *Let I, J be non-empty sets, $I \cap J = \varnothing$. Given an R-duality on E and an R'-duality on E' as above, where E, E' are the sets of partitions of I and J respectively (with duality relations $\perp$ and $\underset{\approx}{\perp}$ respectively), let $X \subseteq E$ and $Y \subseteq E'$ be facts.*
Then the locative product $X \mathbin{\cup\!\!\!\!\times} Y \overset{def}{=} \{\mathcal{P} \cup \mathcal{Q} \mid \mathcal{P} \in X, \mathcal{Q} \in Y\}$ is a fact with respect to the R''-duality on E'', where E'' is the set of partitions of $I \cup J$ (given by the duality relation $\underset{\approxeq}{\perp}$).

Proof : It suffices to show $(X \mathbin{\cup\!\!\!\!\times} Y)^{\underset{\approxeq}{\perp}\underset{\approxeq}{\perp}} \subseteq X \mathbin{\cup\!\!\!\!\times} Y$. Let $\mathcal{R} \in (X \mathbin{\cup\!\!\!\!\times} Y)^{\underset{\approxeq}{\perp}\underset{\approxeq}{\perp}}$ be a partition of $I \cup J$. Consider arbitrary partitions $\mathcal{S} \in X^{\perp}$ of I and $\mathcal{T} \in Y^{\underset{\approx}{\perp}}$ of J. Let $A_i \in \mathcal{S}$ and $B_j \in \mathcal{T}$ be the sets that contain $i \in I$ and $j \in J$ respectively. Define $\mathcal{S} \sqcup_{ij} \mathcal{T} := (\mathcal{S} \backslash \{A_i\}) \cup (\mathcal{T} \backslash \{B_j\}) \cup \{A_i \cup B_j\}$. In words, $\mathcal{S} \sqcup_{ij} \mathcal{T}$ is a partition of $I \cup J$ that consists of elements of $\mathcal{S}$ and $\mathcal{T}$ except for A_i and B_j that are "glued" together. Clearly, $\mathcal{S} \sqcup_{ij} \mathcal{T} \subseteq (X \mathbin{\cup\!\!\!\!\times} Y)^{\underset{\approxeq}{\perp}}$. Also, for all $i \in I, j \in J$, we have $\mathcal{R} \underset{\approxeq}{\perp} \mathcal{S} \sqcup_{ij} \mathcal{T}$. Now $\mathcal{R}$ cannot contain a set (a *part*) that contains elements both from I and J, say $i \in I$ and $j \in J$, since it would create a cycle with the "glued" part of $\mathcal{S} \sqcup_{ij} \mathcal{T}$. Therefore $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$, with $\mathcal{P} \in X$ and $\mathcal{Q} \in Y$. It follows that $\mathcal{R} \in X \mathbin{\cup\!\!\!\!\times} Y$. $\square$

This result is very interesting: it allows for an interpretation of the *multiplicatives* for linear logic.

$$\begin{array}{rcl} \sim A & \mapsto & [\![A]\!]^{\perp} \\ A \otimes B & \mapsto & [\![A]\!] \mathbin{\cup\!\!\!\!\times} [\![B]\!] \end{array}$$

with $A \parr B := \sim (\sim A \otimes \sim B)$ and $A \multimap B := \sim (A \otimes \sim B)$.

However, we need to keep in mind that when we have $[\![A]\!]^{\perp}$, the R-duality is defined on the set of partitions of the same set I as those of $[\![A]\!]$.

The last example gives us an important illustration of the notion of *tests* or *trials* ("*épreuves*"), also known as *paraproofs*, which are a generalization of *proofs*. Partitions are *tests*: in particular $\mathcal{P}$ and $\mathcal{P}'$ are *trials* that succeeded when tested with

$\mathcal{Q}$ and $\mathcal{Q}'$ (the choices of switches) with respect to the duality. Notice that none of the sequent calculus derivations is necessarily a *proof*; each branch starts with a rule $\vdash \Gamma$ which may not be of the *axiom* form $\vdash A, \sim A$. This *liberalized* sequent calculus makes use of a rule similar to the $\maltese$-rule in Ludics (see Section 4.2.1).

Example 4.2.3 (Permutations as epistates)

Epistates originally arose in the form of *permutations* in Girard's work [9]. Let $I \neq \varnothing$ be a finite set. Define a duality on $Sym(I)$, the set of permutations of I, by letting

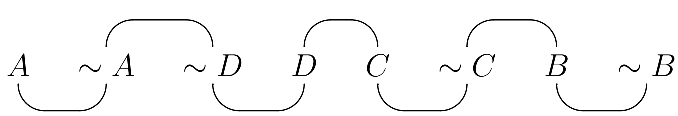
$$\begin{array}{rcl} \ll - \mid - \gg : Sym(I) \times Sym(I) & \longrightarrow & Sym(I) \\ (\sigma, \tau) & \mapsto & \sigma\tau \end{array}$$

and the pole $P = \{$cyclic permutations$\}$. In other words, we have

$$\sigma \mathrel{\overset{|}{\approx}} \tau \Leftrightarrow \sigma\tau \text{ is cyclic}$$

where $\mathrel{\overset{|}{\approx}} \subseteq Sym(I)^2$ is the duality relation. But we need to check that $\mathrel{\overset{|}{\approx}}$ is symmetric, so suppose $\sigma \mathrel{\overset{|}{\approx}} \tau$. This means for any $i \in I$, $\sigma\tau(i), (\sigma\tau)^2(i), ..., (\sigma\tau)^{|I|}(i)$ are pairwise distinct. We conclude that $\tau \mathrel{\overset{|}{\approx}} \sigma$, *i.e.* $\tau\sigma(i), (\tau\sigma)^2(i), ..., (\tau\sigma)^{|I|}(i)$ are also pairwise distinct. Indeed, suppose $(\tau\sigma)^k(i) = (\tau\sigma)^{k'}(i)$ for $k \neq k'$. Then $\sigma(\tau\sigma)^k(i) = \sigma(\tau\sigma)^{k'}(i)$, which implies $(\sigma\tau)^k\sigma(i) = (\sigma\tau)^{k'}\sigma(i)$, *i.e.* $(\sigma\tau)^k(j) = (\sigma\tau)^{k'}(j)$ for some $j \in I$, which contradicts our hypothesis.

In [9], a cyclic permutation, *i.e.* with exactly one cycle, was called a *longtrip*. This idea can be illustrated as follows: Consider the *cut* of the proof-nets for φ (corresponding to the partition $\mathcal{P}$) and $\sim \varphi$ (corresponding to $\mathcal{Q}'$) in the previous example. The *normal* form (see [15] for the procedure used for *normalization*) is then given by



If we let $\sigma = (1)(2)(3\ 4)$ and $\tau = (1\ 4)(2\ 3)$, and recall that A, B, C, D (and $\sim A, \sim B, \sim C, \sim D$) have respective locations $1, 2, 3, 4$, then we could imagine a

particle traveling throughout the proof-net and coming back to its starting location by following the cyclic permutation given by $\sigma\tau = (1\ 3\ 2\ 4)$.

An interesting feature of the last example is that the duality on the set of permutations takes into consideration the *normalization* process. Now *Geometry of Interaction* (GoI) can be viewed as a generalization of the previous approach to full linear logic, *i.e.* MALL with exponentials [21]. Epistates take the form of *unitary operators* on a Hilbert space. The definition of the duality *evolved* along with different versions of GoI. For example, the original definition

$$U \perp V \Leftrightarrow UV \ \ nilpotent$$

later became

$$U \perp V \Leftrightarrow det(I - UV) \neq 0, 1$$

As we end this Chapter, it is worth mentioning that proof-nets originally opened the way to GoI, and they seem to remain a good starting point for the quest of a *transcendental syntax* (see Girard's latest considerations in [21]).

Chapter 5

Discussion

In this work, we have exemplified the use of *polarity* in logic, first introduced by Girard in phase semantics and in a *desessentialized* viewpoint on coherence spaces. We have focused on the idea that this technique not only provides an alternative approach to the meaning of *negation* in logic, but it can also lead to a change of paradigm in the study of logic.

For future work, it would be interesting to investigate how Girard's *transcendental syntax* would fit in categorical logic. Also, the idea of *testing* as a fundamental meaning of *negation* invites us to the possibility of multiple negations emerging from different testing procedures (see Appendix B). Moreover, maybe we can hope that some of the notions we have introduced are related in a way to Peter Dybjer's *testing* [4].

We have encountered some concepts that seem general enough to suggest the emergence of other kinds of "transcendental syntaxes". Indeed, as Girard pointed out in [20] (p.18), the coexistence of Ludics and GoI opens the door to the possibility of many such syntaxes. Maybe some ideas related to testing - *e.g.* the interpretation of & and $\oplus$ of Section 3.2 - may even serve as a starting point (see Appendix A).

Appendix A

Filters as a set of epistates

This example tries to give a simplified version for the interpretation of the *additive* fragment of linear logic where epistates may represent *proofs*.

Example A.0.4

Consider $(X, (-)^\circ)$, where X is an arbitrary non-empty set and $(-)^\circ : X \longrightarrow X$ is a bijection such that $x^{\circ\circ} = x$ for all $x \in X$, *i.e.* an *involution*. Define a duality on $E = \mathcal{P}(X)$ such that $\ll A \mid B \gg_E = A^\circ \cap B$, where $A^\circ := \{a^\circ \mid a \in A\}$, and the pole is given by $P = E \backslash \{\varnothing\}$. Said differently, $A \perp B \Leftrightarrow A^\circ \cap B \neq \varnothing$. It is easy to see that $\perp$ is symmetric: if $A^\circ \cap B \neq \varnothing$, then there exists $b \in B$ such that $b = a^\circ$ for some $a \in A$. Hence $b^\circ = a^{\circ\circ} = a$, which means $B^\circ \cap A \neq \varnothing$.

Proposition A.0.5 *With respect to the above duality on $\mathcal{P}(X)$, the facts are unions of filters, more precisely of the form $\bigcup_{i \in I} (\Uparrow(A_i))$, for some $A_i \subseteq X$, where $\Uparrow(A_i)$ is the principal filter generated by A_i (see Definition 3.1.22).*

Proof : Let $\mathcal{A} \subseteq \mathcal{P}(X)$. Since facts are polar sets, we will look at $\mathcal{A}^\perp$. Consider a distinguished element $a_i \in A_i$ for every $A_i \in \mathcal{A}$. Then $B = \{a_i^\circ \mid \text{ for every } i\}$ is in $\mathcal{A}^\perp$. Moreover, every $B' \supseteq B$ is also in $\mathcal{A}^\perp$. Therefore $\Uparrow(B) \in \mathcal{A}^\perp$. $\square$

It follows that the union of facts is a fact, which is quite convenient.

Now consider $(X', (-)^{\bullet})$ with $(-)^{\bullet}$ a bijection such that $x^{\bullet\bullet} = x$ and a duality on $E' = \mathcal{P}(X')$ as above, with duality relation $\overset{\downarrow}{\sim}$. Consider also a duality on $E'' = \mathcal{P}(X \cup X')$ such that $\ll A \mid B \gg_{E''} := A^{\odot} \cap B$, where $A^{\odot} := \{a^{\odot} \mid a \in A\}$ with

$$a^{\odot} := \begin{cases} a^{\circ} & \text{if } a \in X \\ a^{\bullet} & \text{if } a \in X' \end{cases}$$

with duality relation $\overset{\downarrow}{\approx}$. We define injections $\iota_E : E \longrightarrow E''$ and $\iota_{E'} : E' \longrightarrow E''$ such that $\iota_E(A) = A$ and $\iota_{E'}(A') = A'$. We can easily see they are duality monomorphisms: if $A \perp B$, then $A^{\circ} \cap B \neq \varnothing$, and since $A^{\odot} = A^{\circ}$, it follows that $A \overset{\downarrow}{\approx} B$. The definition of the additives goes as follows: let $\mathcal{A} \subseteq E$ and $\mathcal{B} \subseteq E'$ be facts for dualities on E and E' respectively. Then

$$\begin{aligned} \text{“}\mathcal{A} \,\&\, \mathcal{B}\text{”} &:= \iota_E\left(\mathcal{A}^{\perp}\right)^{\overset{\downarrow}{\approx}} \cap \iota_{E'}\left(\mathcal{B}^{\overset{\downarrow}{\sim}}\right)^{\overset{\downarrow}{\approx}} \\ &= \mathcal{A}^{\perp\,\overset{\downarrow}{\approx}} \cap \mathcal{B}^{\overset{\downarrow}{\sim}\,\overset{\downarrow}{\approx}} \\ \text{“}\mathcal{A} \oplus \mathcal{B}\text{”} &:= \left(\iota_E(\mathcal{A}) \cup \iota_{E'}(\mathcal{B})\right)^{\overset{\downarrow}{\approx}\,\overset{\downarrow}{\approx}} \\ &= (\mathcal{A} \cup \mathcal{B})^{\overset{\downarrow}{\approx}\,\overset{\downarrow}{\approx}} \end{aligned}$$

An interesting feature of this interpretation is given by the following property:

Proposition A.0.6 *Let $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ be facts in the duality defined above. Then $\mathcal{A} \mathbin{\cup\!\!\!\!\times} \mathcal{B} = \mathcal{A} \cap \mathcal{B}$.*

Proof : Let $\mathcal{A} = \bigcup_{i \in I} \uparrow\!(A_i)$ and $\mathcal{B} = \bigcup_{j \in J} \uparrow\!(B_j)$ where $A_i, B_j \subseteq X$. Clearly $\mathcal{A} \mathbin{\cup\!\!\!\!\times} \mathcal{B} = \bigcup_{i \in I, j \in J} \uparrow\!(A_i \cup B_j)$, and $\mathcal{A} \cap \mathcal{B} = \{C \in \mathcal{P}(X) \mid A_i \cup B_j \subseteq C \text{ for some } i, j\}$. $\square$

This result is related to Girard's *Mystery of Incarnation* [17].

Definition A.0.7 (Incarnation) Given a set of sets $\mathcal{A}$, the *incarnation* of $\mathcal{A}$, noted $\mathcal{A}^{\lightning}$, is given by

$$\mathcal{A}^{\lightning} := \{A \in \mathcal{A} \mid A = \bigcap \{A' \in \mathcal{A} \mid A' \subseteq A\}\}$$

For *Ludics*, Girard showed the following result ([17], p.346):

Theorem A.0.8 (Mystery of Incarnation) *For "negative" and "disjoint" behaviours* $\boldsymbol{A}$ *and* $\boldsymbol{B}$*, we have*

$$(\boldsymbol{A} \cap \boldsymbol{B})^{\lightning} = \boldsymbol{A}^{\lightning} \mathbin{\cup\!\!\!\!\times} \boldsymbol{B}^{\lightning}$$

We recall that in this situation, $\mathbf{A} \cap \mathbf{B} =: \mathbf{A} \,\&\, \mathbf{B}$ (see Section 4.2.1). An analogous result can be shown in our case:

Corollary A.0.9 (of Proposition A.0.6) *Let* $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ *be facts in the duality defined above. Then*

$$(\mathcal{A} \cap \mathcal{B})^{\lightning} = \mathcal{A}^{\lightning} \mathbin{\cup\!\!\!\!\times} \mathcal{B}^{\lightning}$$

Proof : By Proposition A.0.6, $(\mathcal{A} \cap \mathcal{B})^{\lightning} = (\mathcal{A} \mathbin{\cup\!\!\!\!\times} \mathcal{B})^{\lightning}$, and by definition of $\mathcal{A}^{\lightning}$, it follows that $(\mathcal{A} \mathbin{\cup\!\!\!\!\times} \mathcal{B})^{\lightning} = \mathcal{A}^{\lightning} \mathbin{\cup\!\!\!\!\times} \mathcal{B}^{\lightning}$. □

Here *epistates* are elements of the filters contained in a dichology $\mathcal{A}$. In particular, the *proofs* of $\mathcal{A}$ consist of the elements of $\mathcal{A}^{\lightning}$. For example, if we let $\mathcal{A} = \Uparrow(\{f\})$ and $\mathcal{B} = \Uparrow(\{g\})$ be dichologies, where $f \in X, g \in X'$ as above, then $\mathcal{A}^{\lightning} = \{\{f\}\}$ and $\mathcal{B}^{\lightning} = \{\{g\}\}$, which means the proofs are $\{f\}$ and $\{g\}$. Moreover, a proof of $\mathcal{A}$ *and* $\mathcal{B}$ would be $\{f, g\}$, since we have $\mathcal{A} \,\&\, \mathcal{B} := \Uparrow(\{f, g\})$, and the proofs of $\mathcal{A}$ *or* $\mathcal{B}$ would be $\{f\}$ and $\{g\}$, since $\mathcal{A} \oplus \mathcal{B} := \Uparrow(\{f\}) \cup \Uparrow(\{g\})$. Notice also that $(\mathcal{A} \,\&\, \mathcal{B})^{\overset{\downarrow}{\approx}} = \Uparrow(\{f^{\circ}\}) \cup \Uparrow(\{g^{\bullet}\}) = \mathcal{A}^{\overset{\downarrow}{\approx}} \cup \mathcal{B}^{\overset{\downarrow}{\approx}} = (\mathcal{A}^{\overset{\downarrow}{\approx}} \cup \mathcal{B}^{\overset{\downarrow}{\approx}})^{\overset{\downarrow}{\approx}\,\overset{\downarrow}{\approx}} =: \mathcal{A}^{\overset{\downarrow}{\approx}} \oplus \mathcal{B}^{\overset{\downarrow}{\approx}}$.

Appendix B

Coexistence of multiple negations

From a set theoretic viewpoint of mathematics, the *negation* of a statement is often perceived as the *complement* of a set. Indeed, in number theory, if "n is *not* even", the first thing that comes to mind is that it is odd. In the framework of polarity, this could be described as a $\{-1,0,1\}$-duality on $\mathbb{N}^* := \{1,2,3,...\}$ where the duality relation is *classical* of the form $\perp_{Id}$. To be precise, we could take the map $\ll - \mid - \gg :$ $\mathbb{N}^* \times \mathbb{N}^* \longrightarrow \{-1,0,1\}$ such that

$$\ll m \mid n \gg = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m, n \neq 1, m \neq n \\ -1 & \text{if } m = 1 \text{ or } n = 1 \text{, but not both} \end{cases}$$

with the pole $P = \{-1, 0\}$, which simply says "$m \neq n$".

The set of even positive integers $\mathbb{E} \subseteq \mathbb{N}^*$ is a fact, since $\mathbb{E}^{\perp_{Id}\perp_{Id}} = \mathbb{E}$. In the same line of thought, when we think of "n is *not* prime", we *intuitively* think of n as a composite number (well, if it is not intuitive, at least it is convenient if we want the *fundamental theorem of arithmetic* to hold). But technically speaking, if $\mathbb{P}$ is the set of prime numbers, $\mathbb{P}^{\perp_{Id}}$ are not composite numbers, since "1" is included in the set. A first solution is to keep our notion of "negation as complement" but restrict our *universe*: we consider the $\{-1,0,1\}$-duality on $\mathbb{N}^* \backslash \{1\}$. But a second approach

would be to define another $\{-1,0,1\}$-duality on $\mathbb{N}^*$ by taking the pole $P' = \{0\}$. With respect to this duality, $\mathbb{P}$ is a fact. We see that the choice of a different pole (or equivalently a duality relation) induces a different negation over the same universe.

It is not clear yet if this is a promising avenue or not, but the idea of multiple negations coexisting looks very *natural*. Here are some properties:

Proposition B.0.10 *Given two R-dualities on E, with duality relations $\perp_1$ and $\perp_2$, we have for all $A \subseteq E$:*

(i) $\perp_1 \subseteq \perp_2 \Rightarrow A^{\perp_1} \subseteq A^{\perp_2}$

(ii) $A^{\perp_1} \cap A^{\perp_2} = A^{\perp_1 \cap \perp_2}$

(iii) $A^{\perp_1} \cup A^{\perp_2} = A^{\perp_1 \cup \perp_2}$

Proof : For (i): if $x \in A^{\perp_1}$, then $x \perp_1 a$ for all $a \in A$ by definition. Hence $x \perp_2 a$ for all $a \in A$ by hypothesis, and $x \in A^{\perp_2}$.
For (ii): $x \in A^{\perp_1} \cap A^{\perp_2}$ iff $x \perp_1 A$ and $x \perp_2 A$, which means $(x,a) \in \perp_1$ and $(x,a) \in \perp_2$ for all $a \in A$. This is equivalent to saying $(x,a) \in \perp_1 \cap \perp_2$ for all $a \in A$, which means $x \in A^{\perp_1 \cap \perp_2}$. A similar reasoning holds for (iii). □

Proposition B.0.11 *Given two R-dualities on a set E, with classical duality relations $\perp_\varphi$ and $\perp_{Id}$, where $\varphi : E \longrightarrow E$ is any bijection, we have*

$$X^{\perp_\varphi \perp_{Id} \perp_\varphi \perp_{Id}} = X$$

for all $X \subseteq E$.

Proof : Use Proposition 2.1.19, combined to Propositions 2.1.23 and 2.1.24. □

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