Cohomology Operations and the Toral Rank Conjecture for Nilpotent Lie Algebras

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Abstract

The action of various operations on the cohomology of nilpotent Lie algebras is studied. In the cohomology of any Lie algebra, we show that the existence of certain nontrivial compositions of higher cohomology operations implies the existence of hypercube-like structures in cohomology, which in turn establishes the Toral Rank Conjecture for that Lie algebra. We provide examples in low dimensions and exhibit an infinite family of nilpotent Lie algebras of arbitrary dimension for which such structures exist. A new proof of the Toral Rank Conjecture is also given for free two-step nilpotent Lie algebras.

Contents

Introduction

A cohomology theory of Lie algebras was introduced by Chevalley and Eilenberg [4] in 1948 in order to place certain questions concerning the topology of Lie groups in an algebraic setting. It was known that the cohomology of a compact connected Lie group G is entirely determined by its associated Lie algebra $\mathfrak{g}$, and Lie algebra cohomology is defined in such a way that $H^*(G;\mathbb{R}) \cong H^*(\mathfrak{g};\mathbb{R})$. This situation is generalized in the case of *nilmanifolds*, a large class of spaces whose cohomological structure remains largely unknown. By a classical theorem of Nomizu [25, Theorem 1], the cohomology of any nilmanifold is isomorphic to that of an associated nilpotent Lie algebra. This thesis treats the cohomology of nilpotent Lie algebras with this interpretation in mind. The main purpose of this research was to examine certain cohomology operations, introduced by Cairns and Jessup [5], and to continue the study of their use in analyzing the cohomological structure of this class of Lie algebras.

Of particular interest is the Toral Rank Conjecture, an outstanding open problem in algebraic topology which aims to relate the toral symmetry of a manifold to its total cohomological dimension. More precisely, it states that the total dimension of the cohomology of every smooth manifold is greater than or equal to that of any torus acting freely upon it. For compact connected Lie groups, this reduces to a well-known result relating the cohomology of G to the dimension of a maximal torus contained in G, but the conjecture is known to be true for many other manifolds as well. In the case of nilmanifolds, the Toral Rank Conjecture can be restated in terms of Lie algebra cohomology (where, in fact, no Lie algebra, nilpotent or otherwise, has been found to provide a counterexample), but while it is known to be true in some special cases, it remains open in general.

After outlining the known results mentioned above concerning the Toral Rank Conjecture and reviewing the necessary algebraic preliminaries in Chapter 1, we give a definition of Lie algebra cohomology in Chapter 2, surveying the main results which will be of use to us in later chapters, such as Poincaré duality and a version of the Hodge decomposition theorem for Lie algebras. In Section 2.4, we describe the Dixmier long exact sequence of cohomology groups and use it to give a new proof of the Toral Rank Conjecture for free two-step nilpotent Lie algebras.

In Chapter 3, we begin our study of the cohomology operations mentioned above. In [5], a representation of the exterior algebra ΛZ in the cohomology of a Lie algebra $\mathfrak{g}$ (where Z is the centre of $\mathfrak{g}$) is shown to be useful in establishing the Toral Rank Conjecture in many cases. A theorem in [5] states that whenever this representation, called the *central representation*, is faithful, one can find a sequence of operations which forms a hypercube-like structure in the cohomology of $\mathfrak{g}$ (viewed as a ΛZ-module) and thereby establishes the Toral Rank Conjecture for $\mathfrak{g}$. The cohomology operations defining the central representation are generalized by so-called *higher operations*, and in Section 3.2 we illustrate the utility of these operations in recovering similar hypercube-like structures in cohomology when the central representation is not faithful. The main result of this chapter is a generalization of the above theorem which states that the existence of certain sequences of operations (including higher operations) on $H^*(\mathfrak{g})$ implies the existence of similar hypercube-like structures which yield the Toral Rank Conjecture for $\mathfrak{g}$. As an application, we find hypercubes in the cohomology of some free two-step nilpotent Lie algebras and exhibit an infinite family of nilpotent Lie algebras attached to graphs where such structures exist in cohomology. We finish with some open questions for future research.

I would like to express my deepest gratitude to my friends and family for their support of me and my studies. In particular, I extend my sincerest thanks to my supervisor, Dr. Barry Jessup, both for introducing me to an exciting research topic and also for his confidence, generosity and unending patience throughout the preparation of this thesis.

Chapter 1

Preliminaries

We begin with a discussion of Lie groups and their homogeneous spaces. These examples of spaces both motivate the Toral Rank Conjecture in its original form and suggest a version of the conjecture rephrased strictly in terms of Lie algebras.

1.1 Lie Groups and the Toral Rank Conjecture

Throughout this section, any mention of the cohomology $H^*(G)$ of a Lie group G refers to the de Rham cohomology $H^*_{dR}(G)$ of the underlying smooth manifold (a good reference for which is [3]).

Let G be a compact connected Lie group. Since the group operation

$$\mu : G \times G \to G$$

is smooth, μ induces a homomorphism of graded algebras

$$\mu^* : H^*(G) \to H^*(G) \otimes H^*(G),$$

where $H^*(G \times G)$ has been identified with $H^*(G) \otimes H^*(G)$ by the Künneth isomorphism. This map gives $H^*(G)$ the structure of a *Hopf algebra*, that is, a graded associative algebra A over a field $\mathbb{F}$ equipped with a homomorphism of graded algebras $\Delta : A \to A \otimes A$ called a *comultiplication* which, for every homogeneous element $x \in A$ with $|x| > 0$, satisfies

$$\Delta(x) = 1 \otimes x + x \otimes 1 + \sum_{i,j} x_i \otimes y_j,$$

where the x_i and y_j are homogeneous elements with $|x_i|, |y_j| > 0$, and $|x_i| + |y_j| = |x|$. A homogeneous element $x \in A$ is called *primitive* if $\Delta(x) = 1 \otimes x + x \otimes 1$. In 1941, Hopf [18] proved that any finite-dimensional connected (meaning $A^0 = \mathbb{F}$) graded-commutative Hopf algebra over a field of characteristic 0 is isomorphic to an exterior algebra generated by elements of odd degree. It was later proved by Samelson [27] that if the comultiplication is coassociative, that is, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\Delta} & A \otimes A \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \Delta \otimes \mathrm{id}} \\ A \otimes A & \xrightarrow[\mathrm{id} \otimes \Delta]{} & A \otimes A \otimes A \end{array}$$

commutes, then A is isomorphic to the exterior algebra ΛP_A over the subspace of primitive elements in A. Note that in the case of a compact connected Lie group G, the associativity of the group multiplication $\mu : G \times G \to G$ implies that

$$\begin{array}{ccc} H^*(G) & \xrightarrow{\mu^*} & H^*(G) \otimes H^*(G) \\ {\scriptstyle \mu^*}\downarrow & & \downarrow{\scriptstyle \mu^* \otimes \mathrm{id}} \\ H^*(G) \otimes H^*(G) & \xrightarrow[\mathrm{id} \otimes \mu^*]{} & H^*(G) \otimes H^*(G) \otimes H^*(G) \end{array}$$

commutes, i.e., $(H^*(G), \mu^*)$ is a coassociative Hopf algebra and the above results of Hopf and Samelson apply.

Definition 1.1.1. *For a compact connected Lie group G, the dimension of a maximal torus in G is called the* rank *of G, denoted* $\mathrm{rank}(G)$.

Moreover, we have the following theorem due to Hopf relating the rank of a compact connected Lie group to the structure of its cohomology, proofs of which can be found in [12] and [19].

Theorem 1.1.2. *Let G be a compact connected Lie group and let P_G denote the subspace of primitive elements of $H^*(G)$. Then there exists an isomorphism of algebras*

$$H^*(G) \cong \Lambda P_G$$

and $\dim P_G = \operatorname{rank}(G)$.

Note that a compact connected Lie group G acts freely on itself by left multiplication and so does any subgroup of G. So if T^r is a maximal torus in G, then G admits a free T^r-action. We are interested in cohomological consequences of such toral symmetries, so we use the following generalization of the notion of rank in compact connected Lie groups.

Definition 1.1.3. *The* toral rank *of a smooth manifold M, denoted* $\operatorname{rk}(M)$, *is the dimension of the largest torus which acts freely on M.*

Now let G be a Lie group and let K be a closed subgroup of G. Then the set G/K of left cosets of K has the structure of a smooth manifold and is called a *homogeneous space* of G. Assuming G is compact and connected and K is a closed connected subgroup, we have by Hopf's theorem that $\dim H^*(K) = 2^{\operatorname{rank}(K)}$. It is shown in [11] that $\operatorname{rk}(G/K) = \operatorname{rank}(G) - \operatorname{rank}(K)$. Hence, taking K to be the trivial subgroup, the toral rank of a compact connected Lie group agrees with the dimension of a maximal torus, $\operatorname{rk}(G) = \operatorname{rank}(G)$.

It follows from the theorem above, then, that $\dim H^*(G) = 2^{\operatorname{rk}(G)}$. Similarly, for homogeneous spaces, by the Serre spectral sequence associated to the fibration $K \to G \to G/K$, one can obtain that $\dim H^*(G) \leq \dim H^*(K) \cdot \dim H^*(G/K)$ [1, page 281], from which it follows that

$$2^{\operatorname{rank}(G)} \leq 2^{\operatorname{rank}(K)} \cdot \dim H^*(G/K).$$

Consequently, $\dim H^*(G/K) \geq 2^{\mathrm{rk}(G/K)}$.

The above examples, and other classes of spaces including Kähler manifolds and products of odd-dimensional spheres [11, page 285], provide evidence for the longstanding Toral Rank Conjecture (TRC). First announced by Steve Halperin [15] in 1968, it states that for any smooth manifold M,

$$\dim H^*(M) \geq 2^{\mathrm{rk}(M)}.$$

In other words, the cohomology of M should be at least as big as that of any torus acting freely on M.

The TRC can be seen as a conjectural bound both on the total cohomological dimension of M and on its toral rank, and the relation is known to hold when either of these values is small. For example, it is known to hold for spaces with toral rank less than or equal to three [1], and in [17] it is shown that $\dim H^*(M) \leq 10$ implies $\dim H^*(M) \geq 2^{\mathrm{rk}(M)}$.

1.2 Nilmanifolds and the TRC for Lie Algebras

While the toral rank of a space is, in general, very difficult to compute, *nilmanifolds* represent a large class of spaces for which the toral rank is known (but for which the TRC is open in general). For these homogeneous spaces the bound suggested by the TRC reduces to a bound related to the centres of the associated Lie algebras. This passage to Lie algebras relies on two results on nilmanifolds, the first on their toral rank and the other concerning their cohomology.

Definition 1.2.1. *If N is a nilpotent Lie group and Γ a discrete co-compact* [1] *subgroup, then the quotient N/Γ is called a* nilmanifold.

[1] A *co-compact* subgroup is a subgroup Γ such that the quotient N/Γ is compact. Hence nilmanifolds are compact by definition. By a theorem of Malcev [24], a simply-connected nilpotent Lie group N contains a discrete co-compact subgroup (often called a *lattice*) if and only if its Lie algebra admits a basis with rational structure constants.

Example 1.2.2. $S^1 = \mathbb{R}/\mathbb{Z}$ is a nilmanifold, as is any torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

Example 1.2.3. The n^{th} *Heisenberg group* is the nilpotent Lie group consisting of upper triangular $n \times n$ matrices with 1's along the diagonal:

$$G_n(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x_{12} & \cdots & x_{1n} \\ 0 & 1 & \cdots & x_{2n} \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 \end{pmatrix} \mid \; x_{ij} \in \mathbb{R} \right\}.$$

Let $G_n(\mathbb{Z})$ denote the subgroup of matrices with integral entries. Then $G_n(\mathbb{R})/G_n(\mathbb{Z})$ is a nilmanifold called the n^{th} *Heisenberg nilmanifold*.

Let N/Γ be a nilmanifold and let $\mathfrak{n}$ be the Lie algebra of N. The first result toward a rephrasing of the TRC in terms of $\mathfrak{n}$ is that the toral rank of N/Γ is equal to the dimension of the centre of $\mathfrak{n}$ (see, for example, Theorem 7.28 in [11]):

$$\mathrm{rk}(N/\Gamma) = \dim Z(\mathfrak{n}).$$

The second is the following classical result due to Nomizu [25, Theorem 1].

Theorem 1.2.4 (Nomizu's Theorem)**.** *Let N/Γ be a nilmanifold and let $\mathfrak{n}$ be the Lie algebra associated to N. Then there exists an isomorphism of algebras*

$$H^*(N/\Gamma) \cong H^*(\mathfrak{n})$$

where $H^*(\mathfrak{n})$ is the Lie algebra cohomology of $\mathfrak{n}$ (with trivial coefficients).

By analogy with nilmanifolds, the toral rank conjecture for Lie algebras can therefore be written as follows.

Conjecture 1.2.5 (TRC for Lie algebras)**.** *If $\mathfrak{g}$ is a finite-dimensional Lie algebra with centre Z, then*

$$\dim H^*(\mathfrak{g}) \geq 2^{\dim Z}.$$

The conjecture is known to hold for two-step nilpotent Lie algebras (see [7]) and reductive Lie algebras (see [13]) but remains open in general. In the nilpotent case, the focus of this thesis, some special cases have been proven in [7]: the TRC holds for a nilpotent Lie algebra $\mathfrak{g}$ if any of the following conditions are satisfied:

a) $\dim Z \leq 5$;

b) $\dim \mathfrak{g}/Z \leq 7$;

c) $\dim \mathfrak{g} \leq 14$.

In particular, all nilmanifolds of dimension at most 14 satisfy the TRC. In addition, it is known to hold for all Lie algebras with a grading $\mathfrak{g} = \oplus_{i=0}^{m} \mathfrak{g}_i$ which satisfies $[\mathfrak{g}_i, \mathfrak{g}_j] \subseteq \mathfrak{g}_{i+j}$ and $Z(\mathfrak{g}) = \mathfrak{g}_m$ [9]. This includes the free k-step nilpotent Lie algebras for all k.

1.3 Algebraic Preliminaries

Here we fix some notation and recall the basic concepts from the theory of Lie algebras, homological algebra and multilinear algebra that we will need before defining Lie algebra cohomology. Standard references on these topics are [20], [23] and [14], respectively.

1.3.1 Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ equipped with a bilinear map $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the following:

- $[x, x] = 0$ for all $x \in \mathfrak{g}$ (skew-symmetry);
- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$ (Jacobi identity).

The map $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is called the *commutator* or *Lie bracket*.

Example 1.3.1. The space of $n \times n$ matrices with entries in a field $\mathbb{F}$ together with the usual commutator bracket $[A, B] = AB - BA$ forms a Lie algebra known as $\mathfrak{gl}_n(\mathbb{F})$. Similarly, the space of endomorphisms of a vector space V form a Lie algebra denoted $\mathfrak{gl}(V)$.

Example 1.3.2. The space of vector fields $\mathrm{Vect}(M)$ on a smooth manifold M is a Lie algebra with the Lie bracket of two vector fields X, Y given by

$$[X, Y]f = X(Yf) - Y(Xf)$$

for any smooth function $f \in C^\infty(M)$.

A *homomorphism of Lie algebras* is a linear map $\phi : \mathfrak{g} \to \mathfrak{h}$ satisfying

$$\phi([x, y]) = [\phi(x), \phi(y)]$$

for all $x, y \in \mathfrak{g}$. For example, for any Lie algebra $\mathfrak{g}$, the linear map $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ sending each $x \in \mathfrak{g}$ to the endomorphism $\mathrm{ad}(x)$ of $\mathfrak{g}$ defined by $\mathrm{ad}(x)(y) = [x, y]$ is a homomorphism of Lie algebras by the Jacobi identity.

A *Lie subalgebra* of $\mathfrak{g}$ is a subspace $\mathfrak{h} \subseteq \mathfrak{g}$ closed under the Lie bracket, and a subspace $I \subseteq \mathfrak{g}$ is called an *ideal* if $[I, \mathfrak{g}] \subseteq I$. Two elements $x, y \in \mathfrak{g}$ are said to *commute* if $[x, y] = 0$. The *centre* of $\mathfrak{g}$, denoted $Z(\mathfrak{g})$, is the ideal consisting of all elements which commute with all of $\mathfrak{g}$:

$$Z(\mathfrak{g}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \text{ for all } y \in \mathfrak{g}\}.$$

A Lie algebra is called *abelian* if the Lie bracket is identically zero, i.e., if $Z(\mathfrak{g}) = \mathfrak{g}$. Conversely, $\mathfrak{g}$ is called *simple* if it is nonabelian and contains no nontrivial ideals (in this case $Z(\mathfrak{g}) = 0$).

If $\{e_1, \dots, e_n\}$ is a basis for a Lie algebra $\mathfrak{g}$ over a field $\mathbb{F}$, then for any pair of basis elements we can write

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k,$$

for some $c_{ij}^k \in \mathbb{F}$ called the *structure constants* of $\mathfrak{g}$ (relative to $\{e_1, \ldots, e_n\}$). By skew-symmetry, $c_{ij}^k = -c_{ji}^k$ for all k, so by bilinearity, the Lie bracket $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is completely determined by the structure constants c_{ij}^k where $i < j$.

For any Lie algebra $\mathfrak{g}$ there is a descending sequence of ideals

$$\mathfrak{g} \supseteq \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^n \supseteq \cdots$$

called the *lower central series* of $\mathfrak{g}$ inductively defined by

$$\mathfrak{g}^0 = \mathfrak{g}, \qquad \mathfrak{g}^k = [\mathfrak{g}, \mathfrak{g}^{k-1}].$$

If $\mathfrak{g}^k = 0$ for some $k \in \mathbb{N}$, then $\mathfrak{g}$ is said to be *nilpotent.*

Example 1.3.3. The strictly upper triangular $n \times n$ matrices with entries in a field $\mathbb{F}$ form a nilpotent Lie subalgebra $\mathfrak{n}_n(\mathbb{F}) \subseteq \mathfrak{gl}_n(\mathbb{F})$. The n^{th} Heisenberg group, defined above, has $\mathfrak{n}_n(\mathbb{R})$ as its associated Lie algebra.

Remark 1.3.4. More precisely, we say $\mathfrak{g}$ is *k-step nilpotent* if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$. Note that $\mathfrak{g}$ is abelian if and only if $[\mathfrak{g}, \mathfrak{g}] = 0$, and furthermore, $\mathfrak{g}$ is associative if and only if $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] = 0$. Hence the one-step nilpotent Lie algebras consist precisely of all the nontrivial abelian Lie algebras while the two-step nilpotent Lie algebras comprise the *nonabelian* associative Lie algebras.

We could also have defined a sequence of ideals by

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \qquad \mathfrak{g}^{(k)} = [\mathfrak{g}^{(k-1)}, \mathfrak{g}^{(k-1)}].$$

This is called the *derived series* of $\mathfrak{g}$, and $\mathfrak{g}$ is called *solvable* if $\mathfrak{g}^{(k)} = 0$ for some k. Note that all nilpotent Lie algebras are solvable but that the converse is not true, as evidenced by the nonabelian two-dimensional Lie algebra with basis $\{x, y\}$ where $[x, y] = y$.

A Lie algebra $\mathfrak{g}$ is called *unimodular* if $\operatorname{tr}(\operatorname{ad} x) = 0$ for all $x \in \mathfrak{g}$. All nilpotent Lie algebras are unimodular, and as this thesis is concerned mostly with nilpotent

Lie algebras, we include here a characterization of nilpotency which will be used in later chapters.

Theorem 1.3.5. *A finite-dimensional Lie algebra* $\mathfrak{g}$ *is nilpotent if and only if there exists a basis* $\{e_1, \ldots, e_n\}$ *of* $\mathfrak{g}$ *such that* $[e_i, e_j] \in \operatorname{span}\{e_{j+1}, \ldots, e_n\}$ *for all* $i < j$.

Proof: One direction is obvious, so assume $\mathfrak{g}$ is nilpotent. By definition, then, the lower central series terminates at some natural number k:

$$\mathfrak{g}^0 = \mathfrak{g} \quad \supset \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}] \quad \supset \quad \mathfrak{g}^2 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]] \quad \supset \cdots \supset \quad \mathfrak{g}^k = 0.$$

Note that each term in the sequence above is a proper ideal of the preceding term. For each $\mathfrak{g}^i \supset \mathfrak{g}^{i+1}$, extend a basis $\{v_1, \ldots, v_p\}$ of $\mathfrak{g}^{i+1}$ to a basis $\{v_1, \ldots, v_{p+q}\}$ of $\mathfrak{g}^i$ and consider the chain of subspaces

$$\mathfrak{g}^i \supset \operatorname{span}\{v_1, \ldots, v_{p+q-1}\} \supset \cdots \supset \operatorname{span}\{v_1, \ldots, v_p, v_{p+1}\} \supset \mathfrak{g}^{i+1}.$$

In this way we obtain a chain of vector spaces

$$\mathfrak{g} = I_0 \supset I_1 \supset I_2 \supset \cdots \supset I_n = 0$$

such that $\dim(I_j/I_{j+1}) = 1$ for all j. For each j, let $j_M = \max\{i \mid \mathfrak{g}^i \supseteq I_j\}$. Note that $\mathfrak{g}^{j_M} \supseteq I_j \supset I_{j+1} \supseteq \mathfrak{g}^{j_M+1}$. Therefore

$$[\mathfrak{g}, I_j] \subseteq [\mathfrak{g}, \mathfrak{g}^{j_M}] = \mathfrak{g}^{j_M+1} \subseteq I_{j+1}. \tag{1.3.1}$$

Now starting with a basis $\{e_n\}$ of I_{n-1}, we extend to a basis $\{e_{n-1}, e_n\}$ of I_{n-2} and continuing in this way (so that $I_j = \operatorname{span}\{e_{j+1}, \ldots, e_n\}$ for all j) we obtain a basis $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ such that, by (1.3.1), $[e_i, e_j] \in \operatorname{span}\{e_{j+1}, \ldots, e_n\}$ for all $i < j$, as desired. ∎

1.3.2 Some homological algebra

In this subsection, all vector spaces and algebras are assumed to be finite-dimensional over a field $\mathbb{F}$ of characteristic zero unless stated otherwise. Although most of what is said here could just as well be said for modules and algebras over other rings, we will not need this generality later on as we are mainly concerned with *real* Lie algebras and cohomology with trivial coefficients.

Recall that a *graded algebra* is a graded vector space

$$A = \bigoplus_{p \geq 0} A^p,$$

together with a bilinear map $\cdot : A \times A \to A$ such that $A^p \cdot A^q \subseteq A^{p+q}$. If $x \in A^p$, then x is said to be of *degree* p, written $|x| = p$.

Example 1.3.6. If V is a vector space, then the tensor algebra of V,

$$T(V) = \bigoplus_{p=0}^{\infty} V^{\otimes p},$$

is an infinite-dimensional graded algebra with multiplication given by the tensor product (i.e., concatenation of words in V).

A *homomorphism* of graded algebras is a linear map $\phi : A \to B$ homogeneous of degree zero (meaning $\phi(A^p) \subseteq B^p$ for all p) such that

$$\phi(xy) = \phi(x)\phi(y)$$

for all $x, y \in A$. A *derivation of degree* k on a graded algebra A is a linear map $\delta : A \to A$ homogeneous of degree k (meaning $\delta(A^p) \subseteq A^{p+k}$ for all p) satisfying the graded Leibniz rule:

$$\delta(xy) = \delta(x)y + (-1)^{k|x|} x\delta(y),$$

for all $x, y \in A$.

A *differential graded algebra*, or DGA, (A, d) is a graded algebra A together with a derivation d of degree 1 such that $d^2 = 0$ (such a derivation is called a *differential*). The kernel and image of d are graded subalgebras, denoted $\ker d = Z^*(A)$ and $\operatorname{im} d = B^*(A)$, and their elements are called *cocycles* and *coboundaries*, respectively. Note that $d^2 = 0$ implies $B^*(A) \subseteq Z^*(A)$, and for any $a \in Z^*(A)$, $b \in B^*(A)$ we have that $b = d(c)$ for some $c \in A$, so

$$d(ac) = d(a)c + (-1)^{|a|}ad(c) = (-1)^{|a|}ab.$$

It follows from the above remarks that $B^*(A)$ is in fact a graded ideal in $Z^*(A)$, which allows us to define the *cohomology algebra* of (A, d) as the quotient

$$H^*(A, d) = Z^*(A)/B^*(A),$$

written $H^*(A)$ when the context is clear. Writing d_p for the restriction of the differential to A^p, we have $\ker d_p = Z^p(A)$ and $\operatorname{im} d_{p-1} = B^p(A)$, so $H^*(A) = \oplus_p H^p(A)$, where $H^p(A) = Z^p(A)/B^p(A)$.

A *homomorphism* of differential graded algebras $\phi : (A, d_A) \to (B, d_B)$ is a homomorphism of graded algebras which commutes with the differentials:

$$\phi \circ d_A = d_B \circ \phi.$$

Note that such a map sends cocycles to cocycles and coboundaries to coboundaries and hence induces a map in cohomology $H^*(\phi) : H^*(A) \to H^*(B)$.

Finally, a *commutative differential graded algebra*, or CDGA, is a differential graded algebra (A, d) which is commutative in the graded sense:

$$xy = (-1)^{pq}yx,$$

for all $x \in A^p$ and $y \in A^q$. The relevant morphisms are the homomorphisms of DGAs defined above. If A is commutative (in the graded sense), then so is $H^*(A)$.

Example 1.3.7. If M is a smooth manifold, then the de Rham complex $\Omega^*(M)$ of differential forms on M together with the exterior derivative $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is a CDGA. Its cohomology is the de Rham cohomology $H^*_{dR}(M)$ of M.

1.3.3 The exterior algebra

The most important example of a commutative graded algebra for us is the exterior algebra. It figures prominently in our later treatment of Lie algebra cohomology so we record some basic results here, all of which can be found in [14].

Recall that the *exterior algebra* of a vector space V over a field $\mathbb{F}$ is defined as the quotient

$$\Lambda V = T(V)/I,$$

where $T(V) = \oplus_{p\geq 0} T^p(V)$ is the tensor algebra of V, and I is the two-sided ideal generated by elements of the form $x \otimes x$ for all $x \in V$. The ideal I inherits a grading from $T(V)$ in the usual way, i.e., $I^p = I \cap T^p(V)$, so we have

$$\Lambda V = \bigoplus_{p\geq 0} \Lambda^p V,$$

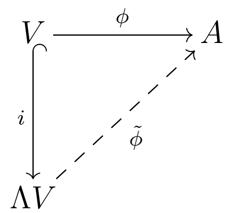
where $\Lambda^p V = T^p(V)/I^p$. Note that since $I^0 = I^1 = \{0\}$, $\Lambda^0 V$ and $\Lambda^1 V$ may be identified with $\mathbb{F}$ and V, respectively. By construction, the induced multiplication in ΛV, which we denote $\wedge$, is graded-commutative. For any basis $\{e_1, \ldots, e_n\}$ of V, the set $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq n\}$ forms a basis of $\Lambda^p V$. It follows that $\{e_1, \ldots, e_n\}$ is a system of generators for ΛV, and

$$\dim \Lambda^p V = \begin{cases} \binom{n}{p} & \text{if } 0 \leq p \leq n \\ 0 & \text{otherwise,} \end{cases}$$

whence $\dim \Lambda V = 2^n$.

The exterior algebra of a vector space V can be viewed as the free graded-commutative (associative, unital) algebra on V (where the elements of V are of degree

1) in the sense that it satisfies the following universal property: given any linear map $\phi : V \to A$ of V into an associative unital algebra A such that $\phi(x)\phi(x) = 0$ for all $x \in V$, there exists a unique homomorphism of algebras $\tilde{\phi} : \Lambda V \to A$ making the following diagram commute:



By the universal property above, for any linear map $f : V \to W$, there exists a unique homomorphism of CGAs $\Lambda(f) : \Lambda V \to \Lambda W$ which restricts to f in degree one. Explicitly, it is defined on decomposable elements by

$$x_1 \wedge \cdots \wedge x_p \mapsto f(x_1) \wedge \cdots \wedge f(x_p)$$

and is extended by linearity. Clearly, $\Lambda(\mathrm{id}_V) = \mathrm{id}_{\Lambda V}$ and $\Lambda(f \circ g) = \Lambda(f) \circ \Lambda(g)$ for any composition $U \xrightarrow{g} V \xrightarrow{f} W$, making $\Lambda(-)$ a functor from the category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over a field $\mathbb{F}$ to the category $\mathbf{CGA}_{\mathbb{F}}$ of commutative graded algebras over $\mathbb{F}$.

Linear maps on a vector space can also be extended to derivations on the exterior algebra. First note that, by the graded Leibniz rule, a derivation on ΛV is completely determined by its values on generators, that is, its values on $\Lambda^1 V = V$. Therefore, for any linear map $\phi : V \to \Lambda^{k+1} V$, there is a unique derivation $\bar{\phi}$ of degree k extending ϕ. It is defined as follows: set

$$\bar{\phi}|_{\mathbb{F}} = 0, \qquad \bar{\phi}|_V = \phi,$$

and extend to elements of higher degree inductively using the graded Leibniz rule:

$$\bar{\phi}(a \wedge b) = \bar{\phi}(a) \wedge b + (-1)^{k|a|} a \wedge \bar{\phi}(b)$$

for homogeneous elements a and b.

Throughout this thesis, we will make use of a certain kind of derivation on ΛV^* called the *interior product*. In Chapter 3, these derivations are used to define cohomology operations. Before defining the interior product, we introduce a bilinear form to put ΛV and ΛV^* in duality.

Proposition 1.3.8 ([14], page 104)**.** *Let V be a finite-dimensional vector space over a field $\mathbb{F}$ and let V^* be its dual. There exists a non-degenerate pairing between ΛV and ΛV^*.*

Proof: We define a bilinear map $\langle\,,\rangle : \Lambda V^* \times \Lambda V \to \mathbb{F}$ by

$$\langle \lambda, \mu \rangle = \lambda\mu \text{ for } \lambda, \mu \in \mathbb{F},$$

$$\langle f_1 \wedge \cdots \wedge f_p, v_1 \wedge \cdots \wedge v_p \rangle = \det(f_i(v_j)) \text{ for } f_1, \ldots, f_p \in V^*, v_1, \ldots, v_p \in V,$$

$$\langle \Lambda^p V^*, \Lambda^q V \rangle = 0 \text{ for } p \neq q,$$

and extend by linearity. To see that the pairing is non-degenerate we observe that for any nonzero monomial $v_1 \wedge \cdots \wedge v_p \in \Lambda V$, we can regard $\{v_1, \ldots, v_p\}$ as a basis of a subspace of V and define a dual basis $\{v_1^*, \ldots, v_p^*\}$ by $v_i^*(v_j) = \delta_{ij}$. Then

$$\langle v_1^* \wedge \cdots \wedge v_p^*, v_1 \wedge \cdots \wedge v_p \rangle = \det I_p = 1,$$

and by linearity, this shows that $\langle\,,\rangle$ is non-degenerate. ∎

Note that the pairing above establishes a natural isomorphism between ΛV^* and $(\Lambda V)^*$ which we use throughout this thesis, and which allows us to identify elements of ΛV^* with skew-symmetric multilinear forms on V. Henceforth, $\langle\,,\rangle$ will always refer to this pairing.

Definition 1.3.9. *For any $x \in V$, the* interior product with x *is defined to be the unique derivation i_x of degree -1 on ΛV^* which acts on generators as*

$$i_x f = f(x),$$

for all $f \in V^*$*.*

In other words, under the dual pairing above, i_x is dual to the operator μ_x in ΛV given by multiplication on the left by x: for all $\alpha \in \Lambda^{p+1}V^*$ and $\beta \in \Lambda^p V$,

$$\langle i_x\alpha, \beta\rangle = \langle \alpha, x \wedge \beta\rangle = \langle \alpha, \mu_x\beta\rangle.$$

Note that

$$\langle (i_x \circ i_y)\alpha, \beta\rangle = \langle \alpha, y \wedge x \wedge \beta\rangle = \langle \alpha, -x \wedge y \wedge \beta\rangle = -\langle (i_y \circ i_x)\alpha, \beta\rangle,$$

so $i_x \circ i_y = -i_y \circ i_x$, and in particular, $i_x^2 = 0$. Similarly, there are derivations i_f acting on ΛV which satisfy $i_f x = f(x)$ for each $f \in V^*$ and $x \in V$.

Lastly, we remark that, although the composition $d \circ \delta$ of two derivations is in general not again a derivation, the graded commutator $d \circ \delta - (-1)^{|d||\delta|}\delta \circ d$ is a derivation of degree $|d| + |\delta|$. For any vector space V, this gives the vector space $\mathrm{Der}(\Lambda V)$ of derivations on ΛV the structure of a *graded Lie algebra.*

Definition 1.3.10. *A* graded Lie algebra *is a graded vector space* $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ *equipped with a bilinear map* $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ *which satisfies the following:*

- $[\mathfrak{g}_p, \mathfrak{g}_q] \subseteq \mathfrak{g}_{p+q}$*;*
- $[x, y] = -(-1)^{pq}[y, x]$ *for all* $x \in \mathfrak{g}_p$ *and* $y \in \mathfrak{g}_q$ *(graded skew-symmetry);*
- $[x, [y, z]] = [[x, y], z] + (-1)^{pq}[y, [x, z]] = 0$ *for all* $x \in \mathfrak{g}_p$*,* $y \in \mathfrak{g}_q$*,* $z \in \mathfrak{g}_r$ *(graded Jacobi identity).*

More generally, if A is any graded algebra, then the set of all derivations on A forms a graded Lie algebra

$$\mathrm{Der}(A) = \bigoplus_{k\in\mathbb{Z}} \mathrm{Der}_k(A),$$

where $\mathrm{Der}_k(A)$ denotes the space of all derivations of degree k on A.

Chapter 2

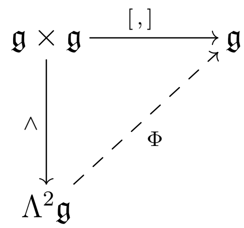
Lie Algebra Cohomology

In this chapter, we develop basic concepts and results surrounding the cohomology of Lie algebras, such as the Koszul complex and the Hodge decomposition theorem which establishes an isomorphism between the cohomology of a Lie algebra $\mathfrak{g}$ and the space of harmonic forms on $\mathfrak{g}$. As an application of the Hodge decomposition theorem, we prove in Section 2.3 that unimodular Lie algebras enjoy Poincaré duality. In Section 2.4, we describe Dixmier's long exact sequence of cohomology groups, a computational tool relating the cohomology of a Lie algebra to that of any codimension one ideal, and use it to give a new proof of the TRC for free two-step nilpotent Lie algebras.

2.1 Definition and Examples

To any Lie algebra $\mathfrak{g}$ there is an associated CDGA, called the *Koszul complex*, whose differential encodes the Lie bracket of $\mathfrak{g}$. The cohomology of a Lie algebra is then defined as the cohomology algebra of the associated Koszul complex.

Let $\mathfrak{g}$ be a Lie algebra. Since the Lie bracket $[\,,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is skew-symmetric and bilinear, it factors through the map $\wedge : \mathfrak{g} \times \mathfrak{g} \to \Lambda^2\mathfrak{g}$, $(x, y) \mapsto x \wedge y$, yielding a unique linear map $\Phi : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ making



commute. Extending the dual of Φ to a derivation $\Lambda\mathfrak{g}^*$ as described in the previous section, we obtain a derivation of degree 1, denoted $d : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$, satisfying

$$\langle df, x \wedge y\rangle = \langle f, [x, y]\rangle,$$

for all $f \in \mathfrak{g}^*$ and $x, y \in \mathfrak{g}$.

Such a derivation can be constructed for any algebra with a skew-symmetric product, but as our next proposition shows, the fact that we started with a Lie algebra ensures that the map d constructed above is, in fact, a *differential* on the CGA $\Lambda\mathfrak{g}^*$, the key being the Jacobi identity.

Proposition 2.1.1. *Let $\mathfrak{g}$ be a Lie algebra and let $d : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ be the derivation extending the dual of the Lie bracket, as described above. Then $d^2 = 0$.*

Proof: Since d is a derivation of degree 1, we know $[d, d] = 2d^2 \in \mathrm{Der}_2(\Lambda\mathfrak{g}^*)$, so d^2 is also a derivation. It therefore suffices to show d^2 vanishes on $\mathfrak{g}^*$.

Let $\Phi : \Lambda^2\mathfrak{g} \to \mathfrak{g}$ be as defined above (i.e., the unique linear map satisfying $\Phi(x \wedge y) = [x, y]$), and define $\Psi : \Lambda^3\mathfrak{g} \to \Lambda^2\mathfrak{g}$ by

$$\Psi(x \wedge y \wedge z) = [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y.$$

Now the Jacobi identity is equivalent to $\Phi \circ \Psi = 0$, which we now show is equivalent to $d^2 = 0$.

For ease of notation, let $d_i = d|_{\Lambda^i\mathfrak{g}^*} : \Lambda^i\mathfrak{g}^* \to \Lambda^{i+1}\mathfrak{g}^*$. Note that, by definition, $\Phi^* = d_1$, so $d_2(u \wedge v) = \Phi^*(u) \wedge v - u \wedge \Phi^*(v)$. Let $u, v \in \mathfrak{g}^*$ and $x, y, z \in \mathfrak{g}$. Then

$$\langle \Psi^*(u \wedge v), x \wedge y \wedge z\rangle = \langle u \wedge v, [x, y] \wedge z + [y, z] \wedge x + [z, x] \wedge y\rangle, \qquad (2.1.1)$$

where

$$\begin{aligned}\langle u\wedge v,[x,y]\wedge z\rangle &= -\langle i_z(u\wedge v),[x,y]\rangle = -\langle i_z(u\wedge v),\Phi(x\wedge y)\rangle\\ &= -\langle u,z\rangle\langle v,\Phi(x\wedge y)\rangle + \langle v,z\rangle\langle u,\Phi(x\wedge y)\rangle,\end{aligned}$$

and similarly,

$$\begin{aligned}\langle u\wedge v,[y,z]\wedge x\rangle &= -\langle u,x\rangle\langle v,\Phi(y\wedge z)\rangle + \langle v,x\rangle\langle u,\Phi(y\wedge z)\rangle,\\ \langle u\wedge v,[z,x]\wedge y\rangle &= -\langle u,y\rangle\langle v,\Phi(z\wedge x)\rangle + \langle v,y\rangle\langle u,\Phi(z\wedge x)\rangle.\end{aligned}$$

Substituting these into (2.1.1), we find that

$$\begin{aligned}\langle\Psi^*(u\wedge v),x\wedge y\wedge z\rangle = &-\langle u,z\rangle\langle v,\Phi(x\wedge y)\rangle + \langle v,z\rangle\langle u,\Phi(x\wedge y)\rangle\\ &-\langle u,x\rangle\langle v,\Phi(y\wedge z)\rangle + \langle v,x\rangle\langle u,\Phi(y\wedge z)\rangle\\ &-\langle u,y\rangle\langle v,\Phi(z\wedge x)\rangle + \langle v,y\rangle\langle u,\Phi(z\wedge x)\rangle,\end{aligned}$$

while

$$\begin{aligned}\langle\Phi^*(u)\wedge v,x\wedge y\wedge z\rangle &= \langle\Phi^*(u),i_v(x\wedge y\wedge z)\rangle\\ &= \langle v,x\rangle\langle\Phi^*(u),y\wedge z\rangle - \langle v,y\rangle\langle\Phi^*(u),x\wedge z\rangle\\ &\quad+\langle v,z\rangle\langle\Phi^*(u),x\wedge y\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle u\wedge\Phi^*(v),x\wedge y\wedge z\rangle &= \langle u,x\rangle\langle\Phi^*(v),y\wedge z\rangle - \langle u,y\rangle\langle\Phi^*(v),x\wedge z\rangle\\ &\quad+\langle u,z\rangle\langle\Phi^*(v),x\wedge y\rangle.\end{aligned}$$

The above two equations show that $\Psi^*(u\wedge v) = \Phi^*(u)\wedge v - u\wedge\Phi^*(v)$. But d_2 is the unique linear map $\Lambda^2\mathfrak{g}^*\to\Lambda^3\mathfrak{g}^*$ satisfying $d_2(u\wedge v) = \Phi^*(u)\wedge v - u\wedge\Phi^*(v)$, so Ψ is the dual of d_2. Finally, $\Phi\circ\Psi = 0$ if and only if $(\Phi\circ\Psi)^* = d_2\circ\Phi^* = d^2|_{\mathfrak{g}^*} = 0$. ∎

Equipping $\Lambda\mathfrak{g}^*$ with the differential d, we obtain a commutative differential graded algebra

$$(\Lambda\mathfrak{g}^*, d) : 0 \longrightarrow \mathbb{F} \longrightarrow \mathfrak{g}^* \longrightarrow \Lambda^2\mathfrak{g}^* \longrightarrow \cdots \longrightarrow \Lambda^{n-1}\mathfrak{g}^* \longrightarrow \Lambda^n\mathfrak{g}^* \longrightarrow 0$$

called the *Koszul complex* of $\mathfrak{g}$. Elements of $\Lambda\mathfrak{g}^*$ are often called forms on $\mathfrak{g}$ and, as in other cohomology theories, elements in the kernel and image of d are called closed and exact forms, respectively. We denote the graded subalgebras of closed and exact forms by $Z^*(\mathfrak{g})$ and $B^*(\mathfrak{g})$. When $\mathfrak{g}^*$ has a basis $\{x_1, \ldots, x_n\}$, we will often write $\Lambda(x_1, \ldots, x_n)$ for $\Lambda\mathfrak{g}^*$.

Definition 2.1.2. *If* $\mathfrak{g}$ *is a Lie algebra, then the* cohomology of $\mathfrak{g}$ *(with trivial coefficients), denoted* $H^*(\mathfrak{g})$*, is defined as the cohomology algebra of the Koszul complex* $(\Lambda\mathfrak{g}^*, d)$*. For each* p*,* $H^p(\mathfrak{g})$ *is called the* p^{th} cohomology group of $\mathfrak{g}$*, and* $b_p(\mathfrak{g}) := \dim H^p(\mathfrak{g})$ *the* p^{th} Betti number *of* $\mathfrak{g}$*.*

For any closed form $\omega \in \Lambda\mathfrak{g}^*$, we write $[\omega]$ for the cohomology class $\omega + B^*(\mathfrak{g})$ in $H^p(\mathfrak{g})$.

Remark 2.1.3. Alternatively, one can define cohomology with coefficients in an arbitrary $\mathfrak{g}$-module M, denoted $H^*(\mathfrak{g}; M)$, by considering a cochain complex

$$C^*(\mathfrak{g}; M) = \bigoplus_{p \geq 0} \operatorname{Hom}(\Lambda^p\mathfrak{g}, M)$$

made up of skew-symmetric multilinear maps $\mathfrak{g} \times \cdots \times \mathfrak{g} \to M$ and a suitably modified differential. See [4] or [30, Chapter 7] for details.

Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$, and note that $\Lambda^p\mathfrak{g}^* = 0$ for $p < 0$. Then $H^0(\mathfrak{g}) = \mathbb{F}$ since $d(\mathbb{F}) = 0$. For the same reason we have $B^1(\mathfrak{g}) = 0$, meaning the first cohomology group $H^1(\mathfrak{g}) = Z^1(\mathfrak{g})/B^1(\mathfrak{g})$ is precisely the space of 1-cocycles $Z^1(\mathfrak{g})$. Since the kernel of $d_1 : \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*$ equals $(\operatorname{im}[\,,\,])^\perp$, we have proved the following:

Proposition 2.1.4. *If $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field $\mathbb{F}$, then*

$$H^0(\mathfrak{g}) = \mathbb{F} \qquad \text{and} \qquad H^1(\mathfrak{g}) \cong \mathfrak{g}/[\mathfrak{g},\mathfrak{g}].$$

We now turn to some examples to show how cohomology can be computed explicitly given a set of generators and relations.

Example 2.1.5. If $\mathfrak{a}$ is an abelian Lie algebra of any dimension, then the corresponding differential is identically zero. Hence $Z^*(\mathfrak{a}) = \Lambda\mathfrak{a}^*$ while $B^*(\mathfrak{a})$ is trivial, so $H^*(\mathfrak{a}) = \Lambda\mathfrak{a}^*$.

Example 2.1.6. Consider the semisimple[1] real Lie algebra $\mathfrak{su}(2)$. This is the Lie algebra of the special unitary group $SU(2)$. It has a basis $\{x, y, z\}$ with cyclic bracket relations

$$[x, y] = z, \qquad [y, z] = x, \qquad [z, x] = y.$$

So, under the dual basis $\{x^*, y^*, z^*\}$, we find that d takes the following values on 1-forms:

$$dx^* = y^* \wedge z^*, \qquad dy^* = z^* \wedge x^*, \qquad dz^* = x^* \wedge y^*.$$

Since these three vectors form a basis of $\Lambda^2\mathfrak{su}(2)^*$, the map $d_1 : \Lambda^1\mathfrak{su}(2)^* \to \Lambda^2\mathfrak{su}(2)^*$ is surjective, and since $d^2 = 0$, the map $d_2 : \Lambda^2\mathfrak{su}(2)^* \to \Lambda^3\mathfrak{su}(2)^*$ is trivial. It follows that

$$\begin{aligned}
H^0(\mathfrak{su}(2)) &= \mathbb{R} \\
H^1(\mathfrak{su}(2)) &= 0 \\
H^2(\mathfrak{su}(2)) &= 0 \\
H^3(\mathfrak{su}(2)) &= \operatorname{span}\{x^* \wedge y^* \wedge z^*\} \cong \mathbb{R},
\end{aligned}$$

since, evidently, there are no nontrivial closed 1-forms, all 2-forms are exact and $x^* \wedge y^* \wedge z^*$ is closed.

[1] A Lie algebra is *semisimple* if it is a direct sum of simple Lie algebras.

Remark 2.1.7. It can be shown that the first two cohomology groups $H^1(\mathfrak{g})$ and $H^2(\mathfrak{g})$ are trivial for *all* semisimple Lie algebras over a field of characteristic zero (indeed, the triviality of $H^1(\mathfrak{g})$ follows from Proposition 2.1.4). This fact is a special case of two results known as the Whitehead lemmas. In fact, if $\mathfrak{g}$ is finite-dimensional over a field of characteristic zero, then $\mathfrak{g}$ is semisimple *if and only if* $H^1(\mathfrak{g}; M) = 0$ for all finite-dimensional $\mathfrak{g}$-modules M [30, Chapter 7].

Example 2.1.8. Consider the solvable real Lie algebra with basis $\{h, x, y\}$ and non-trivial brackets given by $[h, x] = y$ and $[h, y] = -x$. This algebra is called $\mathfrak{se}(2)$ and is the Lie algebra of the special Euclidean group $SE(2)$ of orientation-preserving isometries of the plane.

Fixing a dual basis $\{h^*, x^*, y^*\}$, the differential acts on 1-forms as follows:

$$dh^* = 0, \qquad dx^* = -h^* \wedge y^*, \qquad dy^* = h^* \wedge x^*. \tag{2.1.2}$$

Since no 1-forms are exact, we have $H^1(\mathfrak{se}(2)) = \text{span}\{h^*\}$ (as predicted by the proposition above). On 2-forms,

$$d(x^* \wedge y^*) = d(h^* \wedge x^*) = d(h^* \wedge y^*) = 0,$$

but $h^* \wedge x^*$ and $h^* \wedge y^*$ are exact by (2.1.2). Thus the image of $d_2 : \Lambda^2\mathfrak{se}(2)^* \to \Lambda^3\mathfrak{se}(2)^*$ is trivial, and since $h^* \wedge x^* \wedge y^*$ is closed, we have

$$\begin{aligned}
H^0(\mathfrak{se}(2)) &= \mathbb{R} \\
H^1(\mathfrak{se}(2)) &= \text{span}\{h^*\} \\
H^2(\mathfrak{se}(2)) &= \text{span}\{x^* \wedge y^*, h^* \wedge x^*, h^* \wedge y^*\}/\,\text{span}\{h^* \wedge x^*, h^* \wedge y^*\} \\
&\cong \text{span}\{x^* \wedge y^*\} \\
H^3(\mathfrak{se}(2)) &= \text{span}\{h^* \wedge x^* \wedge y^*\}.
\end{aligned}$$

2.2 Ideals and Koszul Fibrations

In this section, we show that any ideal of a lie algebra $\mathfrak{g}$ gives rise to a sub-CDGA of the Koszul complex $(\Lambda\mathfrak{g}^*, d)$ and describe the associated Koszul fibration, a tool we will use often, either implicitly or explicitly, in later chapters. Throughout this section, $\mathfrak{g}$ denotes a finite-dimensional Lie algebra over a field $\mathbb{F}$ of characteristic zero.

Let $I \subseteq \mathfrak{g}$ be an ideal. We denote by $I^\perp$ the *annihilator* of I in $\mathfrak{g}^*$:

$$I^\perp = \{f \in \mathfrak{g}^* \mid f(x) = 0 \quad \forall x \in I\}.$$

We want to show that $(\Lambda I^\perp, d)$ is a sub-CDGA of the Koszul complex of $\mathfrak{g}$, so since $\Lambda I^\perp$ is clearly a subalgebra of $\Lambda\mathfrak{g}^*$, we need to show that $\Lambda I^\perp$ is invariant under the differential d. We first prove a lemma.

Lemma 2.2.1. *Let I be an ideal of $\mathfrak{g}$. If $\omega \in \Lambda^2\mathfrak{g}^*$, then $\omega \in \Lambda^2 I^\perp$ if and only if $\omega(u \wedge v) = 0$ for every $u \in I$ and $v \in \mathfrak{g}$.*

Proof: Suppose $\omega(u \wedge v) = 0$ for all $u \in I, v \in \mathfrak{g}$. Choose a basis $\{x_1, \dots, x_m\}$ of I and extend it to a basis $\{x_1, \dots, x_m, x_{m+1}, \dots, x_n\}$ of $\mathfrak{g}$. Let $\{x_1^*, \dots, x_n^*\}$ denote the dual basis of $\mathfrak{g}^*$ where $x_i^*(x_j) = \delta_{ij}$. We write

$$\omega = \sum_{i<j} c_{ij} x_i^* \wedge x_j^*$$

for some constants $c_{ij} \in \mathbb{F}$ satisfying $1 \le i < j \le n$; so if $x_a \in I$ and $x_b \in \mathfrak{g}$, then without loss of generality $a < b$ and

$$0 = \langle \omega, x_a \wedge x_b \rangle = \sum_{i<j} c_{ij} \langle x_i^* \wedge x_j^*, x_a \wedge x_b \rangle = c_{ab}.$$

This shows $c_{ij} = 0$ for all $i \in \{1, \dots, m\}$ in the decomposition of ω above, and so $\omega \in \Lambda^2 I^\perp$.

Conversely, suppose $\omega \in \Lambda^2 I^\perp$. Then

$$\omega = \sum_{m<i<j} c_{ij} x_i^* \wedge x_j^*$$

for some $c_{ij} \in \mathbb{F}$ since $\{x^*_{m+1}, \dots, x^*_n\}$ forms a basis of $I^\perp$. But then for any $u \in I$ and $v \in \mathfrak{g}$,

$$\begin{aligned}\langle \omega, u \wedge v\rangle &= \sum_{m<i<j} c_{ij} \langle x^*_i \wedge x^*_j, u \wedge v\rangle \\ &= \sum_{m<i<j} c_{ij} \begin{vmatrix} \langle x^*_i, u\rangle & \langle x^*_i, v\rangle \\ \langle x^*_j, u\rangle & \langle x^*_j, v\rangle \end{vmatrix} \\ &= \sum_{m<i<j} c_{ij} \begin{vmatrix} 0 & \langle x^*_i, v\rangle \\ 0 & \langle x^*_j, v\rangle \end{vmatrix} \\ &= 0,\end{aligned}$$

so by linearity ω is zero on $\{u \wedge v \,|\, u \in I, v \in \mathfrak{g}\}$. ∎

We are now ready to show that, for $I \subseteq \mathfrak{g}$ an ideal, $\Lambda I^\perp$ is invariant under d.

Proposition 2.2.2. *Let I be a subspace of $\mathfrak{g}$. Then I is an ideal of $\mathfrak{g}$ if and only if $d : I^\perp \to \Lambda^2 I^\perp$.*

Proof: Assuming I is an ideal, let $f \in I^\perp$. Then $\langle df, u \wedge v\rangle = \langle f, [u, v]\rangle = 0$ whenever $u \in I$ or $v \in I$. So by the lemma above, $df \in \Lambda^2 I^\perp$.

Conversely, suppose $d : I^\perp \to \Lambda^2 I^\perp$ and let $u \in I$ and $v \in \mathfrak{g}$. By Lemma 2.2.1, $\langle f, [u, v]\rangle = \langle df, u \wedge v\rangle = 0$ for all $f \in I^\perp$. But, choosing a complement $\overline{C}$ of I in $\mathfrak{g}$, we can write $[u, v] = i + \overline{c}$ where $i \in I$ and $\overline{c} \in \overline{C}$. Choose a basis $\{y_1, \dots, y_m\}$ of $\overline{C}$, extend it to a basis of $\mathfrak{g}$ and form the usual dual basis so that $\{y^*_1, \dots, y^*_m\}$ spans $I^\perp$. So

$$\overline{c} = \sum_{k=1}^{m} c_k y_k$$

for some $c_k \in \mathbb{F}$. But for any $c_k \neq 0$ we have $\langle y^*_k, [u, v]\rangle = y^*_k(i + \overline{c}) = y^*_k(\overline{c}) = c_k \neq 0$, a contradiction since $y^*_k \in I^\perp$. Therefore, $c_k = 0$ for all k, i.e., $\overline{c} = 0$ and $[u, v] \in I$, which shows that I is an ideal. ∎

Consequently, if I is an ideal of $\mathfrak{g}$, then $(\Lambda I^\perp, d|_{\Lambda I^\perp})$ is a sub-CDGA of the Koszul complex $(\Lambda\mathfrak{g}^*, d)$. Henceforth, let I be an ideal of $\mathfrak{g}$.

Next we describe the *Koszul fibration*, a sequence of CDGAs corresponding to a choice of complement C of $I^\perp$ in $\mathfrak{g}^*$. If we define K to be the functor $\mathbf{Lie}_\mathbb{F} \to \mathbf{CDGA}_\mathbb{F}$ taking a Lie algebra over $\mathbb{F}$ to its Koszul complex, then up to isomorphism the Koszul fibration is the image of the short exact sequence

$$I \hookrightarrow \mathfrak{g} \to \mathfrak{g}/I$$

under the functor K.

Definition 2.2.3. *Let C be a complement of $I^\perp$ in $\mathfrak{g}^*$ so that $(\Lambda I^\perp \otimes \Lambda C, d_\mathfrak{g})$ is the Koszul complex of $\mathfrak{g}$. The associated* Koszul fibration *is the sequence of commutative differential graded algebras:*

$$(\Lambda I^\perp, d) \xrightarrow{j} (\Lambda I^\perp \otimes \Lambda C, d_\mathfrak{g}) \xrightarrow{\rho} (\Lambda C, \overline{d}),$$

where $j(\alpha) = \alpha \otimes 1$, $\rho(1 \otimes \gamma) = \gamma$, and $\rho(\Lambda^+ I^\perp \otimes \Lambda C) = 0$. The differentials on the base and fibre[2] *are defined by $d = d_\mathfrak{g}|_{\Lambda I^\perp}$ and $\overline{d}\gamma = \rho(d_\mathfrak{g}(1 \otimes \gamma))$, respectively.*

The notation $\Lambda^+ I^\perp$ means $\oplus_{p\geq 1}\Lambda^p I^\perp$. Note that the total space $(\Lambda I^\perp \otimes \Lambda C, d_\mathfrak{g})$ of the fibration has been identified with the Koszul complex $(\Lambda\mathfrak{g}^*, d_\mathfrak{g})$ of $\mathfrak{g}$ via the natural Künneth isomorphism

$$\Lambda I^\perp \otimes \Lambda C \cong \Lambda(I^\perp \oplus C)$$

defined by $\alpha \otimes \gamma \mapsto \alpha \wedge \gamma$. Here, and elsewhere in this thesis, the tensor product $\Lambda I^\perp \otimes \Lambda C$ means the *skew tensor product* where multiplication is defined by

$$(\alpha \otimes \gamma)(\alpha' \otimes \gamma') = (-1)^{|\alpha'||\gamma|}\alpha \wedge \alpha' \otimes \gamma \wedge \gamma'.$$

[2] The apparent reversal of roles of the base and fibre in a fibration here is due to the fact that $(\Lambda I^\perp, d)$ is the cochain complex of the base of the topological fibration of Lie groups corresponding to the short exact sequence of Lie algebras $I \hookrightarrow \mathfrak{g} \to \mathfrak{g}/I$, the functor K being contravariant.

Next we show that the fibre $(\Lambda C, \overline{d})$ and base $(\Lambda I^{\perp}, d)$ of the Koszul fibration are the Koszul complexes of the Lie algebras I and $\mathfrak{g}/I$, respectively. As noted above, we know that the latter is a CDGA because its differential is just the restriction of the differential in $(\Lambda \mathfrak{g}^*, d_{\mathfrak{g}})$. As for the former, we must verify that $\overline{d}^2 = 0$.

First we observe that ρ is a morphism of CDGAs: on $\Lambda^+ I^{\perp} \otimes \Lambda C$, we have

$$(\overline{d} \circ \rho)(\Lambda^+ I^{\perp} \otimes \Lambda C) = 0 = (\rho \circ d_{\mathfrak{g}})(\Lambda^+ I^{\perp} \otimes \Lambda C),$$

and on $\Lambda^0 I^{\perp} \otimes \Lambda C$ we have

$$\overline{d}(\rho(1 \otimes \gamma)) = \overline{d}\gamma = \rho(d_{\mathfrak{g}}(1 \otimes \gamma)).$$

So $\overline{d} \circ \rho = \rho \circ d_{\mathfrak{g}}$. Now for all $\gamma \in \Lambda C$,

$$\overline{d}^2 \gamma = \overline{d}(\rho(d_{\mathfrak{g}}(1 \otimes \gamma))) = (\rho \circ d_{\mathfrak{g}} \circ d_{\mathfrak{g}})(1 \otimes \gamma) = 0$$

because $d_{\mathfrak{g}}^2 = 0$. Therefore, $(\Lambda C, \overline{d})$ is a CDGA.

Theorem 2.2.4. *The base $(\Lambda I^{\perp}, d)$ of the Koszul fibration is isomorphic to the Koszul complex of $\mathfrak{g}/I$.*

Proof: Let $(\Lambda(\mathfrak{g}/I)^*, D)$ be the Koszul complex of $\mathfrak{g}/I$. Define $\phi : I^{\perp} \to (\mathfrak{g}/I)^*$ by $\phi(f)(x+I) = f(x)$, for all $f \in I^{\perp}$. To see that $\phi(f)$ is well-defined, note that for equal cosets $x+I = y+I$, we have $\phi(f)(x+I) - \phi(f)(y+I) = f(x) - f(y) = f(x-y) = 0$ since $x - y \in I$. If $\phi(f) = 0$, then clearly $f = 0$, so ϕ is injective. For dimension reasons then, ϕ is an isomorphism and so extends to an isomorphism of graded algebras $\Lambda I^{\perp} \xrightarrow{\sim} \Lambda(\mathfrak{g}/I)^*$ (which we will also call ϕ). Lastly, an isomorphism of CDGAs must commute with differentials, so we need to show $\phi \circ d = D \circ \phi$. To this end, suppose $f \in I^{\perp}$ and let $x + I,\ y + I \in \mathfrak{g}/I$. Since $d : I^{\perp} \to \Lambda^2 I^{\perp}$, we can write

$df = \sum_i f_{i_1} \wedge f_{i_2} \in \Lambda^2 I^\perp$. Therefore,

$$\begin{aligned}
\langle(\phi \circ d)(f), (x+I) \wedge (y+I)\rangle &= \sum_i \langle \phi(f_{i_1}) \wedge \phi(f_{i_2}), (x+I) \wedge (y+I)\rangle \\
&= \sum_i \begin{vmatrix} \phi(f_{i_1})(x+I) & \phi(f_{i_1})(y+I) \\ \phi(f_{i_2})(x+I) & \phi(f_{i_2})(y+I) \end{vmatrix} \\
&= \sum_i \begin{vmatrix} f_{i_1}(x) & f_{i_1}(y) \\ f_{i_2}(x) & f_{i_2}(y) \end{vmatrix} \\
&= \sum_i \langle f_{i_1} \wedge f_{i_2}, x \wedge y\rangle \\
&= \langle df, x \wedge y\rangle \\
&= \langle f, [x,y]\rangle \\
&= \langle \phi(f), [x,y] + I\rangle \\
&= \langle \phi(f), [x+I, y+I]\rangle \\
&= \langle (D \circ \phi)(f), (x+I) \wedge (y+I)\rangle.
\end{aligned}$$

Since ϕ, d and D are uniquely determined by their values on generators, it follows that the diagram

$$\begin{array}{ccc}
\Lambda^{p+1} I^\perp & \xrightarrow{\phi} & \Lambda^{p+1}(\mathfrak{g}/I)^* \\
\uparrow{\scriptstyle d} & & \uparrow{\scriptstyle D} \\
\Lambda^p I^\perp & \xrightarrow[\phi]{} & \Lambda^p(\mathfrak{g}/I)^*
\end{array}$$

commutes for all p. Thus $(\Lambda I^\perp, d) \cong (\Lambda(\mathfrak{g}/I)^*, D)$. ∎

Theorem 2.2.5. *The fibre $(\Lambda C, \overline{d})$ of the Koszul fibration is isomorphic to the Koszul complex of I.*

Proof: Let $(\Lambda I^*, \delta)$ be the Koszul complex of I. Define a linear map $\psi : C \to I^*$ by $f \mapsto f|_I$. If $\psi(f) = 0$, then $f(x) = 0$ for all $x \in I$, i.e., $f \in I^\perp$. But then

$f = 0$ because $C \cap I^{\perp} = \{0\}$, making ψ injective. Since $\mathfrak{g}^* = I^{\perp} \oplus C$, we have $\dim C = \dim I^*$, so ψ is an isomorphism. This induces an isomorphism $\Lambda C \xrightarrow{\sim} \Lambda I^*$ (which we will also call ψ). It remains to show $\psi \circ \overline{d} = \delta \circ \psi$, so let $c \in C$ and let $x, y \in I$. If $d_{\mathfrak{g}}(1 \otimes c) \in \Lambda^+ I^{\perp} \otimes \Lambda C$, then

$$\langle (\psi \circ \overline{d})(c), x \wedge y \rangle = \langle \psi(\rho(d_{\mathfrak{g}}(1 \otimes c))), x \wedge y \rangle = 0$$

by definition of ρ, and

$$\begin{aligned} \langle (\delta \circ \psi)(c), x \wedge y \rangle &= \langle c|_I, [x, y] \rangle \\ &= \langle 1 \otimes c, [x, y] \rangle \qquad (\text{in } (\Lambda I^{\perp} \otimes \Lambda C, d_{\mathfrak{g}})) \\ &= \langle d_{\mathfrak{g}}(1 \otimes c), x \wedge y \rangle \\ &= 0 \end{aligned}$$

since $x, y \in I$. Supposing now that $d_{\mathfrak{g}}(1 \otimes c)$ is an arbitrary element of $\Lambda I^{\perp} \otimes \Lambda C$, we decompose $d_{\mathfrak{g}}(1 \otimes c)$ as $d_{\mathfrak{g}}(1 \otimes c) = u + \sum_i a_i \otimes (c_{i_1} \wedge c_{i_2})$, where $u \in \Lambda^+ I^{\perp} \otimes \Lambda C$ and $\sum_i a_i \otimes (c_{i_1} \wedge c_{i_2}) \in \Lambda^0 I^{\perp} \otimes \Lambda^2 C$. Then

$$\begin{aligned} \langle (\delta \circ \psi)(c), x \wedge y \rangle &= \langle d_{\mathfrak{g}}(1 \otimes c), x \wedge y \rangle \\ &= \langle u, x \wedge y \rangle + \langle \sum_i a_i \otimes (c_{i_1} \wedge c_{i_2}), x \wedge y \rangle \\ &= \sum_i a_i \langle c_{i_1} \wedge c_{i_2}, x \wedge y \rangle. \end{aligned} \tag{2.2.1}$$

On the other hand,

$$\begin{aligned}
\langle(\psi\circ\overline{d})(c), x\wedge y\rangle &= \langle\psi(\rho(d_{\mathfrak{g}}(1\otimes c))), x\wedge y\rangle \\
&= \langle\psi(\rho(u+\sum_i a_i\otimes(c_{i_1}\wedge c_{i_2}))), x\wedge y\rangle \\
&= \langle\psi(\rho(\sum_i a_i\otimes(c_{i_1}\wedge c_{i_2}))), x\wedge y\rangle \\
&= \langle\psi(\sum_i a_i(c_{i_1}\wedge c_{i_2})), x\wedge y\rangle \\
&= \sum_i a_i \begin{vmatrix} \psi(c_{i_1})(x) & \psi(c_{i_1})(y) \\ \psi(c_{i_2})(x) & \psi(c_{i_2})(y) \end{vmatrix} \\
&= \sum_i a_i \begin{vmatrix} c_{i_1}(x) & c_{i_1}(y) \\ c_{i_2}(x) & c_{i_2}(y) \end{vmatrix},
\end{aligned}$$

which equals (2.2.1). Since ψ, $\overline{d}$ and δ are uniquely determined by their values on generators, it follows that

$$\begin{array}{ccc}
\Lambda^{p+1}C & \xrightarrow{\psi} & \Lambda^{p+1}I^* \\
\uparrow{\scriptstyle \overline{d}} & & \uparrow{\scriptstyle \delta} \\
\Lambda^p C & \xrightarrow[\psi]{} & \Lambda^p I^*
\end{array}$$

commutes for all p. Thus $(\Lambda C, \overline{d}) \cong (\Lambda I^*, \delta)$. ∎

We finish this section with a remark on the Koszul complexes of nilpotent Lie algebras. Recall from Theorem 1.3.5 that if $\mathfrak{g}$ is nilpotent, then there exists a basis $\{e_1,\ldots,e_n\}$ of $\mathfrak{g}$ such that $[e_i,e_j]\in\operatorname{span}\{e_{j+1},\ldots,e_n\}$ for all $i<j$. Dualizing this, we find that the Koszul complex of a nilpotent Lie algebra can always be written in terms of a basis as $(\Lambda(x_1,\ldots,x_n),d)$ where $dx_i\in\Lambda^2(x_1,\ldots,x_{i-1})$ for $1\le i\le n$. In particular, $dx_1=dx_2=0$. Note that this implies $b_1(\mathfrak{g})\ge 2$. We will see in Section 2.4 that this bound holds for all Betti numbers $b_p(\mathfrak{g})$ where $0<p<n$.

2.3 Twisted Cohomology and Poincaré Duality

The main purpose of this section is to prove the *Hodge decomposition theorem*, which provides an orthogonal decomposition of the Koszul complex into the kernel and image of an important operator Δ, called the *Laplacian*, and establishes an isomorphism $\ker \Delta \cong H^*(\mathfrak{g})$. We first recall the definition of another operator called the *Hodge star* which, together with the Hodge decomposition, will allow us to prove a Poincaré duality theorem for Lie algebra cohomology. The proofs of results in this section are taken largely from [2]. Henceforth, we will take $\mathbb{F}$ to be a *formally real* field. [3] In particular, we are interested in the cases $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{Q}$. As in the last section, $\mathfrak{g}$ denotes a finite-dimensional Lie algebra over $\mathbb{F}$.

2.3.1 The Hodge star

Fix an ordered basis $\mathcal{B} = \{e_1, \ldots, e_n\}$ of $\mathfrak{g}^*$ and recall how this extends to a basis of $\Lambda\mathfrak{g}^*$: for each $p \in \{0, \ldots, n\}$, the set $\{e_{i_1} \wedge \cdots \wedge e_{i_p} | 1 \leq i_1 < \cdots < i_p \leq n\}$ forms a basis of $\Lambda^p\mathfrak{g}^*$. In particular, $\mathcal{B}$ selects the distinguished basis element $e_1 \wedge \cdots \wedge e_n$ of $\Lambda^n\mathfrak{g}^*$, which we call the *top form* and denote by τ.

Definition 2.3.1. *The* Hodge star operator $* : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ *is the unique linear map defined on basis elements as follows:*

$$*(e_{i_1} \wedge \cdots \wedge e_{i_p}) = (-1)^{\mathrm{sgn}(\sigma)} e_{j_1} \wedge \cdots \wedge e_{j_q},$$

where $e_{j_1} \wedge \cdots \wedge e_{j_q}$ is the ordered q-fold wedge product of all basis elements in $\mathcal{B} \setminus \{e_{i_1}, \ldots, e_{i_p}\}$ (so $q = n - p$), and $\sigma \in S_n$ is the permutation

$$\sigma = \begin{pmatrix} 1 & \cdots & p & p+1 & \cdots & n \\ i_1 & \cdots & i_p & j_1 & \cdots & j_q \end{pmatrix}.$$

[3] A field $\mathbb{F}$ is called *formally real* if $\sum_i c_i^2 = 0$, $c_i \in \mathbb{F}$, implies each $c_i = 0$. Such fields admit an ordering and are necessarily of characteristic zero (see [21, page 271]).

Note that $*$ is injective and is therefore an isomorphism (of vector spaces), as are the restrictions $*_p : \Lambda^p\mathfrak{g}^* \to \Lambda^{n-p}\mathfrak{g}^*$ for each p. In particular, $*_n : \Lambda^n\mathfrak{g}^* \to \mathbb{F}$ is the isomorphism sending τ to 1. Moreover, it is easy to see that $\alpha \wedge *\alpha = \tau$ for every monomial α in the basis $\mathcal{B}$, and that $** = (-1)^{p(n-p)}\,\mathrm{id}$ on $\Lambda^p\mathfrak{g}^*$.

We can define a non-degenerate bilinear form $\langle\, , \rangle_* : \Lambda\mathfrak{g}^* \times \Lambda\mathfrak{g}^* \to \mathbb{F}$ on $\Lambda\mathfrak{g}^*$ as follow: for homogeneous elements $\alpha \in \Lambda^p\mathfrak{g}^*$ and $\beta \in \Lambda^q\mathfrak{g}^*$, set

$$\langle \alpha, \beta\rangle_* = \begin{cases} *(\alpha \wedge *\beta) & \text{if } p = q \\ 0 & \text{otherwise.} \end{cases}$$

Consider two monomials $\alpha = e_{i_1} \wedge \cdots \wedge e_{i_p}$ and $\beta = e_{j_1} \wedge \cdots \wedge e_{j_p}$ in $\mathcal{B}$. Clearly, if $\alpha \neq \beta$, then $\langle \alpha, \beta\rangle_* = 0$, while by the remarks above, $\alpha = \beta$ implies $\langle \alpha, \beta\rangle_* = *\tau = 1$. Hence the set of monomials in $\mathcal{B}$ forms an orthonormal basis for $(\Lambda\mathfrak{g}^*, \langle\, , \rangle_*)$, which establishes that $\langle\, , \rangle_*$ is indeed non-degenerate. It also follows that $\langle\, , \rangle_*$ is positive-definite since $\mathbb{F}$ is formally real.

We will denote by $\partial : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ the adjoint of the differential d with respect to $\langle\, , \rangle_*$. In the next subsection, we write ∂ explicitly in terms of the Hodge star.

Remark 2.3.2. Clearly the definitions above depend on our choice of ordered basis. Indeed, let $\mathcal{A} = \{a_i\}_{i=1}^n$ and $\mathcal{B} = \{b_i\}_{i=1}^n$ be two ordered bases of $\mathfrak{g}^*$, and let $*_\mathcal{A}$ and $*_\mathcal{B}$ denote the respective Hodge star operators defined according to the definition above. Then it can be shown that $*_\mathcal{A} = *_\mathcal{B}$ if and only if the change of basis $T : (\mathfrak{g}^*, \langle\, , \rangle_\mathcal{A}) \to (\mathfrak{g}^*, \langle\, , \rangle_\mathcal{A})$ given by $a_i \mapsto b_i$ is an orientation-preserving orthogonal transformation. In other words, the ordered bases $\mathcal{A}$ and $\mathcal{B}$ give rise to the same Hodge star precisely when the matrices $[a_1a_2\cdots a_n]$ and $[b_1b_2\cdots b_n]$ lie in the same orbit under the action of $SO(n)$.

2.3.2 Twisted cohomology

For any Lie algebra $\mathfrak{g}$, define $\gamma \in \mathfrak{g}^*$ by $\langle \gamma, x\rangle = \mathrm{tr}(\mathrm{ad}\, x)$ for all $x \in \mathfrak{g}$. Then γ is a closed 1-form. Indeed, as both $\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ and $\mathrm{tr} : \mathfrak{gl}(\mathfrak{g}) \to \mathbb{R}$ are Lie algebra homomorphisms, for any $x, y \in \mathfrak{g}$ we find that

$$\langle d\gamma, x \wedge y \rangle = \langle \gamma, [x,y] \rangle = \operatorname{tr}(\operatorname{ad}[x,y]) = \operatorname{tr}([\operatorname{ad} x, \operatorname{ad} y]) = 0.$$

The next lemma shows that d restricted to $\Lambda^{\dim \mathfrak{g}-1}\mathfrak{g}^*$ is simply left multiplication by γ.

Lemma 2.3.3 ([26])**.** *If* $\dim \mathfrak{g} = n$, *then* $d\omega = \gamma \wedge \omega$ *for all* $\omega \in \Lambda^{n-1}\mathfrak{g}^*$.

Proof: Let $\{x_1, \ldots, x_n\}$ and $\{x_1^*, \ldots, x_n^*\}$ be dual bases of $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively. Then we can write

$$dx_k^* = \sum_{i<j} c_{ij}^k x_i^* \wedge x_j^*$$

for all $k = 1, \ldots, n$ where the c_{ij}^k are the structure constants of $\mathfrak{g}$ relative to the basis $\{x_1, \ldots, x_n\}$.

Note that for any linear map $T : \mathfrak{g}^* \to \mathfrak{g}^*$, if $T(x_k^*) = a_1 x_1^* + \cdots + a_n x_n^*$ for some $a_1, \ldots, a_n \in \mathbb{F}$, then

$$i_{x_k} T(x_k^*) = \langle i_{x_k} T(x_k^*), 1 \rangle = \langle T(x_k^*), x_k \rangle = a_k.$$

Thus $\operatorname{tr} T = \sum_{k=1}^n i_{x_k} T(x_k^*)$.

It follows from the above remarks that for each $x \in \mathfrak{g}$,

$$\begin{aligned}
\operatorname{tr}(i_x d) &= \sum_{k=1}^n i_{x_k}(i_x dx_k^*) \\
&= \sum_{k=1}^n \sum_{i<j} i_{x_k} i_x (c_{ij}^k x_i^* \wedge x_j^*) \\
&= -\sum_{k=1}^n \sum_{i<j} i_x i_{x_k} (c_{ij}^k x_i^* \wedge x_j^*) \\
&= -\sum_{k=1}^n i_x \left(\sum_{k<j} c_{kj}^k x_j^* + \sum_{i<k} c_{ik}^k (-x_i^*) \right) \\
&= -\sum_{k=1}^n \sum_{j=1}^n i_x (c_{kj}^k x_j^*), \qquad (2.3.1)
\end{aligned}$$

and since $\mathrm{ad}\, x$ and $i_x d$ are clearly adjoint (with respect to the dual pairing $\langle\, ,\rangle$), we have that $\gamma(x) = \mathrm{tr}(\mathrm{ad}\, x)$ is equal to (2.3.1) for each $x \in \mathfrak{g}$. Now, since $\Lambda^{n-1}\mathfrak{g}^*$ has a basis of monomials of the form $x_1^* \wedge \cdots \wedge \widehat{x_i^*} \wedge \cdots \wedge x_n^*$, it suffices to show that d acts on this basis by left multiplication by γ. So let $\omega = x_1^* \wedge \cdots \wedge x_{n-1}^*$ (the following calculation is similar for the other basis elements). Then

$$\begin{aligned} d\omega &= \sum_{k=1}^{n-1} (-1)^{k+1} dx_k^* \wedge x_1^* \wedge \cdots \wedge \widehat{x_k^*} \wedge \cdots \wedge x_{n-1}^* \\ &= \sum_{k=1}^{n-1} (-1)^{k+1} (c_{kn}^k x_k^* \wedge x_n^*) \wedge x_1^* \wedge \cdots \wedge \widehat{x_k^*} \wedge \cdots \wedge x_{n-1}^* \\ &= -\left(\sum_{k=1}^{n-1} c_{kn}^k x_n^* \right) \wedge x_1^* \wedge \cdots \wedge x_{n-1}^* \\ &= \gamma \wedge \omega. \end{aligned}$$

∎

As in [2], we define a map $D : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ by $D\omega = d\omega - \gamma \wedge \omega$. We organize some useful properties of D into the following proposition:

Proposition 2.3.4.

1. *The restriction of D to $\Lambda^{n-1}\mathfrak{g}^*$ is zero;*
2. *$D(\alpha \wedge \beta) = D\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = d\alpha \wedge \beta + (-1)^p \alpha \wedge D\beta$, for all $\alpha \in \Lambda^p\mathfrak{g}^*$, $\beta \in \Lambda\mathfrak{g}^*$;*
3. *$D^2 = 0$.*

Proof: (1) follows immediately from Lemma 2.3.3. Let $\alpha \in \Lambda^p\mathfrak{g}^*$ and $\beta \in \Lambda\mathfrak{g}^*$. Then

$$\begin{aligned} D(\alpha \wedge \beta) &= d(\alpha \wedge \beta) - \gamma \wedge \alpha \wedge \beta = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta - \gamma \wedge \alpha \wedge \beta \\ &= D\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta = d\alpha \wedge \beta + (-1)^p \alpha \wedge D\beta, \end{aligned}$$

proving (2), and

$$\begin{aligned} D^2\alpha &= d^2\alpha - \gamma \wedge d\alpha - (d(\gamma \wedge \alpha) - \gamma \wedge \gamma \wedge \alpha) \\ &= -\gamma \wedge d\alpha + \gamma \wedge d\alpha \\ &= 0, \end{aligned}$$

which proves (3). ∎

In light of part (3) of the proposition above, we are now able to define the *twisted cohomology of* $\mathfrak{g}$ as the cohomology of the cochain complex $(\Lambda\mathfrak{g}^*, D)$.

Definition 2.3.5. *Let* $D_p = D|_{\Lambda^p\mathfrak{g}^*}$*. Then for each* p*, the* p^{th} twisted cohomology group of $\mathfrak{g}$ *is defined as* $H^p_D(\mathfrak{g}) = \ker D_p / \operatorname{im} D_{p-1}$*.*

Note that since D is not a derivation, the ring structure on $\Lambda\mathfrak{g}^*$ afforded by the wedge product does not descend to twisted cohomology as it does with usual cohomology, and so we regard the direct sum $H^*_D(\mathfrak{g}) = \oplus_p H^p_D(\mathfrak{g})$ only as a graded vector space.

Recall that a Lie algebra $\mathfrak{g}$ is *unimodular* if $\operatorname{tr}(\operatorname{ad} x) = 0$ for all $x \in \mathfrak{g}$. For such algebras we clearly have $D = d$. In particular, if $\mathfrak{g}$ is nilpotent, then $H^*(\mathfrak{g}) = H^*_D(\mathfrak{g})$.

Lemma 2.3.6. *The adjoint* ∂ *of the differential* d *is given by* $\partial = (-1)^{n(p+1)+1} * D*$ *on* $\Lambda^p\mathfrak{g}^*$*.*

Proof: Let $\alpha \in \Lambda^p\mathfrak{g}^*$ and $\beta \in \Lambda^{p+1}\mathfrak{g}^*$. Since $*$ is an isomorphism, to prove that $\langle d\alpha, \beta\rangle_* = \langle \alpha, (-1)^{n(p+2)+1} * D * \beta\rangle_*$, it suffices to show

$$d\alpha \wedge *\beta = \alpha \wedge *(-1)^{n(p+2)+1} * D * \beta.$$

Since $\alpha \wedge *\beta \in \Lambda^{n-1}\mathfrak{g}^*$, by Proposition 2.3.4, $D(\alpha \wedge *\beta) = 0$. Thus

$$\begin{aligned} \alpha \wedge *(-1)^{n(p+2)+1} * D * \beta &= (-1)^{n(p+2)+1} \alpha \wedge * * D * \beta \\ &= (-1)^{n(p+2)+1+p(n-p)} \alpha \wedge D * \beta \\ &= -(-1)^{p} \alpha \wedge D * \beta \\ &= d\alpha \wedge *\beta, \end{aligned}$$

where the last equality follows from part (2) of Proposition 2.3.4. Since adjoints are unique, this proves the restriction of ∂ to $\Lambda^p\mathfrak{g}^*$ is $(-1)^{n(p+1)+1} * D *$. ∎

2.3.3 The Hodge decomposition theorem

We have established that the adjoint ∂ of d is, up to a sign, given by $*D*$ (or $*d*$ for unimodular Lie algebras). We now define the *Laplacian operator* $\Delta : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ as the linear map of degree zero defined by $\Delta = d\partial + \partial d$. Elements in the kernel of Δ are called *harmonic forms* on $\mathfrak{g}$.

Lemma 2.3.7 ([2], Lemma 2.2)**.** *The Laplacian operator satisfies the following:*

1. *Δ is self-adjoint;*
2. $\ker \Delta = \ker d \cap \ker \partial$.

Proof: (1) Let α, $\beta \in \Lambda\mathfrak{g}^*$. Then clearly

$$\langle \Delta\alpha, \beta \rangle_* = \langle d\partial\alpha, \beta \rangle_* + \langle \partial d\alpha, \beta \rangle_* = \langle \alpha, d\partial\beta \rangle_* + \langle \alpha, \partial d\beta \rangle_* = \langle \alpha, \Delta\beta \rangle_*.$$

(2) The inclusion $\ker d \cap \ker \partial \subseteq \ker \Delta$ is obvious. To see the other inclusion, let $\omega \in \ker \Delta$. Then

$$0 = \langle \Delta\omega, \omega \rangle_* = \langle d\partial\omega + \partial d\omega, \omega \rangle_* = \langle \partial\omega, \partial\omega \rangle_* + \langle d\omega, d\omega \rangle_*,$$

which, by positive-definiteness, implies $d\omega = \partial\omega = 0$, so $\ker\Delta \subseteq \ker d \cap \ker \partial$. ∎

Dually, if we define $\delta : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ to be the linear map of degree -1 given by $\delta = (-1)^{n(p+1)+1} * d*$ on $\Lambda^p\mathfrak{g}^*$ and set $\nabla = D\delta + \delta D$, then in an analogous way we find the following:

Lemma 2.3.8. *The adjoint of the twisted differential D is δ.*

Proof: Let $\alpha \in \Lambda^p\mathfrak{g}^*$ and $\beta \in \Lambda^{p+1}\mathfrak{g}^*$. Then, as in Lemma 2.3.6, $D(\alpha \wedge *\beta) = 0$ and it suffices to show $D\alpha \wedge *\beta = \alpha \wedge *\delta\beta$. We compute

$$\begin{aligned}
\alpha \wedge *\delta\beta &= \alpha \wedge *(-1)^{n(p+2)+1} * d * \beta \\
&= (-1)^{n(p+2)+1+p(n-p)} \alpha \wedge d * \beta \\
&= -(-1)^p \alpha \wedge d * \beta \\
&= D\alpha \wedge *\beta,
\end{aligned}$$

where the last equality follows from part (2) of Proposition 2.3.4. ∎

Just as with the Laplacian, ∇ is self-adjoint and $\ker\nabla = \ker D \cap \ker\delta$ (cf. Lemma 2.3.7).

With the results above concerning the operators d, ∂ and Δ, the Hodge decomposition theorem for Lie algebras will follow immediately from the following theorem. Its proof is a straightforward exercise in linear algebra which we include for completeness. We will make use of the corresponding results concerning the operators D, δ and ∇ in proving Poincaré duality.

Theorem 2.3.9 ([2], [5]). *Let ϕ be a linear operator on $\Lambda\mathfrak{g}^*$ such that $\phi^2 = 0$ and let $\bar{\phi}$ denote its adjoint with respect to $\langle\,,\rangle_*$. Then for $\Delta_\phi = \phi\bar{\phi} + \bar{\phi}\phi$, one has orthogonal decompositions:*

1. $\Lambda\mathfrak{g}^* = \ker\Delta_\phi \oplus \operatorname{im}\Delta_\phi$;

2. $\operatorname{im}\Delta_\phi = \operatorname{im}\phi \oplus \operatorname{im}\bar{\phi}$;

3. $\ker\phi = \ker\Delta_\phi \oplus \operatorname{im}\phi$.

Proof: (1) If $\alpha \in \ker\Delta_\phi \cap \operatorname{im}\Delta_\phi$, then $\alpha = \Delta_\phi\beta$ for some $\beta \in \Lambda\mathfrak{g}^*$ and

$$0 = \langle \Delta_\phi\alpha, \beta\rangle_* = \langle \Delta_\phi\beta, \Delta_\phi\beta\rangle_*,$$

so $\Delta_\phi\beta = \alpha = 0$. Therefore, $\ker\Delta_\phi \cap \operatorname{im}\Delta_\phi = 0$ and $\Lambda\mathfrak{g}^* = \ker\Delta_\phi \oplus \operatorname{im}\Delta_\phi$ follows for dimension reasons. This decomposition is orthogonal since Δ_ϕ is self-adjoint.

(2) Clearly $\operatorname{im}\Delta_\phi \subseteq \operatorname{im}\phi \oplus \operatorname{im}\bar{\phi}$, so by (1) it suffices to show

$$\operatorname{im}\phi \cap \operatorname{im}\bar{\phi} = \ker\Delta_\phi \cap \operatorname{im}\phi = \ker\Delta_\phi \cap \operatorname{im}\bar{\phi} = 0.$$

So suppose $\omega = \phi(\alpha) = \bar{\phi}(\beta)$ for some α and β. But then

$$\langle \omega, \omega\rangle_* = \langle \phi(\alpha), \bar{\phi}(\beta)\rangle_* = \langle \phi^2(\alpha), \beta\rangle_* = 0,$$

so $\omega = 0$. Thus $\operatorname{im}\phi \cap \operatorname{im}\bar{\phi} = 0$. Now suppose $\phi(\omega) \in \ker\Delta_\phi$. Then $\ker\Delta_\phi = \ker\phi \cap \ker\bar{\phi}$ implies $\bar{\phi}\phi(\omega) = 0$, but then

$$0 = \langle \bar{\phi}\phi(\omega), \omega\rangle_* = \langle \phi(\omega), \phi(\omega)\rangle_*,$$

so $\phi(\omega) = 0$ and it follows that $\ker\Delta_\phi \cap \operatorname{im}\phi = 0$. Similarly, $\ker\Delta_\phi \cap \operatorname{im}\bar{\phi} = 0$.

This shows $\operatorname{im}\Delta_\phi = \operatorname{im}\phi \oplus \operatorname{im}\bar{\phi}$. Moreover, this decomposition is orthogonal since ϕ and $\bar{\phi}$ are adjoint and square to zero.

(3) Clearly $\ker\Delta_\phi \oplus \operatorname{im}\phi \subseteq \ker\phi$ since each summand is contained in $\ker\phi$. Conversely, suppose $\omega \in \ker\phi$. By (2), $\omega \in \Lambda\mathfrak{g}^* = \ker\Delta_\phi \oplus \operatorname{im}\phi \oplus \operatorname{im}\bar{\phi}$, so it suffices to show $\ker\phi \cap \operatorname{im}\bar{\phi} = 0$. So assume $\omega = \bar{\phi}(\alpha)$ for some α. Then

$$0 = \langle \phi(\omega), \alpha\rangle_* = \langle \phi\bar{\phi}(\alpha), \alpha\rangle_* = \langle \bar{\phi}(\alpha), \bar{\phi}(\alpha)\rangle,$$

so $\bar{\phi}(\alpha) = \omega = 0$, as desired, and clearly $\operatorname{im}\phi \perp \ker \Delta_\phi$ so the decomposition in (3) is orthogonal. ∎

Corollary 2.3.10 (Hodge decomposition theorem for Lie algebras)**.** *There are orthogonal decompositions:*

1. $\Lambda\mathfrak{g}^* = \ker\Delta \oplus \operatorname{im} d \oplus \operatorname{im}\partial$*;*

2. $\ker d = \ker\Delta \oplus \operatorname{im} d$*.*

Therefore, $H^*(\mathfrak{g}) \cong \ker\Delta$*.*

The above result shows that every cohomology class has a unique harmonic representative and thus provides an expression of the cohomology of $\mathfrak{g}$ as the space of harmonic forms on $\mathfrak{g}$, that is, those forms which are both d-closed and ∂-closed. Note that the wedge product of harmonic forms need not be harmonic, so the isomorphism above is only an isomorphism of vector spaces. We will write $\mathcal{H}^*(\mathfrak{g})$ when we are thinking of cohomology as a space of harmonic forms.

By Theorem 2.3.9, we also have a linear isomorphism $H^*_D(\mathfrak{g}) \cong \ker\nabla$. Note that if $\omega \in \ker\Delta = \ker d \cap \ker\partial$, then $*\omega \in \ker D \cap \ker\delta = \ker\nabla$, since $\delta = \pm * d*$ and $\partial = \pm * D*$. We therefore have a linear map $* : \ker\Delta \to \ker\nabla$. Similarly, the Hodge star sends p-forms in $\ker\nabla$ to $(n-p)$-forms in $\ker\Delta$. Furthermore, if η is a p-form in $\ker\nabla$, then $*(*\eta) = (-1)^{p(n-p)}\eta = \eta$, unless p is odd and n is even, in which case $*(-*\eta) = \eta$, so $* : \ker\Delta \to \ker\nabla$ is surjective. And since $*$ is clearly injective, we have proven the following:

Theorem 2.3.11 (Poincaré duality for Lie algebras)**.** *For any n-dimensional Lie algebra $\mathfrak{g}$ and any integer p,*

$$H^p(\mathfrak{g}) \cong H^{n-p}_D(\mathfrak{g}).$$

Recall that if $\mathfrak{g}$ is unimodular, twisted cohomology agrees with the usual cohomology of $\mathfrak{g}$. We therefore have the following useful corollary:

Corollary 2.3.12. *For any n-dimensional unimodular Lie algebra $\mathfrak{g}$ and any integer p,*

$$H^p(\mathfrak{g}) \cong H^{n-p}(\mathfrak{g}).$$

2.4 The Dixmier Long Exact Sequence

In this section we recall a well-known result of J. Dixmier which gives the following lower bound on the Betti numbers of nilpotent Lie algebras:

$$b_p \geq 2 \text{ for } 0 < p < \dim \mathfrak{g},$$

where $b_p = \dim H^p(\mathfrak{g})$ denotes the p^{th} Betti number of $\mathfrak{g}$. The proof makes use of a long exact sequence in cohomology induced by the short exact sequence

$$\mathfrak{h} \hookrightarrow \mathfrak{g} \to \mathfrak{g}/\mathfrak{h},$$

where $\mathfrak{h}$ is an ideal of codimension 1 in $\mathfrak{g}$. In this case, any element $X \in \mathfrak{g}\backslash\mathfrak{h}$ determines a 1-form $\omega \in \mathfrak{g}^*$ by $\langle \omega, X\rangle = 1$ and $\langle \omega, Y\rangle = 0$ for all $Y \in \mathfrak{h}$, so that $\Lambda\mathfrak{h}^\perp = \Lambda(\omega)$. This 1-form is closed since $\mathfrak{h}$ is an ideal:

$$\langle d\omega, Y_1 \wedge Y_2\rangle = \langle \omega, [Y_1, Y_2]\rangle = 0$$

for all $Y_1, Y_2 \in \mathfrak{g}$ since $[Y_1, Y_2] \in \mathfrak{h}$.

Theorem 2.4.1 (Dixmier [10])**.** *If $\mathfrak{h}$ is an ideal of codimension 1 in a Lie algebra $\mathfrak{g}$, then there exists a long exact sequence in cohomology*

$$\cdots \xrightarrow{\theta^*} H^{p-1}(\mathfrak{h}) \xrightarrow{\omega\wedge -} H^p(\mathfrak{g}) \xrightarrow{\rho} H^p(\mathfrak{h}) \xrightarrow{\theta^*} H^p(\mathfrak{h}) \longrightarrow \cdots,$$

where ω is the closed 1-form determined by some $X \in \mathfrak{g}\backslash\mathfrak{h}$ as above, ρ is the homomorphism induced by the inclusion $\mathfrak{h} \hookrightarrow \mathfrak{g}$, and θ^* is the homomorphism induced by the adjoint action of X on $\mathfrak{h}$ extended to a derivation of $\Lambda\mathfrak{h}^*$.

Proof: As in Section 2.2, we identify the Koszul complex of $\mathfrak{g}$ with $(\Lambda(\omega)\otimes\Lambda\mathfrak{h}^*, d)$ and the Koszul complex of $\mathfrak{h}$ with $(\Lambda\mathfrak{h}^*, \overline{d})$, where $\overline{d} = \rho\circ d$ and ρ is defined as in the Koszul fibration: $\rho(1\otimes\alpha) = \alpha$ for $\alpha\in\Lambda\mathfrak{h}^*$ and $\rho(\Lambda^+(\omega)\otimes\Lambda\mathfrak{h}^*) = 0$.

Since $\mathfrak{h}$ is invariant under the adjoint action of X, we can extend the map $\operatorname{ad}(X) : \mathfrak{h}\to\mathfrak{h}$ to a derivation $\theta : \Lambda\mathfrak{h}^*\to\Lambda\mathfrak{h}^*$ of degree 0 in the usual way. Now θ and $\overline{d}$ are derivations of degrees 0 and 1, respectively. Recall that the set of all derivations of $\Lambda\mathfrak{h}^*$ form the graded Lie algebra $\operatorname{Der}(\Lambda\mathfrak{h}^*)$, so to show θ commutes with the differential $\overline{d}$, it suffices to show the commutator $[\overline{d},\theta]$, a derivation of degree 1, vanishes on generators of $\Lambda\mathfrak{h}^*$. So letting $\alpha\in\mathfrak{h}^*$ and $a, b\in\mathfrak{h}$, we find that

$$\begin{aligned}\langle[\overline{d},\theta]\alpha, a\wedge b\rangle &= \langle\overline{d}\theta(\alpha)-\theta\overline{d}(\alpha), a\wedge b\rangle\\ &= \langle\theta(\alpha),[a,b]\rangle - \langle d\alpha, \operatorname{ad}(X)(a)\wedge b + a\wedge\operatorname{ad}(X)(b)\rangle\\ &= \langle\alpha,[X,[a,b]]\rangle - \langle\alpha,[[X,a],b]+[a,[X,b]]\rangle\\ &= 0\end{aligned}$$

by the Jacobi identity. Therefore, the derivation $\theta : \Lambda\mathfrak{h}^*\to\Lambda\mathfrak{h}^*$ commutes with the differential in $(\Lambda\mathfrak{h}^*, \overline{d})$ and induces a derivation of graded algebras $\theta^* : H^*(\mathfrak{h})\to H^*(\mathfrak{h})$.

Consider the short exact sequence

$$0\longrightarrow(\Lambda^{p-1}\mathfrak{h}^*, -\overline{d})\xrightarrow{\omega\wedge -}(\Lambda^p\mathfrak{g}^*, d)\xrightarrow{\rho}(\Lambda^p\mathfrak{h}^*, \overline{d})\longrightarrow 0,$$

where $\omega\wedge -$ is the homomorphism induced by left multiplication by ω, and the minus sign in the leftmost complex is needed since

$$d(\omega\wedge\alpha) = -\omega\wedge d\alpha = -\omega\wedge\overline{d}\alpha.$$

This induces the long exact sequence in the statement of the theorem where the connecting homomorphism is the map θ^* described above. ∎

Note that, for each p, we can condense the long exact sequence in Theorem 2.4.1 to the following short exact sequence:

$$0 \longrightarrow H^{p-1}(\mathfrak{h})/\operatorname{im}\theta^*_{p-1} \xrightarrow{\omega\wedge -} H^p(\mathfrak{g}) \xrightarrow{\rho} \ker\theta^*_p \longrightarrow 0,$$

where θ^*_p denotes the restriction of θ^* to $H^p(\mathfrak{h})$. This sequence splits, yielding

$$H^p(\mathfrak{g}) \cong \ker\theta^*_p \oplus \operatorname{coker}\theta^*_{p-1}. \tag{2.4.1}$$

Note that $\ker\theta^*_p$ is precisely the invariant subalgebra $H^p(\mathfrak{h})^X$ for the adjoint action of X on the cohomology of $\mathfrak{h}$ described above, so we will sometimes write

$$H^p(\mathfrak{g}) \cong H^p(\mathfrak{h})^X \oplus H^{p-1}(\mathfrak{h})^X.$$

Now suppose $\mathfrak{g}$ is nilpotent and recall that every nilpotent Lie algebra contains a codimension 1 ideal $\mathfrak{h}$ (this follows from Theorem 1.3.5, for example). In this case, the map $\operatorname{ad}(X) : \mathfrak{h} \to \mathfrak{h}$ from the proof of the theorem is a nilpotent operator. Consequently, θ^* is nilpotent and so θ^*_p has a nontrivial kernel for each p. Dixmier's result now follows:

Corollary 2.4.2. *If $\mathfrak{g}$ is a nilpotent Lie algebra, then $\dim H^p(\mathfrak{g}) \geq 2$ for $1 < p < \dim\mathfrak{g}$.*

Since, by Poincaré duality, $\dim H^0(\mathfrak{g}) = \dim H^{\dim\mathfrak{g}}(\mathfrak{g}) = 1$, the above result gives the lower bound $\dim H^*(\mathfrak{g}) \geq 2\dim\mathfrak{g}$ for all nilpotent Lie algebras $\mathfrak{g}$. Note that this bound establishes the TRC for $\mathfrak{g}$ when $2\dim\mathfrak{g} \geq 2^{\dim Z(\mathfrak{g})}$. In particular, since the TRC is true for algebras with a centre Z of codimension 0 (i.e., abelian Lie algebras) and Z never has codimension 1, the TRC is true for all nilpotent Lie algebras $\mathfrak{g}$ with $\dim\mathfrak{g} \leq 5$.

2.4.1 A proof of the TRC for free two-step nilpotent Lie algebras

Recall that a Lie algebra $\mathfrak{g}$ is called two-step nilpotent when $\mathfrak{g}^2 = 0$ and $\mathfrak{g}^1 \neq 0$, where $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{g}^2 = [\mathfrak{g}, [\mathfrak{g}, \mathfrak{g}]]$ are the first and second terms in the lower central series of $\mathfrak{g}$.

Definition 2.4.3. *The* free two-step nilpotent Lie algebra on r generators $\mathfrak{f}_2(r)$ *is the real Lie algebra with underlying vector space*

$$\mathfrak{f}_2(r) = E \oplus F,$$

where E has a basis $\{e_i \,|\, 1 \leq i \leq r\}$, F has a basis $\{f_{ij} \,|\, 1 \leq i < j \leq r\}$, and nonzero brackets are given by $[e_i, e_j] = f_{ij}$ for all $i < j$.

Example 2.4.4. The free two-step on 2 generators $\mathfrak{f}_2(2) = \langle e_1, e_2, f_{12} \,|\, [e_1, e_2] = f_{12} \rangle$ is the first Heisenberg Lie algebra. The free two-step on 3 generators is the 6-dimensional Lie algebra $\mathfrak{f}_2(3) = \langle e_1, e_2, e_3, f_{12}, f_{13}, f_{23} \,|\, [e_i, e_j] = f_{ij} \text{ for } i < j \rangle$.

The cohomology of the free two-step nilpotent Lie algebras is well-studied. Their Betti numbers are calculated in [28], for example, and several proofs can be found in the literature that the TRC holds for this family of Lie algebras (see [7], [9] or [29]). Using the Dixmier long exact sequence, we give a new proof of the TRC for all free two-step nilpotent Lie algebras.

Note that each $\mathfrak{f}_2(r)$ embeds into $\mathfrak{f}_2(r+1)$ via $e_i \mapsto e_i$ and $f_{ij} \mapsto f_{ij}$. Consider the chain of inclusions

$$\begin{array}{ccccccccc}
\mathfrak{f}_2(2) & \hookrightarrow & \mathfrak{f}_2(3) & \hookrightarrow & \mathfrak{f}_2(4) & \hookrightarrow & \cdots & \hookrightarrow & \mathfrak{f}_2(r) \hookrightarrow \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow \\
 & & \mathfrak{h}_3 & & \mathfrak{h}_4 & & & & \mathfrak{h}_r
\end{array}$$

where each $\mathfrak{h}_r \subseteq \mathfrak{f}_2(r)$ is the ideal of codimension one defined as

$$\begin{aligned}\mathfrak{h}_r &= \mathfrak{f}_2(r-1) \oplus \mathfrak{a}_{r-1} \\ &= \langle e_1, \ldots, e_{r-1}, f_{12}, f_{13}, \ldots, f_{r-2,r-1} \,|\, [e_i, e_j] = f_{ij}, 1 \le i < j \le r-1 \rangle \\ &\quad \oplus \langle f_{1r}, f_{2r}, \ldots, f_{r-1,r} \rangle.\end{aligned}$$

So each $\mathfrak{h}_r$ is the direct sum of the free two-step on $r-1$ generators and the abelian Lie algebra $\mathfrak{a}_{r-1}$ of dimension $r-1$. Note also that $\Lambda\mathfrak{h}_r^*$ appears as the fibre in the Koszul fibration

$$(\Lambda(e_r^*), 0) \xrightarrow{j} (\Lambda(e_r^*) \otimes \Lambda\mathfrak{h}_r^*, d) \xrightarrow{\rho} (\Lambda\mathfrak{h}_r^*, \overline{d}).$$

By the Künneth formula, we have

$$H^*(\mathfrak{h}_r) \cong H^*(\mathfrak{f}_2(r-1)) \otimes H^*(\mathfrak{a}_{r-1}),$$

so that $\dim H^*(\mathfrak{h}_r) = 2^{r-1} \dim H^*(\mathfrak{f}_2(r-1))$. Let $\theta^* : H^*(\mathfrak{h}_r) \to H^*(\mathfrak{h}_r)$ denote the map induced by the adjoint action of $e_r \in \mathfrak{f}_2(r) \backslash \mathfrak{h}_r$ on $\mathfrak{h}_r$, as in Theorem 2.4.1. We would like to show that this action kills at least half of the cohomology of $\mathfrak{h}_r$, i.e., that $\dim \ker \theta^* = \dim H^*(\mathfrak{h}_r)^{e_r} \ge \frac{1}{2} \dim H^*(\mathfrak{h}_r)$.

Lemma 2.4.5. *Let θ_k^* denote the restriction of θ^* to $H^k(\mathfrak{h}_r)$. Then ${\theta_k^*}^2 = 0$ for all k.*

Proof: For ease of notation we will drop the stars from the notation of a dual basis and identify $\{e_1^*, \ldots, e_{r-1}^*, f_{12}^*, \ldots, f_{r-2,r-1}^*\}$ with $\{e_1, \ldots, e_{r-1}, f_{12}, \ldots, f_{r-2,r-1}\}$. Since $\theta : \Lambda\mathfrak{h}_r^* \to \Lambda\mathfrak{h}_r^*$ is a derivation given by the transpose of $\mathrm{ad}(e_r) : \mathfrak{h}_r \to \mathfrak{h}_r$ on elements of degree 1, we have $\theta_1 = i_{e_r} d$ (where d is the differential on $\Lambda\mathfrak{f}_2(r)^*$). Therefore, $\theta(e_i) = 0$ for all $e_i \in \mathfrak{h}_r^*$, $\theta(f_{ij}) = 0$ for all $i < j \ne r$ and

$$\theta(f_{ir}) = i_{e_r} d(f_{ir}) = i_{e_r}(e_i \wedge e_r) = -e_i$$

for all $i < r$.

Let $[\omega] \in H^k(\mathfrak{h}_r)$. Then any representative of the cohomology class $[\omega]$ is of the form $\omega = \sum_i \alpha_i \otimes \beta_i$, where $[\alpha_i] \in H^{p_i}(\mathfrak{f}_2(r-1))$ and $[\beta_i] \in H^{q_i}(\mathfrak{a}_{r-1})$, and $p_i + q_i = k$. So it suffices to assume $\omega = \alpha \otimes \beta$ where $\alpha \in H^p(\mathfrak{f}_2(r-1))$, $\beta \in H^q(\mathfrak{a}_{r-1})$ and $p+q = k$. Furthermore, we can assume $\beta = f_{j_1 r} \wedge \cdots \wedge f_{j_q r}$ for some $1 \leq j_1 < \cdots < j_q \leq r-1$ since $H^q(\mathfrak{a}_{r-1}) = \Lambda^q \mathfrak{a}^*_{r-1}$. Thus

$$\begin{aligned}
\theta_k^2(\omega) &= \theta_k^2(\alpha \otimes f_{j_1 r} \cdots f_{j_q r}) \\
&= \theta_k(\sum_{l=1}^{q} \pm e_{j_l} \alpha \otimes f_{j_1 r} \cdots \widehat{f_{j_l r}} \cdots f_{j_q r}) \\
&= \sum_{l=1}^{q} \sum_{m \in \{1,\ldots,q\} \setminus \{l\}} \pm e_{j_l} e_{j_m} \alpha \otimes f_{j_1 r} \cdots \widehat{f_{j_l r}} \cdots \widehat{f_{j_m r}} \cdots f_{j_q r} \\
&= \sum_{l=1}^{q} \sum_{m \neq l} \pm d(f_{j_l j_m} \alpha \otimes f_{j_1 r} \cdots \widehat{f_{j_l r}} \cdots \widehat{f_{j_m r}} \cdots f_{j_q r})
\end{aligned}$$

is a boundary in $\Lambda \mathfrak{h}_r^*$. Consequently, $\theta_k^{*2}([\omega]) = 0$ and hence $\theta_k^{*2} = 0$ for all k. ∎

It follows from the lemma above that $\theta^{*2} = 0$ and thus

$$\dim \ker \theta^* \geq \frac{1}{2} \dim H^*(\mathfrak{h}_r).$$

With this result we can now prove the TRC for the free two-step nilpotent Lie algebras.

Theorem 2.4.6. *The Toral Rank Conjecture is true for all free two-step nilpotent Lie algebras* $\mathfrak{f}_2(r)$.

Proof: The centre of $\mathfrak{f}_2(r)$ is given by $\operatorname{span}\{f_{ij} \,|\, 1 \leq i < j \leq r\}$, so $\dim Z(\mathfrak{f}_2(r)) = \binom{r}{2}$. For $r = 2$ a simple calculation shows that

$$\dim H^*(\mathfrak{f}_2(2)) = 6 > 2^{\dim Z(\mathfrak{f}_2(2))} = 2.$$

We proceed by induction. Suppose the TRC holds for all free two-steps on less than or equal to r generators for some $r \geq 2$. For the codimension 1 ideal $\mathfrak{h}_{r+1} \subseteq \mathfrak{f}_2(r+1)$

we have

$$H^*(\mathfrak{h}_{r+1}) \cong H^*(\mathfrak{f}_2(r)) \otimes H^*(\mathfrak{a}_r),$$

so by the induction hypothesis,

$$\dim H^*(\mathfrak{h}_{r+1}) = \dim H^*(\mathfrak{f}_2(r)) \cdot 2^r \geq 2^{\binom{r}{2}} \cdot 2^r = 2^{\binom{r+1}{2}}.$$

Letting $\theta^* : H^*(\mathfrak{h}_{r+1}) \to H^*(\mathfrak{h}_{r+1})$ denote the map induced by the adjoint action of $e_{r+1} \in \mathfrak{f}_2(r+1) \backslash \mathfrak{h}_{r+1}$, we have by (2.4.1) that

$$\dim H^*(\mathfrak{f}_2(r+1)) = 2 \dim(\ker \theta^*).$$

But it follows from Lemma 2.4.5 that

$$\dim(\ker \theta^*) \geq \frac{1}{2} \dim H^*(\mathfrak{h}_{r+1}).$$

Therefore $\dim H^*(\mathfrak{f}_2(r+1)) \geq \dim H^*(\mathfrak{h}_{r+1}) \geq 2^{\binom{r+1}{2}}$, completing the induction. ∎

The key to the above proof was in improving the bound $\dim H^p(\mathfrak{g}) \geq 2$ for $1 < p < \dim \mathfrak{g}$ (given by Dixmier's result) by showing that the kernel of θ^* was sufficiently large. Unfortunately, given a nilpotent Lie algebra, one cannot always find a codimension 1 ideal $\mathfrak{h}$ for which the induced derivation $\theta^* : H^*(\mathfrak{h}) \to H^*(\mathfrak{h})$ satisfies $\theta^{*2} = 0$. For example, every ideal of codimension 1 in the free three-step nilpotent Lie algebra

$$\mathfrak{f}_3(2) = \langle x, y, z, u, v \,|\, [x, y] = z, [x, z] = u, [y, z] = v \rangle$$

is isomorphic to $\mathfrak{h} = \langle x, z, u, v \,|\, [x, z] = u \rangle$, and in this case the derivation θ^* (induced by $\operatorname{ad} y : \mathfrak{h} \to \mathfrak{h}$) does not square to zero since $\theta^{*2}([v^*]) = \theta^*([z^*]) = [x^*] \neq 0$ in $H^1(\mathfrak{h})$. So the method employed above does not generalize to the case of the free k-step nilpotent Lie algebras for $k \geq 3$.

Chapter 3

Cohomology Operations

We continue our study of Lie algebra cohomology in this chapter by considering the algebra $H^*(\mathfrak{g})$ under the action of various operations. Our attention will be focused primarily on nilpotent Lie algebras. Despite the research of several authors, relatively little is known about the cohomological structure of nilpotent Lie algebras compared to other classes of algebras, and it is hoped that the operations discussed in this chapter will shed light on this topic.

In Section 1 we define *primary operations* and the *central representation*, as introduced in [5]. These are operations on cohomology that depend on elements of the centre Z of a Lie algebra $\mathfrak{g}$ and equip $H^*(\mathfrak{g})$ with the structure of a module over ΛZ. The overall theme of this chapter will be the attempt to use cohomology operations to construct "cube-like" structures in $H^*(\mathfrak{g})$ which yield the TRC for $\mathfrak{g}$, as will happen when the central representation is faithful, for example. When the central representation of a nilpotent Lie algebra $\mathfrak{g}$ is not faithful, we will find that a "cube" in $H^*(\mathfrak{g})$ can often be repaired with the use of operations generalizing the primary operations. These *higher operations* and the module structure they afford for $H^*(\mathfrak{g})$ will be the subject of Section 2.

In this chapter, all vector spaces and algebras are assumed to be finite-dimensional over a formally real field $\mathbb{F}$.

3.1 Primary Operations and the Central Representation

A *representation* of an (associative) algebra A in a vector space V is a homomorphism of algebras $\rho : A \to \mathrm{End}(V)$. Such a homomorphism gives V the structure of an A-module and, conversely, any A-module structure on V determines a representation of A in V. A representation is called *faithful* if ρ is injective, and a subspace $W \subseteq V$ is called *stable under* A if $\rho(a)(W) \subseteq W$ for all $a \in A$.

If (R, d_R) is a differential graded algebra and $\rho : A \to \mathrm{End}(R)$ is a representation of A in the underlying vector space of R such that

$$\rho(a) \circ d_R = d_R \circ \rho(a)$$

for all $a \in A$, then the subalgebras $Z^*(R)$ and $B^*(R)$ of cocycles and coboundaries, respectively, are stable under A. It follows that each map $\rho(a)$ induces a linear map

$$\rho^*(a) : H^*(R) \to H^*(R),$$

and the assignment $a \mapsto \rho^*(a)$ defines a representation ρ^* of A in $H^*(R)$ called the *induced representation in cohomology.*

Recall from Section 1.3 that for any vector space V, each $x \in V$ gives rise to a derivation of degree -1 on ΛV^* called the interior product with x. It is denoted by $i_x : \Lambda V^* \to \Lambda V^*$ and defined on generators $f \in V^*$ by

$$i_x f = f(x).$$

Since the assignment $x \mapsto i_x$ is clearly linear and $i_x \circ i_y = -i_y \circ i_x$, we can set

$$i_{x_1 \wedge \cdots \wedge x_p} = i_{x_p} \circ \cdots \circ i_{x_1}$$

and extend by linearity to obtain a homomorphism of algebras

$$\Lambda V \to \mathrm{End}(\Lambda V^*)$$

which we will call the *standard representation* of ΛV in ΛV^*, as in [22].

Remark 3.1.1. Note that this definition extends the duality between the interior product i_x and exterior product μ_x (given by left multiplication by x) in the sense that

$$\langle i_{x_1\wedge\cdots\wedge x_p}\alpha, \beta\rangle = \langle (i_{x_p}\circ\cdots\circ i_{x_1})\alpha, \beta\rangle = \langle \alpha, x_1\wedge\cdots\wedge x_p\wedge\beta\rangle = \langle \alpha, \mu_{x_1\wedge\cdots\wedge x_p}\beta\rangle,$$

for all $\alpha \in \Lambda V^*$ and $\beta \in \Lambda V$.

We now replace V with a Lie algebra $\mathfrak{g}$. In this case, since each i_x is a derivation on the underlying algebra of the Koszul complex of $\mathfrak{g}$, it is natural to ask if any $x \in \mathfrak{g}$ give rise to maps which commute with the differential.

Proposition 3.1.2. *Let $\mathfrak{g}$ be a Lie algebra with centre Z. Then $i_x d + d i_x = 0$ if and only if $x \in Z$.*

Proof: Since $\mathrm{Der}(\Lambda\mathfrak{g}^*)$ is a graded Lie algebra, it suffices to show $[i_x, d] = i_x d + d i_x$ is zero on generators if and only if $x \in Z$. Letting $f \in \mathfrak{g}^*$ and $y \in \mathfrak{g}$, we find that

$$\begin{aligned}\langle [i_x, d]f, y\rangle &= \langle i_x df, y\rangle \\ &= \langle df, x\wedge y\rangle \\ &= \langle f, [x, y]\rangle.\end{aligned}$$

So $[i_x, d]f = 0$ for all $f \in \mathfrak{g}^*$ if and only if $[x, y] = 0$ for all $y \in \mathfrak{g}$, i.e., $x \in Z$. ∎

Therefore, if we denote by $i : \Lambda Z \to \mathrm{End}(\Lambda\mathfrak{g}^*)$ the representation defined on Z by $z \mapsto i_z$ and extended to ΛZ as above, then $i_\zeta \circ d = \pm d \circ i_\zeta$ for all $\zeta \in \Lambda Z$. The induced representation in cohomology

$$i^* : \Lambda Z \to \mathrm{End}\, H^*(\mathfrak{g})$$

gives $H^*(\mathfrak{g})$ the structure of a ΛZ-module with action defined by $\zeta \cdot \omega = i_\zeta^* \omega$ for all $\omega \in H^*(\mathfrak{g})$.

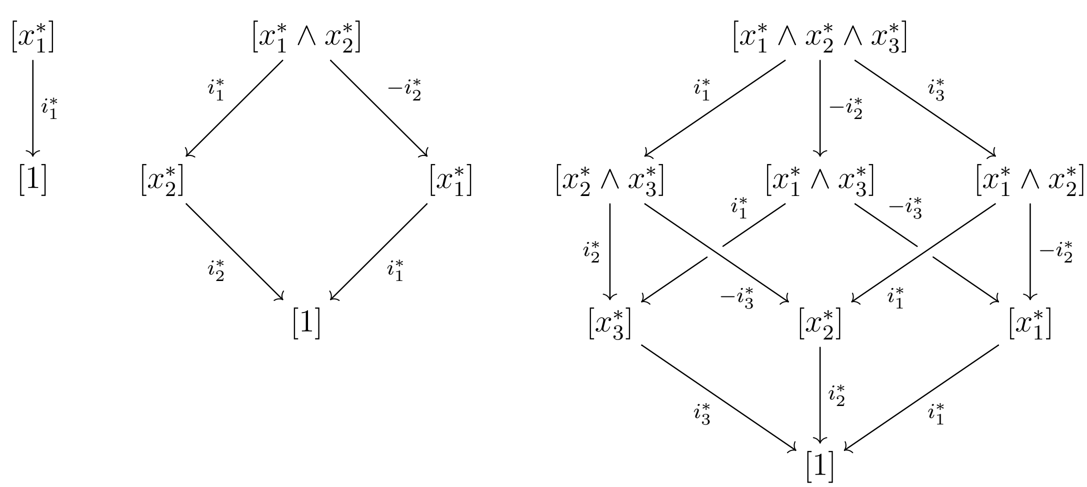
Definition 3.1.3. *The algebra homomorphism*

$$\begin{aligned} i^* : \Lambda Z &\longrightarrow \operatorname{End} H^*(\mathfrak{g}) \\ \zeta &\longmapsto i_\zeta^* \end{aligned}$$

is called the central representation *and each* i_z^**, where* $z \in Z$*, is called a* primary operation *on* $H^*(\mathfrak{g})$*. The cohomology algebra* $H^*(\mathfrak{g})$ *is said to have a* nontrivial ΛZ-module structure *if* $i_z^* \neq 0$ *for some* $z \in Z$*.*

In the following examples, dual bases will be denoted using stars.

Example 3.1.4. For the n-dimensional abelian Lie algebra $\mathfrak{a}_n$, since $Z(\mathfrak{a}_n) = \mathfrak{a}_n$ and $H^*(\mathfrak{a}_n) = \Lambda \mathfrak{a}_n^*$, the central representation clearly coincides with the standard representation of $\Lambda \mathfrak{a}_n$ in $\Lambda \mathfrak{a}_n^*$. Let $\{x_1, \ldots, x_n\}$ be a basis of $\Lambda \mathfrak{a}_n$ and for ease of notation let i_m^* denote $i_{x_m}^*$ for each m. The ΛZ-module structure on the cohomology of $\mathfrak{a}_1$, $\mathfrak{a}_2$ and $\mathfrak{a}_3$ can be written concisely with the following diagrams:



Remark 3.1.5. We remark that regarding $H^*(\mathfrak{g})$ as a ΛZ-module results in a stronger invariant than cohomology alone. As a simple example, let $\mathfrak{g}$ denote the 2-dimensional

solvable Lie algebra $\langle x, y \,|\, [x, y] = y\rangle$. Since $dx^* = 0$ and $dy^* = x^* \wedge y^*$, we have that

$$\begin{aligned} H^0(\mathfrak{g}) &= \operatorname{span}\{1\} \\ H^1(\mathfrak{g}) &= \operatorname{span}\{x^*\} \\ H^2(\mathfrak{g}) &= 0, \end{aligned}$$

and the Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{a}_1$ given by $x \mapsto x_1$ and $y \mapsto 0$ induces an isomorphism (of algebras) $H^*(\mathfrak{g}) \xrightarrow{\sim} H^*(\mathfrak{a}_1)$ in cohomology. Since $\mathfrak{g}$ has a trivial centre, $\mathfrak{g}$ has a trivial ΛZ-module structure, but as seen in the previous example the abelian Lie algebra $\mathfrak{a}_1$ does not. Hence the central representation can distinguish between Lie algebras which cohomology cannot.

Example 3.1.6. Consider the 4-dimensional *standard filiform* Lie algebra

$$\mathfrak{g} = \langle x_1, x_2, x_3, x_4 \,|\, [x_1, x_2] = x_3, [x_1, x_3] = x_4\rangle$$

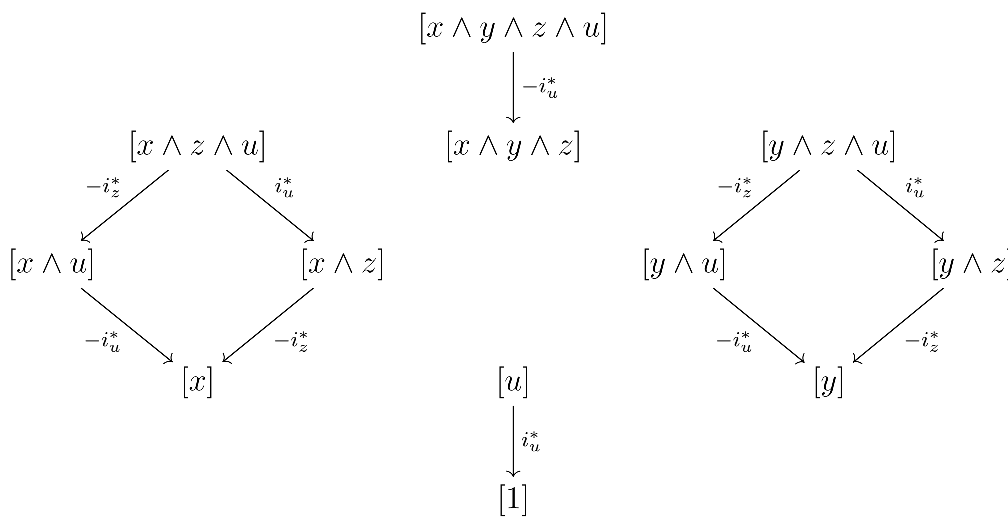
with 1-dimensional centre $Z = \langle x_4\rangle$. By Proposition 2.1.4 and Poincaré duality, $H^1(\mathfrak{g}) = \operatorname{span}\{[x_1^*],\, [x_2^*]\}$ and $H^3(\mathfrak{g}) = \operatorname{span}\{[x_1^* \wedge x_3^* \wedge x_4^*],\, [x_2^* \wedge x_3^* \wedge x_4^*]\}$. Since the closed 2-forms have a basis $\{x_1^* \wedge x_2^*,\, x_1^* \wedge x_3^*,\, x_1^* \wedge x_4^*,\, x_2^* \wedge x_3^*\}$ with $x_1^* \wedge x_2^* = dx_3^*$ and $x_1^* \wedge x_3^* = dx_4^*$ exact, we have $H^2(\mathfrak{g}) = \operatorname{span}\{[x_1^* \wedge x_4^*],\, [x_2^* \wedge x_3^*]\}$. Dropping the stars from our notation, the central representation looks as follows:

$$\begin{array}{ccc} & [x_1 \wedge x_2 \wedge x_3 \wedge x_4] & \\ [x_1 \wedge x_3 \wedge x_4] & & [x_2 \wedge x_3 \wedge x_4] \\ & & \downarrow{\scriptstyle i_4^*} \\ [x_1 \wedge x_4] & & [x_2 \wedge x_3] \\ {\scriptstyle -i_4^*}\downarrow & & \\ [x_1] & & [x_2] \\ & [1] & \end{array}$$

Example 3.1.7. Let $\mathfrak{g}$ be the 4-dimensional nilpotent Lie algebra

$$\langle x, y, z, u \,|\, [x, y] = z \rangle$$

with centre $Z = \langle z, u \rangle$ and dual basis $\{x^*, y^*, z^*, u^*\}$. Note that $\mathfrak{g}$ is isomorphic to the direct sum $\mathfrak{f}_2(2) \oplus \mathfrak{a}_1$ of the free two-step nilpotent Lie algebra on 2 generators and a 1-dimensional abelian Lie algebra. A calculation similar to the one above yields the following diagram for the central representation:



In each of the examples above, $H^*(\mathfrak{g})$ has a nontrivial ΛZ-module structure. Moreover, in each case one can find in the corresponding diagram that $H^*(\mathfrak{g})$ contains an isomorphic copy of ΛZ^*. Indeed, in each of the nonabelian examples we see two dual ΛZ-submodules isomorphic to ΛZ^* embedded in $H^*(\mathfrak{g})$. The cubes in the diagrams formed by these submodules exhibit that the central representations of the nilpotent algebras above are faithful. In fact, as the following theorem from [5] shows, the faithfulness of the central representation $i^* : \Lambda Z \to \operatorname{End} H^*(\mathfrak{g})$ is equivalent to the existence of at least one such free ΛZ-submodule in $H^*(\mathfrak{g})$.

Recall that for any vector space V of dimension n, an *orientation* on V is a choice of a nonzero element $\tau \in \Lambda^n V$.

Theorem 3.1.8 ([5]). *Let $\mathfrak{g}$ be a Lie algebra with centre Z and let τ be an orientation on Z. The following are equivalent:*

1. *The central representation $i^* : \Lambda Z \to \operatorname{End} H^*(\mathfrak{g})$ is faithful;*

2. *There exists some $\omega \in H^*(\mathfrak{g})$ such that $i^*_\tau \omega \neq 0$;*

3. *$H^*(\mathfrak{g})$ contains a free ΛZ-submodule.*

If any of these conditions hold, the TRC is true for $\mathfrak{g}$.

Proof: (2) $\Rightarrow$ (1): For any nonzero $\zeta \in \Lambda Z$, we have that $\zeta \wedge *\zeta$ is a nonzero element of $\Lambda^{\dim Z} Z$, so $\zeta \wedge *\zeta = c\tau$ for some nonzero $c \in \mathbb{F}$. Now

$$0 \neq c i^*_\tau \omega = i^*_{c\tau}\omega = i^*_{\zeta \wedge *\zeta}\omega = (i^*_{*\zeta} \circ i^*_\zeta)\omega,$$

so in particular $i^*_\zeta \neq 0$, which shows that i^* is injective.

(1) $\Rightarrow$ (3): If i^* is injective, then $i^*_\tau \neq 0$, so there exists some $\omega \in H^*(\mathfrak{g})$ with $i^*_\tau \omega \neq 0$. Then by the same argument as above, $i^*_\zeta \omega \neq 0$ for every nonzero $\zeta \in \Lambda Z$. In other words, the linear map $e_\omega : \Lambda Z \to H^*(\mathfrak{g})$ defined by

$$\zeta \longmapsto \zeta \cdot \omega = i^*_\zeta \omega$$

is injective, and therefore the orbit of ω under the action of ΛZ is a free ΛZ-submodule of $H^*(\mathfrak{g})$ isomorphic to ΛZ^*.

(3) $\Rightarrow$ (2): If $M \subseteq H^*(\mathfrak{g})$ is a free ΛZ-submodule, then, choosing a basis element $\omega \in M$, we must have $i^*_\tau \omega \neq 0$.

If the equivalent conditions above hold, then as $H^*(\mathfrak{g})$ contains a free ΛZ-submodule, $\dim H^*(\mathfrak{g}) \geq \dim \Lambda Z$, so the TRC is true for $\mathfrak{g}$. ∎

With this result, it is now clear that each of the examples above admit faithful central representations. If $\{z_1, \ldots, z_k\}$ is a basis of the centre Z of an arbitrary Lie algebra $\mathfrak{g}$, then to show that the central representation of $\mathfrak{g}$ is faithful, one need only find some $\omega \in H^*(\mathfrak{g})$ where the sequence of primary operations

$$i_{z_1}^* \circ i_{z_2}^* \circ \cdots \circ i_{z_k}^*$$

is nonzero on ω. Since primary operations anti-commute, the order above does not matter, and clearly if $i_\tau^*\omega \neq 0$ for some orientation τ on Z, then the same is true for *all* orientations on Z.

Following the terminology in [5], if $\omega \in H^*(\mathfrak{g})$ satisfies $i_\tau^*\omega \neq 0$ for some (and therefore every) orientation τ on Z, we will say that the central representation is *faithful on* ω. By the theorem above, the orbit of such a cohomology class forms an isomorphic copy of ΛZ^* in $H^*(\mathfrak{g})$ which will appear as a "cube" of dimension $\dim Z$ in the diagram of the central representation. For example, the central representation of an abelian Lie algebra $\mathfrak{a}_n$ is faithful on any (nonzero) form in $\Lambda^n\mathfrak{a}_n = H^n(\mathfrak{a}_n)$. Similarly, the central representation in Example 3.1.6 is faithful on $[x_2^* \wedge x_3^* \wedge x_4^*]$ and $[x_1^* \wedge x_4^*]$, while the central representation in Example 3.1.7 is faithful on $[x^* \wedge z^* \wedge u^*]$ and $[y^* \wedge z^* \wedge u^*]$.

As we will see in later examples, the property of having a faithful central representation is far from typical for general Lie algebras. A non-nilpotent example is the 6-dimensional real Lie algebra

$$\begin{aligned}\mathfrak{g} = \langle x, y, z, u, v, w \mid [x, y] = 2y, [x, z] = -2z, [x, u] = u, [x, v] = -v,\\ [y, z] = x, [y, v] = u, [z, u] = v, [u, v] = w\rangle\end{aligned}$$

whose centre is $Z = \langle w\rangle$ and whose Betti numbers are given by

$$(b_0, b_1, b_2, b_3, b_4, b_5, b_6) = (1, 0, 0, 2, 0, 0, 1).$$

Since $i_w^*(H^p(\mathfrak{g})) \subseteq H^{p-1}(\mathfrak{g})$ and at least one of the cohomology groups $H^p(\mathfrak{g})$ and $H^{p-1}(\mathfrak{g})$ are trivial for all p, $\mathfrak{g}$ has a trivial ΛZ-module structure.

Other algebras which have a nontrivial centre yet whose central representation is trivial can already be found in low dimensions (see [5] for a solvable example). In the nilpotent case, however, while non-faithful central representations abound (see Section 3.2), no nilpotent Lie algebra has been found to have a trivial ΛZ-module structure to date. The authors of [5] conclude that article with the conjecture that no such algebras exist, i.e., that every nilpotent Lie algebra has a nontrivial central representation. This is obvious for abelian Lie algebras and has been shown to hold for all two-step nilpotent algebras (see [26]), but remains open in general.

3.2 Higher Operations

In this section, we define higher cohomology operations for arbitrary Lie algebras with nontrivial centre, as introduced in [5]. The main use of these operations for us is the construction of cube-like structures in the cohomology algebras of nilpotent Lie algebras similar to those found in the cohomology of Lie algebras with a faithful central representation. The basic idea is to define secondary operations where primary operations are trivial, tertiary operations where these secondary operations are trivial, and so on. For example, if $z \in Z(\mathfrak{g})$ and $[\eta]$ is a cohomology class with representative $\eta \in \Lambda^p\mathfrak{g}^*$ and $i_z^*[\eta] = 0$, then $i_z\eta = d\mu$ for some $\mu \in \Lambda^{p-1}\mathfrak{g}^*$, and we would like to define a secondary operation which acts on the cohomology class of η by

$$[\eta] \longmapsto [i_z\mu].$$

In other words, the secondary operation would be defined on forms, roughly, as "$i_z \circ d^{-1} \circ i_z$". Note that since $i_z\eta$ may be a boundary in more than one way — i.e., it is possible that $i_z\eta = d\mu_1 = d\mu_2$ for distinct forms μ_1 and μ_2 — such an operation would only be defined modulo $\operatorname{im} i_z^*$. So rather than defining higher operations as functions from kernels to cokernels of primary operations, we work with harmonic forms instead, identifying $H^*(\mathfrak{g})$ with the space $\mathcal{H}^*(\mathfrak{g})$ of harmonic forms on $\mathfrak{g}$ (cf.

Section 2.3). As we will see below, this approach allows us to replace the "d^{-1}" above with a well-defined linear map and leads to a definition of all higher operations as linear maps on $\mathcal{H}^*(\mathfrak{g})$.

Recall from Section 2.3 the Hodge decomposition

$$\Lambda\mathfrak{g}^* = \ker\Delta \oplus \operatorname{im}\Delta$$

and the canonical (graded vector space) isomorphism

$$\begin{array}{ccc} \mathcal{H}^*(\mathfrak{g}) & \xrightarrow{\sim} & H^*(\mathfrak{g}) \\ \gamma & \mapsto & [\gamma] \end{array} \quad ,$$

where $\mathcal{H}^*(\mathfrak{g}) = \ker\Delta$ and $\Delta : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ is the Laplacian operator. Since we would like to define all cohomology operations on $\mathcal{H}^*(\mathfrak{g})$, we first trace the primary operations (i.e., the central representation) through the isomorphism above before defining higher operations.

Although the space of harmonic forms on $\mathfrak{g}$ is in general not closed under the wedge product, we can of course equip $\mathcal{H}^*(\mathfrak{g})$ with the structure of a commutative graded algebra (CGA) by carrying over the product in $H^*(\mathfrak{g})$. More precisely, let

$$\pi : \Lambda\mathfrak{g}^* \to \mathcal{H}^*(\mathfrak{g})$$

denote the orthogonal projection onto the kernel of Δ. Then by composing the wedge product of forms with π we obtain a product in $\mathcal{H}^*(\mathfrak{g})$ making the isomorphism above an isomorphism of CGAs. Similarly, we can equip $\mathcal{H}^*(\mathfrak{g})$ with a ΛZ-module structure compatible with the module structure on $H^*(\mathfrak{g})$ as follows: for each $\zeta \in \Lambda Z$, let πi_ζ denote the restriction of the composition

$$\Lambda\mathfrak{g}^* \xrightarrow{i_\zeta} \Lambda\mathfrak{g}^* \xrightarrow{\pi} \mathcal{H}^*(\mathfrak{g})$$

to the subspace $\mathcal{H}^*(\mathfrak{g}) \subseteq \Lambda\mathfrak{g}^*$. Then $\zeta \cdot \gamma = \pi i_\zeta(\gamma)$ for each $\gamma \in \mathcal{H}^*(\mathfrak{g})$ defines a ΛZ-module structure on $\mathcal{H}^*(\mathfrak{g})$, and

$$[\zeta \cdot \gamma] = [\pi i_\zeta(\gamma)] = [i_\zeta(\gamma)] = i_\zeta^*[\gamma] = \zeta \cdot [\gamma]$$

shows that the isomorphism above also defines an isomorphism of ΛZ-modules.

The isomorphisms discussed above justify the following terminology. In this section, we will call $\pi i : \Lambda Z \to \operatorname{End}\mathcal{H}^*(\mathfrak{g})$, $\zeta \mapsto \pi i_\zeta$, the *central representation* and each πi_z, for $z \in Z$, a *primary operation.*

Proposition 3.2.1. *The Laplacian* $\Delta : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$ *restricts to a linear isomorphism on* $\operatorname{im}\Delta$*.*

Proof: Suppose $\alpha \in \operatorname{im}\Delta$ and $\Delta\alpha = 0$. Then $\alpha = 0$ since $\ker\Delta \cap \operatorname{im}\Delta = 0$. So $\Delta|_{\operatorname{im}\Delta}$ is injective, and is therefore an isomorphism for dimension reasons. ∎

Definition 3.2.2. Green's function $G : \Lambda\mathfrak{g}^* \to \operatorname{im}\Delta$ *is defined as the linear map which is zero on* $\ker\Delta$ *and equals* $(\Delta|_{\operatorname{im}\Delta})^{-1}$ *on* $\operatorname{im}\Delta$*.*

Note that Δ and π are splitting maps in the short exact sequence

$$0 \longrightarrow \ker\Delta \underset{\curvearrowleft \pi}{\hookrightarrow} \Lambda\mathfrak{g}^* \underset{G}{\overset{\curvearrowleft \Delta}{\longrightarrow}} \operatorname{im}\Delta \longrightarrow 0$$

and that $\mathrm{id}_{\Lambda\mathfrak{g}^*} = \pi + \Delta G$. We are now ready to define the higher operations.

Let $\mathfrak{g}$ be a Lie algebra (over $\mathbb{F}$) with centre Z. For each $z \in Z$, let P_z denote the primary operation πi_z and let $Q_z = \partial G i_z$. Now for each $p \in \mathbb{N}$ and p-tuple $\bar{z} = (z_1, \ldots, z_p)$ of elements in Z, let

$$\delta_{\bar{z}} = \frac{1}{p!} \sum_{\sigma \in S_p} P_{z_{\sigma 1}} Q_{z_{\sigma 2}} Q_{z_{\sigma 3}} \cdots Q_{z_{\sigma p}}$$

and define a map

$$\begin{array}{ccc} Z \times \cdots \times Z & \longrightarrow & \operatorname{End}\mathcal{H}^*(\mathfrak{g}) \\ \bar{z} & \longmapsto & \delta_{\bar{z}} \end{array} .$$

Since, by the definition of $\delta_{\bar{z}}$, this is a symmetric multilinear map for each p, it induces a linear map

$$\begin{array}{ccc} \delta : S(Z) & \longrightarrow & \operatorname{End}\mathcal{H}^*(\mathfrak{g}) \\ s & \longmapsto & \delta_s \end{array} ,$$

where $S(Z) = \bigoplus_{p \geq 0} S^p(Z)$ is the symmetric algebra on Z, making

$$\begin{array}{ccc} \overbrace{Z \times \cdots \times Z}^{p \text{ times}} & \longrightarrow & \operatorname{End} \mathcal{H}^*(\mathfrak{g}) \\ \big\downarrow & \nearrow_{\delta} & \\ S^p(Z) & & \end{array}$$

commute for all p.

Definition 3.2.3. *For each $s \in S(Z)$, the linear map δ_s is called the* higher operation *associated to s.*

When $\{z_1, \ldots, z_m\}$ is a basis for Z, we identify the symmetric algebra $S(Z)$ with the polynomial ring $\mathbb{F}[z_1, \ldots, z_m]$ over $\mathbb{F}$ with indeterminates in $\{z_1, \ldots, z_m\}$. This is the convention we will adopt in all explicit examples, and in this case for any monomial $z_1^{t_1} \cdots z_m^{t_m} \in \mathbb{F}[z_1, \ldots, z_m]$, we write

$$\delta_{z_1^{t_1} \cdots z_m^{t_m}} = \frac{1}{p!} \sum_{\sigma \in S_p} P_{y_{\sigma 1}} Q_{y_{\sigma 2}} Q_{y_{\sigma 3}} \cdots Q_{y_{\sigma p}},$$

where $p = \sum_{i=1}^m t_i$ and $\{y_1, \ldots, y_p\} = \{\overbrace{z_1, \ldots, z_1}^{t_1 \text{ times}}, \ldots, \overbrace{z_m, \ldots, z_m}^{t_m \text{ times}}\}$.

For each $z \in Z$, since i_z and ∂ are maps of degree -1 and G is a map of degree 0, the maps $P_z = \pi i_z$ and $Q_z = \partial G i_z$ are of degree -1 and -2, respectively. It follows that if $s \in \mathbb{F}[z_1, \ldots, z_m]$ is a homogeneous polynomial of degree $p \geq 1$, then the higher operation δ_s associated to s is of degree $1 - 2p$. Note that the primary operations are recovered as the higher operations associated to monomials of degree 1: $\delta_z = P_z = \pi i_z$ for each $z \in Z$.

Remark 3.2.4. Unlike the central representation, the linear map $\delta : s \mapsto \delta_s$ is not an algebra homomorphism. For example, if $z, z' \in Z$, then $\delta_z \delta_{z'}$ is a map of degree -2 while $\delta_{zz'}$ is of degree -3, so $\delta_z \delta_{z'} \neq \delta_{zz'}$.

3.2.1 Hypercubes in cohomology

We turn now to some examples where the central representation is not faithful to illustrate how higher operations can be used to construct cubes in the cohomology of nilpotent Lie algebras. By an *n-cube* we mean the n-dimensional hypercube graph $\mathcal{Q}_n$. This is a connected, regular graph with 2^n vertices, each incident to exactly n edges. The hypercube graphs we consider will be directed and the vertices grouped into levels (as in [22]) according to the following construction:

- Level n of $\mathcal{Q}_n$ consists of one vertex with n edges directed from it to n distinct vertices in level $n-1$;
- For $i = 0, \ldots, n-1$, level i consists of $\binom{n}{i}$ vertices, each with i edges directed from it to i distinct vertices in level $i-1$, and $n-i$ edges directed to it from $n-i$ distinct vertices in level $i+1$.

Thus a 0-cube is just a single vertex $\bullet$, a 1-cube is two vertices connected by a directed edge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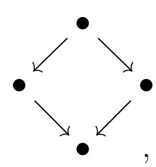
a 2-cube is the square graph with edges directed as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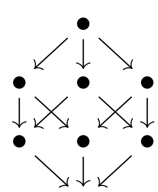
and a 3-cube is given by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Regard the directed graph $\mathcal{Q}_n$ described above as a category with objects given by the vertices and morphisms given by paths in the graph (in addition to identity morphisms for every object). Let $v_1, \ldots, v_{2^n}$ denote the objects in $\mathcal{Q}_n$. Then for a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, we define an *n-cube in the cohomology of* $\mathfrak{g}$ as a functor $F : \mathcal{Q}_n \to \mathbf{Vect}_{\mathbb{F}}$ satisfying:

1. $\{F(v_1), \ldots, F(v_{2^n})\}$ is a collection of nontrivial independent subspaces of $H^*(\mathfrak{g})$ (or $\mathcal{H}^*(\mathfrak{g})$), meaning $F(v_i) \cap (F(v_1) \oplus \cdots \oplus F(v_{i-1}) \oplus F(v_{i+1}) \oplus \cdots \oplus F(v_{2^n})) = 0$ for all $i = 1, \ldots, 2^n$;

2. For each morphism $v_i \xrightarrow{e} v_j$ in $\mathcal{Q}_n$, the linear map $F(e) : F(v_1) \to F(v_2)$ is nontrivial.

By abuse of terminology, we will usually refer to the image of F as the n-cube in $H^*(\mathfrak{g})$. Note that, by axiom 1, the presence of an n-cube in the cohomology of $\mathfrak{g}$ ensures that $\dim H^*(\mathfrak{g}) \geq 2^n$.

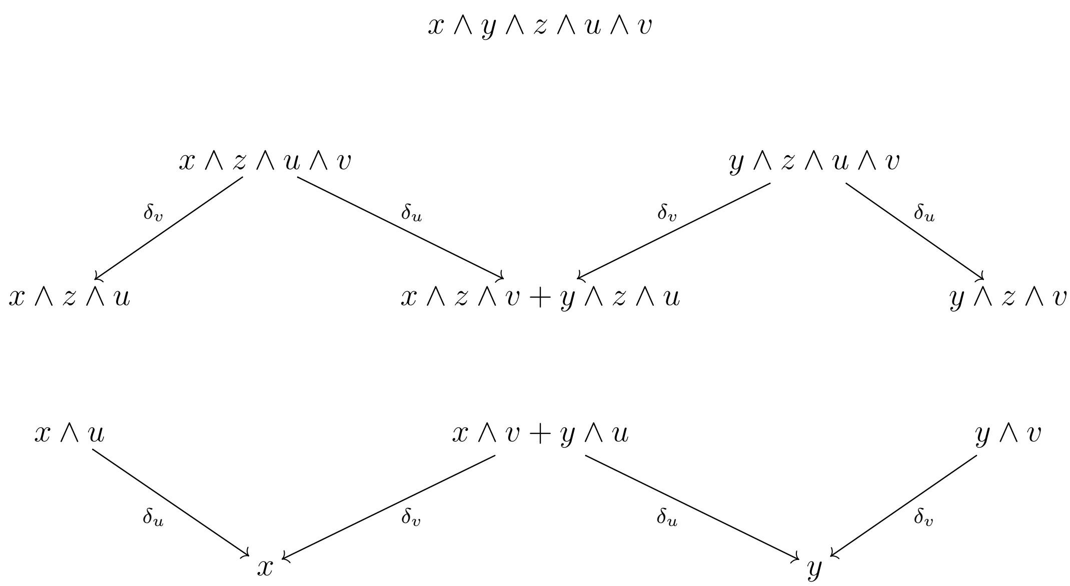
Example 3.1.4 exhibits n-cubes in $H^*(\mathfrak{a}_n)$ for $n = 1, 2$ and 3 (take each vertex to be the span of the cohomology class defining that vertex). In this case, the objects in level i of $\mathcal{Q}_n$ are mapped to subspaces of $H^i(\mathfrak{a}_n)$. As we have already seen, it follows from Theorem 3.1.8 that there exists a z-cube in $H^*(\mathfrak{g})$, where $z = \dim Z(\mathfrak{g})$, whenever $\mathfrak{g}$ has a faithful central representation. In the following example, higher operations form z-cubes where the primary operations fall short, a phenomenon which seems to be typical for nilpotent Lie algebras in low dimensions.

Henceforth, we will drop the stars from the notation of a dual basis, and all instances of $\alpha \xrightarrow{\delta_s} \beta$ in cohomology diagrams mean $0 \neq \delta_s(\alpha) \in \text{span}\{\beta\}$.

Example 3.2.5. Recall from the end of the previous chapter the free three-step nilpotent Lie algebra on 2 generators

$$\mathfrak{f}_3(2) = \langle x, y, z, u, v \mid [x, y] = z, [x, z] = u, [y, z] = v \rangle.$$

This algebra has a 2-dimensional centre $Z = \langle u, v\rangle$, and the ΛZ-module structure on $\mathcal{H}^*(\mathfrak{g})$ is pictured below. Recall that the primary operations πi_u and πi_v are equivalent to higher operations associated to monomials in $\{u, v\}$ of degree 1, i.e., $\delta_u = \pi i_u$ and $\delta_v = \pi i_v$.



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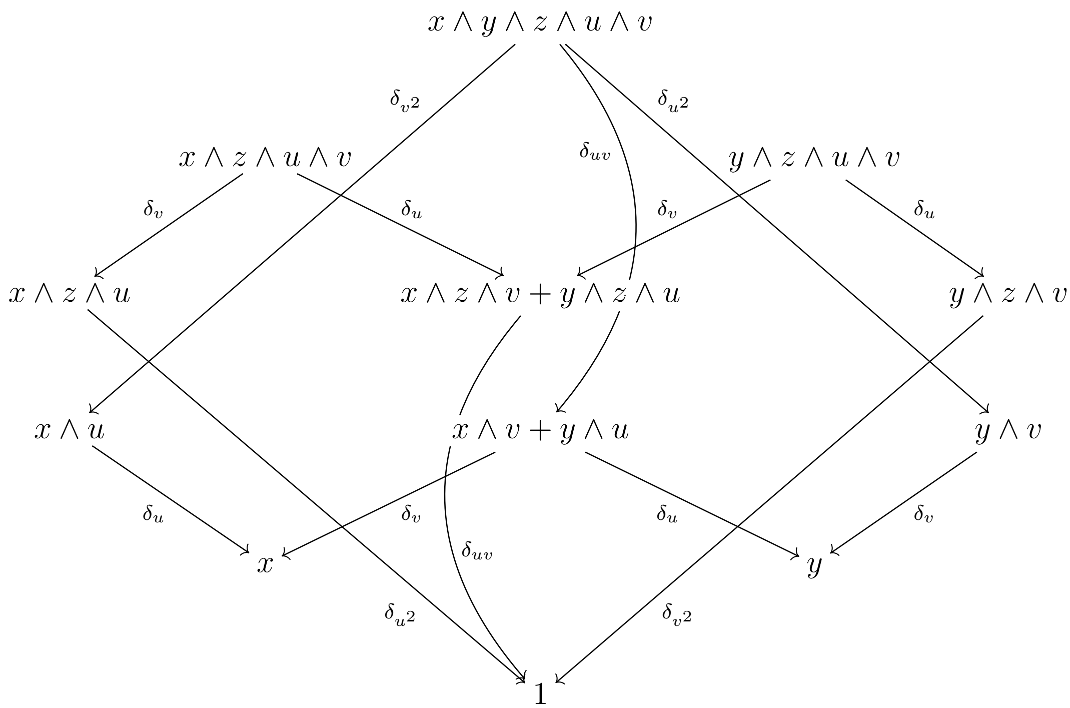
The harmonic forms in the diagram form a basis for $\mathcal{H}^*(\mathfrak{f}_3(2))$. Because $i^*_{u\wedge v} = i^*_v \circ i^*_u \in \operatorname{End} H^*(\mathfrak{f}_3(2))$ corresponds to $\pi i_v \circ \pi i_u = \delta_v \circ \delta_u \in \operatorname{End} \mathcal{H}^*(\mathfrak{f}_3(2))$ under the isomorphism between $H^*(\mathfrak{f}_3(2))$ and $\mathcal{H}^*(\mathfrak{f}_3(2))$, and since $\delta_v \circ \delta_u$ is zero on every basis element of $\mathcal{H}^*(\mathfrak{f}_3(2))$, it follows from Theorem 3.1.8 that $\mathfrak{f}_3(2)$ does not have a faithful central representation. However, since $\delta_u(x \wedge z \wedge u) = 0$, we evaluate the higher operation associated to the monomial u^2 on $x \wedge z \wedge u$ and find

$$\begin{aligned}
\delta_{u^2}(x \wedge z \wedge u) &= P_u Q_u (x \wedge z \wedge u) \\
&= \pi i_u \partial G i_u (x \wedge z \wedge u) \\
&= \pi i_u \partial G(du) \\
&= \pi i_u u \\
&= 1
\end{aligned}$$

since $x \wedge z = du$. Similarly,

$$\begin{aligned}
\delta_{uv}(x \wedge z \wedge v + y \wedge z \wedge u) &= \left(\frac{1}{2} P_u Q_v + \frac{1}{2} P_v Q_u\right)(x \wedge z \wedge v + y \wedge z \wedge u) \\
&= \left(\frac{1}{2} \pi i_u \partial G i_v + \frac{1}{2} \pi i_v \partial G i_u\right)(x \wedge z \wedge v + y \wedge z \wedge u) \\
&= \frac{1}{2} \pi i_u \partial G(du) + \frac{1}{2} \pi i_v \partial G(dv) \\
&= \frac{1}{2} \pi i_u u + \frac{1}{2} \pi i_v v \\
&= 1,
\end{aligned}$$

so we have a 2-cube in $\mathcal{H}^*(\mathfrak{f}_3(2))$ hanging off the vertex given by $x \wedge z \wedge u \wedge v$. The cohomology diagram from the previous page with all higher operations included is given below.



Here we see four 2-cubes in $\mathcal{H}^*(\mathfrak{f}_3(2))$. Notice that although primary and secondary operations don't commute, by permuting the monomials defining a composition of operations we obtain a pair of maps which commute up to a scalar multiple. For example, $\delta_{u^2}\delta_v \neq \delta_v\delta_{u^2}$, but $\delta_{u^2}\delta_v$ and $\delta_{uv}\delta_u$ commute up to a nonzero scalar multiple.

3.2.2 The free two-step nilpotent Lie algebras

We now consider the action of higher operations on the cohomology of the free two-step nilpotent Lie algebras and exhibit $\binom{r}{2}$-cubes in $\mathcal{H}^*(\mathfrak{f}_2(r))$ for $r = 2, 3$ and 4 (recall that $\dim Z(\mathfrak{f}_2(r)) = \binom{r}{2}$ for all r).

Beginning with the free two-step on 2 generators

$$\mathfrak{f}_2(2) = \langle e_1, e_2, f_{12} \,|\, [e_1, e_2] = f_{12} \rangle,$$

we fix the dual basis $\{e_1^*, e_2^*, f_{12}^*\}$ and calculate $de_1^* = de_2^* = 0$ and $df_{12}^* = e_1^* \wedge e_2^*$, from which it follows by Poincaré duality that

$$\begin{aligned}
\mathcal{H}^0(\mathfrak{f}_2(2)) &= \mathbb{F} \\
\mathcal{H}^1(\mathfrak{f}_2(2)) &= \operatorname{span}\{e_1^*,\, e_2^*\} \\
\mathcal{H}^2(\mathfrak{f}_2(2)) &= \operatorname{span}\{e_1^* \wedge f_{12}^*,\, e_2^* \wedge f_{12}^*\} \\
\mathcal{H}^3(\mathfrak{f}_2(2)) &= \operatorname{span}\{e_1^* \wedge e_2^* \wedge f_{12}^*\}.
\end{aligned}$$

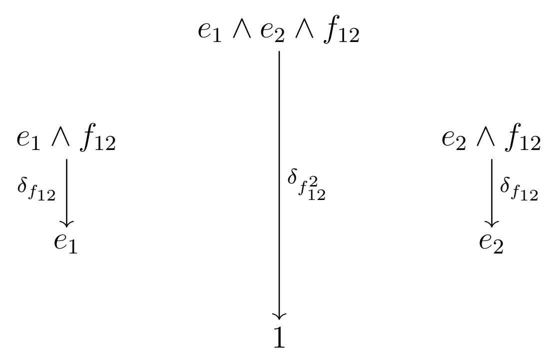
Evidently, the central representation is faithful on $e_1^* \wedge f_{12}^*$ and $e_2^* \wedge f_{12}^*$. The primary operation $\delta_{f_{12}}$ is trivial on the top form $e_1^* \wedge e_2^* \wedge f_{12}^*$, but since

$$i_{f_{12}}(e_1^* \wedge e_2^* \wedge f_{12}^*) = df_{12}^* = d\partial(e_1^* \wedge e_2^*),$$

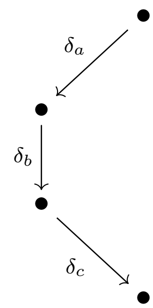
we have

$$\begin{aligned}
\delta_{f_{12}^2}(e_1^* \wedge e_2^* \wedge f_{12}^*) &= \pi i_{f_{12}} \partial G i_{f_{12}}(e_1^* \wedge e_2^* \wedge f_{12}^*) \\
&= \pi i_{f_{12}} \partial G(d\partial(e_1^* \wedge e_2^*)) \\
&= \pi i_{f_{12}} f_{12}^* \\
&= 1.
\end{aligned}$$

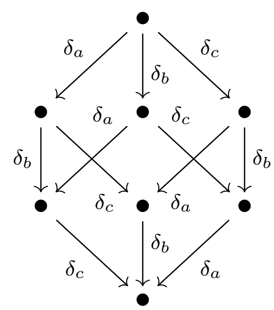
The cohomology diagram for $\mathfrak{f}_2(2)$ therefore contains three 1-cubes and looks as follows:



Before moving on to $\mathfrak{f}_2(r)$ for $r > 2$, we make a few remarks which will make our search for hypercubes in cohomology easier. We would like to show that, in certain cases, to find an n-cube in cohomology it suffices to find a single nonzero composition of n cohomology operations. For example, any non-zero path of primary operations of length three



automatically "fills out" a 3-cube since we can construct a 3-cube simply by considering the paths given by all permutations of δ_a, δ_b and δ_c of lengths 1, 2 and 3:



We know these paths are non-zero because primary operations anti-commute. The higher operations are, by comparison, more delicate: we cannot freely swap the positions of secondary operations in a path like we can with the primary ones. Instead, we show that on closed forms in the kernel of primary operations δ_a and δ_b, we have that $\delta_b\delta_{a^2} = -2\delta_a\delta_{ab}$.

Lemma 3.2.6. *Let $a, b \in Z(\mathfrak{g})$ and let $\omega \in \mathcal{H}^*(\mathfrak{g})$. If $i_a\omega$ and $i_b\omega$ are boundaries, then*

$$\frac{1}{2}i_a\partial G i_b(\omega) + \frac{1}{2}i_b\partial G i_a(\omega)$$

is closed.

Proof: By the Hodge decomposition theorem we have $i_a\omega = d\partial\alpha$ and $i_b\omega = d\partial\beta$

for some forms α and β. Therefore

$$\begin{aligned} d\left(\frac{1}{2}i_a\partial G i_b(\omega)+\frac{1}{2}i_b\partial G i_a(\omega)\right) &= -\frac{1}{2}(i_a d\partial G i_b(\omega)+i_b d\partial G i_a(\omega)) \\ &= -\frac{1}{2}(i_a d\partial G(d\partial\beta)+i_b d\partial G(d\partial\alpha)) \\ &= -\frac{1}{2}(i_a(d\partial\beta)+i_b(d\partial\alpha)) \\ &= -\frac{1}{2}(i_a i_b\omega+i_b i_a\omega) \\ &= 0. \end{aligned}$$

∎

Note that any closed form ω is cohomologous to $\pi\omega$ so that, under the hypotheses above, $\delta_{ab}(\omega)$ and $\frac{1}{2}i_a\partial G i_b(\omega)+\frac{1}{2}i_b\partial G i_a(\omega)$ only differ by a boundary.

Lemma 3.2.7. *Let $a, b, c \in Z(\mathfrak{g})$ and let $\omega \in \mathcal{H}^*(\mathfrak{g})$. If $\delta_a(\omega)=\delta_b(\omega)=\delta_c(\omega)=0$, then*

$$(\delta_a\delta_{bc}+\delta_b\delta_{ac}+\delta_c\delta_{ab})(\omega)=0.$$

Proof: We will show that $(\delta_a\delta_{bc}+\delta_b\delta_{ac})(\omega)=-\delta_c\delta_{ab}(\omega)$. We have

$$\begin{aligned} &(\delta_a\delta_{bc}+\delta_b\delta_{ac})(\omega) \\ &= \pi i_a\left(\frac{1}{2}\pi i_b\partial G i_c(\omega)+\frac{1}{2}\pi i_c\partial G i_b(\omega)\right)+\pi i_b\left(\frac{1}{2}\pi i_a\partial G i_c(\omega)+\frac{1}{2}\pi i_c\partial G i_a(\omega)\right) \\ &= \pi i_a\left(\frac{1}{2}i_b\partial G i_c(\omega)+\frac{1}{2}i_c\partial G i_b(\omega)\right)+\pi i_b\left(\frac{1}{2}i_a\partial G i_c(\omega)+\frac{1}{2}i_c\partial G i_a(\omega)\right) \\ &= -\frac{1}{2}\pi i_b i_a\partial G i_c(\omega)-\frac{1}{2}\pi i_c i_a\partial G i_b(\omega)+\frac{1}{2}\pi i_b i_a\partial G i_c(\omega)-\frac{1}{2}\pi i_c i_b\partial G i_a(\omega) \\ &= -\pi i_c\left(\frac{1}{2}i_a\partial G i_b(\omega)+\frac{1}{2}i_b\partial G i_a(\omega)\right) \\ &= -\delta_c\delta_{ab}(\omega), \end{aligned}$$

where the second and last equalities follow from Lemma 3.2.6. ∎

It now follows from the Jacobi-like identity above that if $\delta_a(\omega) = \delta_b(\omega) = 0$, then $\delta_b\delta_{a^2}(\omega) = -2\delta_a\delta_{ab}(\omega)$.

Theorem 3.2.8. *Let* $\dim \mathfrak{g} = n$ *and let* $0 \neq \tau \in \mathcal{H}^n(\mathfrak{g})$. *If all primary operations are zero on* τ *and* $\delta_{z_k}\delta_{z_{k-1}} \cdots \delta_{z_2}\delta_{z_1^2}(\tau) \neq 0$ *for some* $z_1, \ldots, z_k \in Z(\mathfrak{g})$, *then there exists a* k*-cube in* $\mathcal{H}^*(\mathfrak{g})$.

Proof: Writing $\delta(s)$ for δ_s, any composition of operations $\delta(z_k)\delta(z_{k-1}) \cdots \delta(z_2)\delta(z_1^2)$ (i.e., a secondary followed by primaries) which is nonzero on τ fills out a k-cube in $\mathcal{H}^*(\mathfrak{g})$ as follows:

- The single vertex at level k is given by $\text{span}\{\tau\}$ with k edges directed from it to level $k-1$ given by the secondary operations $\delta(z_1^2)$, $\delta(z_1z_2)$, $\delta(z_1z_3), \ldots, \delta(z_1z_{k-1})$ and $\delta(z_1z_k)$;

- Level $k-1$ consists of the $\binom{k}{k-1} = k$ vertices

  $\text{span}\{\delta(z_1^2)(\tau)\}$, $\text{span}\{\delta(z_1z_2)(\tau)\}, \ldots, \text{span}\{\delta(z_1z_{k-1})(\tau)\}$ and $\text{span}\{\delta(z_1z_k)(\tau)\}$.

  Each of these clearly has 1 edge directed to it from the level above, as required, and we define $k-1$ edges directed from each vertex $\text{span}\{\delta(z_1z_j)(\tau)\}$ to level $k-2$ to be the primary operations $\delta(z_1)$, $\delta(z_2), \ldots, \widehat{\delta(z_j)}, \ldots, \delta(z_k)$;

- Similarly, level i consists of $\binom{k}{i}$ vertices, each given by

  $$\text{span}\{\delta(z_{j_{\sigma(k-i)}})\delta(z_{j_{\sigma(k-i-1)}}) \cdots \delta(z_{j_{\sigma 2}})\delta(z_1z_{j_{\sigma 1}})(\tau) \mid \sigma \in S_i\}$$

  for some choice of $k-i$ elements $j_1, \ldots, j_{k-i} \in \{1, \ldots, k\}$. Each of these has $k-i$ edges directed to it from $k-i$ distinct vertices in the level above it, and each has i edges directed from it to i distinct vertices below given by the primary operations $\delta(z_j)$ for $j \notin \{j_1, \ldots, j_{k-i}\}$.

To see that all of the vertices thus obtained are nontrivial subspaces of $\mathcal{H}^*(\mathfrak{g})$, note that $\delta(z_k)\delta(z_{k-1})\cdots\delta(z_2)\delta(z_1^2)(\tau) \neq 0$ implies that

$$\delta(z_k)\delta(z_{k-1})\cdots\delta(z_{j+1})\delta(z_1)\delta(z_{j-1})\cdots\delta(z_2)\delta(z_1 z_j)(\tau) \neq 0,$$

for all $j = 1, \ldots, k$, since primary operations anti-commute and $\delta(z_j)\delta(z_1^2)(\tau) = -2\delta(z_1)\delta(z_1 z_j)(\tau)$ by Lemma 3.2.7.

Clearly, it suffices to check for independence only among vertices at the same level of the k-cube since vertices at distinct levels are of distinct homogeneous degree and the cohomology groups $\mathcal{H}^0(\mathfrak{g}), \mathcal{H}^1(\mathfrak{g}), \ldots, \mathcal{H}^n(\mathfrak{g})$ are independent subspaces of $\mathcal{H}^*(\mathfrak{g})$. So suppose $c_1\gamma_1 + \cdots + c_m\gamma_m = 0$ for $c_1, \ldots, c_m \in \mathbb{F}$ and $\gamma_1, \ldots, \gamma_m$ nonzero harmonic forms from m distinct vertices at some level of the k-cube. But by construction of the vertices, there exists a sequence of primary operations which sends some γ_i, $1 \leq i \leq m$, to a nonzero element of the single vertex at level 0 of the cube, and which kills each γ_j, $j \neq i$ (this follows from the fact that primary operations square to zero and that $\delta(z_1)\delta(z_1^2) = 0$). Repeating this we obtain $c_1 = \cdots = c_m = 0$, as required. ∎

Returning to the free two-step nilpotent Lie algebras, consider now the case of 3 generators:

$$\mathfrak{f}_2(3) = \langle e_1, e_2, e_3, f_{12}, f_{13}, f_{23} \,|\, [e_i, e_j] = f_{ij}, 1 \leq i < j \leq 3\rangle.$$

We have that $Z(\mathfrak{f}_2(3)) = \langle f_{12}, f_{13}, f_{23}\rangle$ and it is easy to see that the central representation is not faithful. Moreover, all primary operations are trivial on $\mathcal{H}^6(\mathfrak{f}_2(3))$, so to find a 3-cube in $\mathcal{H}^*(\mathfrak{f}_2(3))$ it suffices to find a sequence of 3 cohomology operations $\delta_c\delta_b\delta_{a^2}$, where $a, b, c \in Z(\mathfrak{f}_2(3))$, which is nonzero on $\tau = e_1^* \wedge e_2^* \wedge e_3^* \wedge f_{12}^* \wedge f_{13}^* \wedge f_{23}^* \in \mathcal{H}^6(\mathfrak{f}_2(3))$.

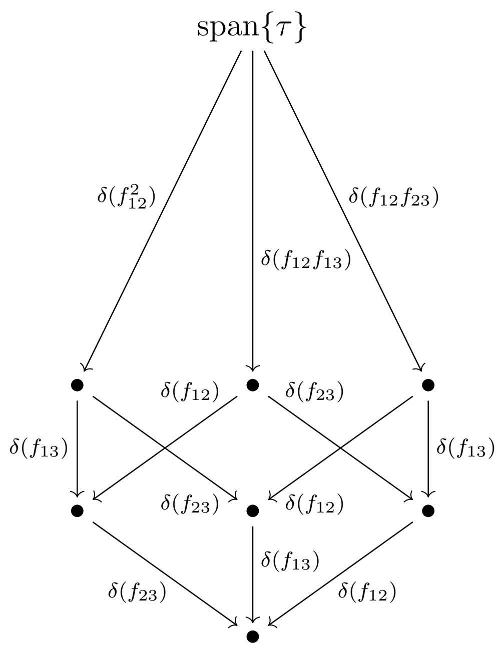
Observe that $i_{f_{12}}\tau$ is a boundary and is therefore in the image of the Laplacian:

$$i_{f_{12}}\tau = -\widehat{f_{12}^*} = d(e_3^* \wedge f_{12}^* \wedge f_{13}^* \wedge f_{23}^*) = d\partial(-\widehat{f_{12}^*}) = \Delta(-\widehat{f_{12}^*}).$$

Here, $\widehat{f_{12}^*}$ means $e_1^* \wedge e_2^* \wedge e_3^* \wedge f_{13}^* \wedge f_{23}^*$. So

$$\begin{aligned}
\delta_{f_{12}^2}(\tau) &= \pi i_{f_{12}} \partial G i_{f_{12}}(\tau) \\
&= \pi i_{f_{12}} \partial G(\Delta(-\widehat{f_{12}^*})) \\
&= \pi i_{f_{12}}(e_3^* \wedge f_{12}^* \wedge f_{13}^* \wedge f_{23}^*) \\
&= \pi(-e_3^* \wedge f_{13}^* \wedge f_{23}^*) \\
&= -e_3^* \wedge f_{13}^* \wedge f_{23}^*
\end{aligned}$$

since $-e_3^* \wedge f_{13}^* \wedge f_{23}^*$ is harmonic. It is now clear that $\delta_{f_{23}}\delta_{f_{13}}\delta_{f_{12}^2}(\tau) = -e_3^*$, which shows that there exists a 3-cube of the form



in the cohomology of $\mathfrak{f}_2(3)$ with vertices defined by the construction in the proof of Theorem 3.2.8.

The free two-step on 4 generators is given by

$$\mathfrak{f}_2(4) = \langle e_1, e_2, e_3, e_4, f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34} \,|\, [e_i, e_j] = f_{ij}, 1 \leq i < j \leq 4 \rangle.$$

Let $\tau = e_1^* \wedge \cdots \wedge e_4^* \wedge f_{12}^* \wedge \cdots \wedge f_{34}^*$. It is again clear that all primary operations are trivial on $\mathcal{H}^{10}(\mathfrak{f}_2(4))$ since $\widehat{f_{ij}^*} = \pm d(\widehat{e_i^* e_j^*})$ is a boundary for all $f_{ij} \in Z(\mathfrak{f}_2(4))$. A calculation similar to the one in the previous example shows that

$$\delta_{f_{12}^*}(\tau) = \widehat{e_1^* e_2^* f_{12}^*} = e_3^* \wedge e_4^* \wedge f_{13}^* \wedge f_{14}^* \wedge f_{23}^* \wedge f_{24}^* \wedge f_{34}^*,$$

from which it follows that

$$\delta_{f_{24}}\delta_{f_{23}}\delta_{f_{14}}\delta_{f_{13}}\delta_{f_{12}^2}(\tau) = \pi(e_3^* \wedge e_4^* \wedge f_{34}^*) = e_3^* \wedge e_4^* \wedge f_{34}^*$$

since $e_3^* \wedge e_4^* \wedge f_{34}^*$ is harmonic. Note that this is only a sequence of 5 operations and $\dim Z(\mathfrak{f}_2(4)) = 6$. Applying $\delta_{f_{34}}$ next gives zero since $e_3^* \wedge e_4^* = df_{34}^*$ is a boundary, so we apply the secondary operation $\delta_{f_{34}^2}$ instead:

$$\delta_{f_{34}^2}(e_3^* \wedge e_4^* \wedge f_{34}^*) = 1.$$

We now have a 6-cube in $\mathcal{H}^*(\mathfrak{f}_2(4))$ whose top six levels are as described in the proof of Theorem 3.2.8, i.e.,

- Level 6 consists of the vertex given by $\operatorname{span}\{\tau\}$. The edges directed from level 6 to level 5 are given by $\delta(f_{12}^2), \delta(f_{12}f_{13}), \ldots, \delta(f_{12}f_{34})$;

- For $i = 1, \ldots, 5$, level i consists of $\binom{6}{i}$ vertices, each given by

$$\operatorname{span}\{\delta(z_{\sigma(6-i)})\delta(z_{\sigma(6-i-1)}) \cdots \delta(z_{\sigma 2})\delta(z_1 z_{\sigma 1})(\tau) \mid \sigma \in S_i\}$$

  for some choice of $6-i$ elements $z_1, \ldots, z_{6-i} \in \{f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}\}$. There is an obvious edge from any given vertex to one below it whenever the subsets of $\{f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}\}$ defining the two vertices differ by exactly one element. For example, the vertex at level 5 given by $\operatorname{span}\{\delta(f_{12}f_{34})(\tau)\}$ is mapped nontrivially by $\delta(f_{13})$ into the vertex at level 4 given by

$$\operatorname{span}\{\delta(f_{34})\delta(f_{12}f_{13})(\tau),\ \delta(f_{13})\delta(f_{12}f_{34})(\tau)\}.$$

Note that Theorem 3.2.8 does not ensure that vertices defined by subsets of $\{f_{12}, f_{13}, f_{14}, f_{23}, f_{24}, f_{34}\}$ containing f_{34} are nontrivial, but this can be verified by checking that all 6 vertices at level 1 are nontrivial since each vertex above level 1 is mapped nontrivially into a vertex at level 1 by a sequence of operations. Similarly, the independence of the 2^6 vertices of the 6-cube can be verified simply by checking that the 6 vertices at level 1 are independent since any dependence relation among elements from distinct vertices at level k for $k > 1$ can be turned into a dependence relation among elements in distinct vertices at level 1 by applying a sequence of primary operations.

The single vertex at level 0 is given by span$\{1\}$. We have already seen that the vertex containing the element $\delta_{f_{24}}\delta_{f_{23}}\delta_{f_{14}}\delta_{f_{13}}\delta_{f_{12}^2}(\tau)$ is mapped nontrivially into level 0 by a secondary operation, and the same can be checked by hand for the other five vertices.

3.2.3 Nilpotent Lie algebras associated to graphs

In [8], to every finite graph is associated a two-step nilpotent Lie algebra in such a way that two graphs are isomorphic if and only if the associated Lie algebras are isomorphic. To illustrate that z-cubes exist in the cohomology of many algebras where the central representation is not faithful (that is, z-cubes made up of higher operations), we examine as an example a class of nilpotent Lie algebras attached to graphs.

A *finite (simple) graph* is a pair (V, E) where V is a finite set called the set of vertices and E is a set of unordered pairs of distinct vertices called the set of edges (so such graphs are undirected). We can associate to any finite graph a nilpotent Lie algebra $\mathcal{L}(V, E)$ in the following way: let X be the real vector space with basis V, and let Y be the subspace of $\Lambda^2 X$ defined by

$$Y = \text{span}\{a \wedge b \,|\, (a, b) \in E\};$$

now let $\mathcal{L}(V, E)$ be the Lie algebra with underlying vector space $X \oplus Y$ and non-trivial brackets on basis elements given by $[a, b] = a \wedge b$ if $a, b \in V$ and $(a, b) \in E$. Therefore,

$\mathcal{L}(V, E)$ is two-step nilpotent when the set of edges E is nonempty.

Example 3.2.9. If $(V, \varnothing)$ is a graph with no edges, then $\mathcal{L}(V, \varnothing)$ is the abelian Lie algebra of dimension $|V|$.

Example 3.2.10. The Lie algebra associated to the complete graph on r vertices is the free two-step nilpotent Lie algebra on r generators $\mathfrak{f}_2(r)$.

If $\mathcal{L}(V, E)$ is the nilpotent Lie algebra associated to a graph (V, E), then it has a basis of the form $\{e_1, \ldots, e_n\} \cup \{f_{ij} = e_i \wedge e_j \,|\, (e_i, e_j) \in E \text{ and } i < j\}$ where n is the cardinality of V. Note that $de_i^* = 0$ for all $i = 1, \ldots, n$ and $df_{ij}^* = e_i^* \wedge e_j^* \in \Lambda^2 \mathcal{L}(V, E)^*$ for all f_{ij}^* in the standard dual basis $\{e_1^*, \ldots, e_n^*\} \cup \{f_{ij}^* \,|\, (e_i, e_j) \in E \text{ and } i < j\}$. For each graph (V, E), let $\tau = e_1^* \wedge \cdots \wedge e_n^* \wedge f^* \in \mathcal{H}^*(\mathcal{L}(V, E))$ where f^* denotes the wedge product of all f_{ij}^* in lexicographic order.

Note that for a disconnected graph (V, E) with connected components $(V_1, E_1), \ldots, (V_k, E_k)$, we have

$$\mathcal{L}(V, E) = \bigoplus_{p=1}^{k} \mathcal{L}(V_p, E_p),$$

so for our purposes we may restrict our attention to connected graphs.

Lemma 3.2.11. *Let (V, E) be a connected graph and let $\mathcal{L}(V, E)$ be its associated nilpotent Lie algebra with centre Z. Then $i_z\tau$ is a boundary and $\Delta i_z(\tau) = \pm i_z\tau$ for all $z \in Z$.*

Proof: Let $\{e_1^*, \ldots, e_n^*\} \cup \{f_{ij}^* \,|\, (e_i, e_j) \in E,\, i < j\}$ be the standard dual basis of $\mathcal{L}(V, E)^*$. Since (V, E) is connected, the centre of $\mathcal{L}(V, E)$ is

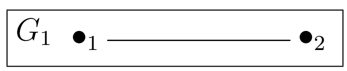
$$Z = [\mathcal{L}(V, E), \mathcal{L}(V, E)] = \operatorname{span}\{f_{ij} \,|\, (e_i, e_j) \in E\}.$$

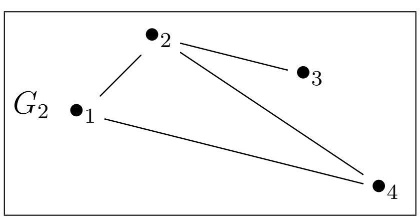
So let $f_{ab} \in Z$. Then $i_{f_{ab}}\tau = \pm\widehat{f_{ab}^*} = d(\pm e_1^* \wedge \cdots \wedge \widehat{e_a^*} \wedge \cdots \wedge \widehat{e_b^*} \wedge \cdots \wedge e_n^* \wedge f^*)$. Hence

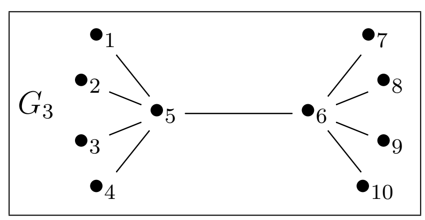
$$\begin{aligned}\Delta i_{f_{ab}}\tau &= (d\partial + \partial d)(i_{f_{ab}}\tau)\\ &= d\partial(\pm\widehat{f_{ab}^*})\\ &= \pm d * d * (\widehat{f_{ab}^*})\\ &= \pm d(e_1^* \wedge \cdots \wedge \widehat{e_a^*} \wedge \cdots \wedge \widehat{e_b^*} \wedge \cdots \wedge e_n^* \wedge f^*)\\ &= \pm i_{f_{ab}}\tau.\end{aligned}$$

∎

Consider the following graphs.







The associated nilpotent Lie algebras admit the following presentations:

$$\mathcal{L}(G_1) = \langle e_1, e_2, f_{12} \,|\, [e_1, e_2] = f_{12}\rangle$$

$$\begin{aligned}\mathcal{L}(G_2) = \langle e_1, e_2, e_3, e_4, f_{12}, f_{14}, f_{23}, f_{24} \,|\, &[e_1, e_2] = f_{12}, [e_1, e_4] = f_{14},\\ &[e_2, e_3] = f_{23}, [e_2, e_4] = f_{24}\rangle\end{aligned}$$

$$\begin{aligned}\mathcal{L}(G_3) = \langle e_1, \ldots, e_{10}, f_{15}, \ldots, f_{45}, f_{56}, f_{67}, \ldots, f_{6,10} \,|\, &[e_1, e_5] = f_{15}, [e_2, e_5] = f_{25},\\ &[e_3, e_5] = f_{35}, [e_4, e_5] = f_{45}, [e_5, e_6] = f_{56}, [e_6, e_7] = f_{67}, [e_6, e_8] = f_{68},\\ &[e_6, e_9] = f_{69}, [e_6, e_{10}] = f_{6,10}\rangle.\end{aligned}$$

In each graph above there is an edge adjacent to all other edges in the graph: $(1,2),(2,4)$ and $(5,6)$ in G_1, G_2 and G_3, respectively. We will prove that for any graph (V,E) with at least one such distinguished edge, the Lie algebra $\mathcal{L}(V,E)$ has a z-cube in its cohomology where $z = \dim Z(\mathcal{L}(V,E))$.

Definition 3.2.12. *Call a graph (V,E) an* edge-star graph *if (V,E) is connected and there exists an edge $(a,b) \in E$ such that $\{a,b\} \cap \{k,l\} \neq \varnothing$ for each edge $(k,l) \in E$. Such an edge (a,b) is called a* central edge*.*

Note that the centre of such a graph has dimension $|E|$.

Theorem 3.2.13. *If (V,E) is an edge-star graph, then there exists a cube of dimension $|E|$ in $\mathcal{H}^*(\mathcal{L}(V,E))$.*

Proof: Let (a,b) be a central edge in (V,E). Then there exists a basis of the form $\{e_1^*, \ldots, e_n^*, f_{ab}^*\} \cup \{f_{aj}^* \,|\, (e_a, e_j) \in E, j \neq b\} \cup \{f_{bj}^* \,|\, (e_b, e_j) \in E, j \neq a\}$ for $\mathcal{L}(V,E)^*$ where $df_{ij}^* = e_i^* \wedge e_j^*$ for each basis vector f_{ij}^*, and $de_i^* = 0$ for $i = 1, \ldots, n$. Let τ denote the top harmonic form $e_1^* \wedge \cdots \wedge e_n^* \wedge f^* \in \mathcal{H}^{|V|+|E|}(\mathcal{L}(V,E))$, where f^* denotes the wedge product of all f_{ij}^* in the dual basis above, and let $\widehat{e_i^* f_{kl}^*}$ denote the wedge product of all basis elements except e_i^* and f_{kl}^*. Then by Lemma 3.2.11, $i_{f_{ab}}\tau$ is a boundary and $\Delta i_{f_{ab}}(\tau) = \pm i_{f_{ab}}\tau$. Recalling that Green's function G acts as the inverse of the Laplacian Δ on $\operatorname{Im}\Delta$, we calculate

$$\begin{aligned}
\delta(f_{ab}^2)(\tau) &= \pi i_{f_{ab}} \partial G i_{f_{ab}}(\tau) \\
&= \pi i_{f_{ab}} \partial(\pm i_{f_{ab}}(\tau)) \\
&= \pm \pi i_{f_{ab}} \star d \star (\widehat{f_{ab}}) \\
&= \pm \pi i_{f_{ab}} (\widehat{e_a^* e_b^*}) \\
&= \pm \pi (\widehat{e_a^* e_b^* f_{ab}^*}) \\
&= \pm \widehat{e_a^* e_b^* f_{ab}^*}.
\end{aligned}$$

Next we apply all primary operations associated to each remaining $f_{ij} \neq f_{ab}$ in the basis and find

$$\delta(f_{aj_1})\cdots\delta(f_{aj_p})\delta(f_{bk_1})\cdots\delta(f_{bk_q})\delta(f_{ab}^2)(\tau) = \pm\pi(\widehat{e_a^* e_b^* f^*}).$$

Since $\widehat{e_a^* e_b^* f^*}$ and $*(\widehat{e_a^* e_b^* f^*}) = \pm e_a^* \wedge e_b^* \wedge f^*$ are both closed, it follows that $\widehat{e_a^* e_b^* f^*}$ is harmonic, so $\delta(f_{aj_1})\cdots\delta(f_{aj_p})\delta(f_{bk_1})\cdots\delta(f_{bk_q})\delta(f_{ab}^2)(\tau) \neq 0$. Furthermore, by Lemma 3.2.11, all primary operations are zero on τ, so by Theorem 3.2.8 there exists a $|E|$-cube in $\mathcal{H}^*(\mathcal{L}(V,E))$. ∎

Chapter 4

Open Questions

Recall the Dixmier long exact sequence from Section 2.4. It is easy to see that the derivation $\theta^* : H^*(\mathfrak{h}) \to H^*(\mathfrak{h})$ on the cohomology of a codimension one ideal $\mathfrak{h} \subseteq \mathfrak{g}$ is equivariant with respect to the action of ΛZ (where Z is the centre of $\mathfrak{g}$ and ΛZ acts on $H^*(\mathfrak{h})$ in the obvious way by primary operations). The Dixmier long exact sequence is in fact a long exact sequence of ΛZ-modules. While the short exact sequence

$$0 \longrightarrow H^{p-1}(\mathfrak{h})/\operatorname{im}\theta^*_{p-1} \xrightarrow{\omega\wedge-} H^p(\mathfrak{g}) \xrightarrow{\rho} \ker\theta^*_p \longrightarrow 0$$

does not split as a sequence of modules, we of course have that a cube in $\ker\theta^*$ implies the existence of a cube in $H^*(\mathfrak{g})$. If $\alpha \in \ker\theta^*$, then the orbit of α under the action of ΛZ is also contained in $\ker\theta^*$. This gives a quick proof of the fact that any Lie algebra with an abelian ideal of codimension one has a faithful central representation. Since nilpotent Lie algebras contain ideals of arbitrary codimension, a potential approach to finding a general proof of the TRC for nilpotent Lie algebras is to try to compare the cohomology of $\mathfrak{g}$ to the cohomology of a chain of codimension one ideals using the induced derivations θ^*. As far as we know, the question of whether the TRC holds for Lie algebras containing an abelian ideal of codimension two is open, for example.

Nilpotent Lie algebras over a field of characteristic not equal to two have been classified up to dimension six. Cubes were found in the cohomology of every real nilpotent Lie algebra of dimension less than or equal to five (the 6-dimensional algebras have not been checked exhaustively). In each case, a sequence of cohomology operations is nonzero on a top cohomology class and this sequence fills out a cube of dimension $\dim Z$ as described in Section 3.2. One can show that there exists a nonzero sequence of $\dim Z$ operations, primary and secondary, in the cohomology of the free two-steps $\mathfrak{f}_2(r)$ for all r, but none were found to form a cube of the desired size in $H^*(\mathfrak{f}_2(5))$. It would be interesting to look for commutativity relations (similar to the one described in Lemma 3.2.7) among higher operations associated to monomials of degree greater than two in order to find other sequences of operations which imply the existence of cubes in cohomology.

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