The Origin of Wave Blocking for a Bistable Reaction-Diffusion Equation : A General Approach

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Abstract

Mathematical models displaying travelling waves appear in a variety of domains. These waves are often faced with different kinds of perturbations. In some cases, these perturbations result in propagation failure, also known as wave-blocking. Wave-blocking has been studied in the case of several specific models, often with the help of numerical tools. In this thesis, we will display a technique that uses symmetry and a center manifold reduction to find a criterion which defines regions in parameter space where a wave will be blocked. We focus on waves with low velocity and small symmetry-breaking perturbations, which is where the blocking initiates; the organising center. The range of the tools used makes the technique easily generalizable to higher dimensions. In order to demonstrate this technique, we apply it to the bistable equation. This allows us to do calculations explicitly. As a result, we show that wave-blocking occurs inside a wedge originating from the organising center and derive an expression for this wedge to leading order. We verify our results with some numerical simulations.

Résumé

Les modèles mathématiques qui admettent des ondes progressives comme solutions apparaissent dans une multitude de domaines. Ces ondes font souvent face à différents genres de perturbations. Dans certains cas, ces perturbations peuvent causer un blocage d'onde. Le phénomène de blocage d'onde à été étudié dans le cas de plusieurs modèles spécifiques, souvent à l'aide d'outils numériques. Dans cette thèse, nous allons démontrer une technique qui utilise la symmétrie ainsi qu'une réduction à une variété du centre afin de trouver un critère qui nous permet de déterminer quand le blocage d'onde prend lieu. Nous dirigeons notre attention vers les ondes à faible vitesse et aux petites perturbations qui brisent la symétrie de translation des ondes, car c'est là où le blocage débute; le centre d'organisation. La grande portée des outils utilisés rend la technique facile à généraliser à des dimensions supérieures. Afin de démontrer cette technique, nous allons l'appliquer à l'équation bistable. Ceci nous permet de faire les calculs de façon explicite. Comme résultat, nous obtenons que le blocage d'onde se produit entre deux courbes qui émanent du centre d'organisation, et nous dérivons des expressions pour ces courbes.

Acknowledgements

I would like to thank my supervisor, Dr. Victor LeBlanc, for his guidance throughout this project. This project would have been impossible without his unparalleled devotion and support. I also want to express my gratitude to the Ontario Graduate Scholarship Program (OGS) and to the Natural Sciences and Engineering Research Council of Canada (NSERC) for financial support.

Dedication

I dedicate this work to my parents. They have always supported and encouraged me, which has led me to become who I am. As such, am very grateful and proud to be their son.

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Chapter 1

Introduction

1.1 Introduction

Picture yourself at beach during sunset. Looking at the horizon, you see a beautiful gradient of yellow and red as the sun is about to disappear. You then turn your eyes to the sea, and watch closely as waves keep coming towards the shore. One after the other they fade away as they reach the sand, only to leave behing them the soothing sound of seaside...

Seaside isn't the only place where we can observe waves, in fact, we can observe travelling waves in a multitude of scenarios, some of which are playing an essential role in our daily lives. For example, electrical signals transmitted through our neurons can be described as travelling waves [1, 2]. When stimulated, cells in neurons undergo an action potential; a brief but significant increase in electrical potential, which is then quickly followed by a decrease. This increase in potential excites nearby cells, making them go through the same procedure. Therefore, we observe an electrical impulse propagating through the neurons as a travelling wave (See Figures (1.1) and (1.2)).

In cardiac electrophysiology, the same phenomenon of action potentials propagating though the electrically excitable cells of the heart is responsible for the pumping

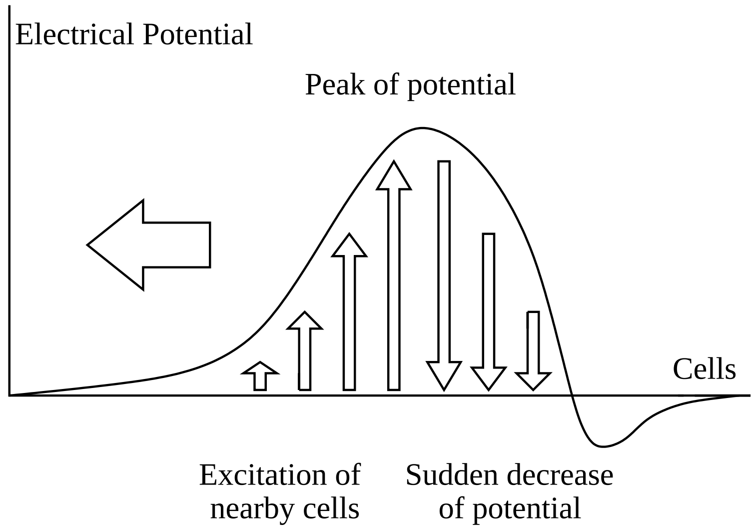

Figure 1.1: Wave formed by the action potential

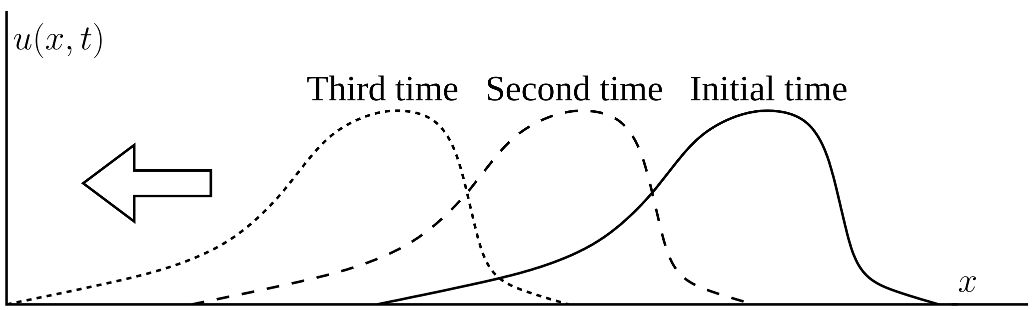

Figure 1.2: Picture of a travelling wave at different times.

of blood through the human body [11]. Travelling waves can also be observed in combustion experiments, where fire spreads to new regions in wave-like patterns [14]. As an other application, travelling waves appear in models of tumor growth [3]. It certainly isn't an overstatement to say that travelling waves are important objects to study, with a multitude of applications.

In this thesis, we will explore the interaction between travelling waves and perturbations. Whether it is a partially blocked artery, a defective neuron cell, or wanting to stop the progression of a fire, it is important to understand under which conditions travelling waves will proceed with their normal course, and the conditions which will halt the waves in their path. In particular, we will look at slow waves and small perturbations, where the blocking initiates. This allows us to obtain a result that is generalizable to many different kinds of travelling waves, and explains a fundamental property of wave blocking.

In order to analyse travelling waves mathematically, we need a model. Reaction-diffusion equations are natural candidates when it comes to modelling travelling waves as their terms relate closely to the phenomena studied. In one-dimension, a typical reaction-diffusion equation would be:

$$u_t(x,t) = Du_{xx}(x,t) + f(u(x,t)) \tag{1.1.1}$$

The function $u(x,t)$ represents a wave in space and time, and can be considered as $u(x)(t)$, where $u(x)$ is the spatial wave profile, which changes with time. As such, $x \in \mathbb{R}$, $t \in \mathbb{R}^+$. The term Du_{xx} is refered to as the "diffusion" term, since it represents well experimentally observed diffusion behaviours. In particular, if $f(u) = 0$, then we obtain the heat equation $u_t = Du_{xx}$, which models the diffusion of heat. The second term in our equation, $f(u(x,t))$ is called the "reaction" term. It represents the autocatalytic process, specific to the situation we are studying. For example, in the case of electrically excitable cells, a punctual stimulation of some cells can be given as an initial state. The diffusion term then transfers some of the potential to nearby cells. Once a treshold value is reached, the reaction term comes in and increases the potential until the cell reaches its peak value. If the cells maintain their peak values, we are in the presence of a special type of travelling wave called a wavefront, whose general form is depicted in Figure (1.3).

Now that we have equation (1.1.1) to model our situation, we need to express mathematically the form of the travelling waves we wish to study. A travelling wave of speed c, is a solution $u(x,t)$ to the differential equation which satisfies $u(x,t) =$

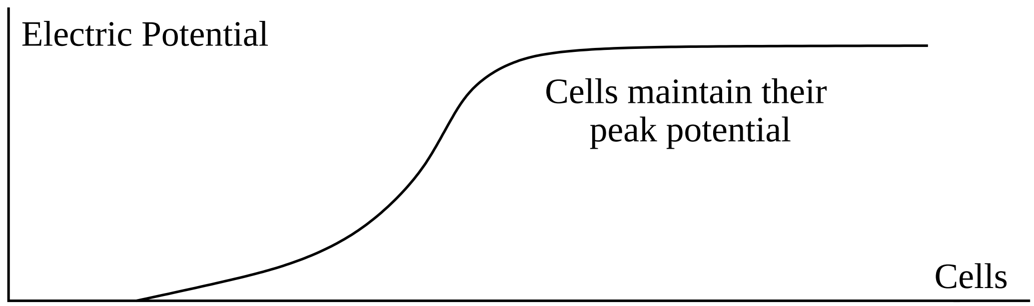

Figure 1.3: A typical wavefront

$u(x+ct, 0)$, where $u(x, 0) = u(x)$ is called the initial profile of the travelling wave. In other words, the evolution of the solution in time can be described by a translation at speed c of the wave profile. Using this parametrisation, u becomes a function of one variable ($\xi = x + ct$), which reduces (1.1.1) to

$$cu'(\xi) = Du''(\xi) + f(u(\xi)) \tag{1.1.2}$$

For a function $u(x)$ to be a wave profile, it must tend to a limit as $x \to \pm\infty$. This gives rise to two types of profiles; A pulse $u(x)$, as represented in Figure (1.1), is characterized by

$$\lim_{x\to-\infty} u(x) = \lim_{x\to\infty} u(x) \ , \ \lim_{x\to-\infty} u'(x) = \lim_{x\to\infty} u'(x) = 0$$

while a wavefront, represented in Figure (1.3) is characterized by

$$\lim_{x\to-\infty} u(x) \neq \lim_{x\to\infty} u(x) \ , \ \lim_{x\to-\infty} u'(x) = \lim_{x\to\infty} u'(x) = 0$$

assuming that the limits exist.

In order to find travelling wave solutions, the technique of phase plane analysis is often used. Looking for a solution to (1.1.2) we introduce a new variable to transform this second order equation into the system:

$$u_\xi = v$$

$$v_\xi = \frac{1}{D}\left(cv - f(u)\right)$$

We can easily draw the vector field for this system in the (u, v)-plane (also called the phase-plane), and therefore, look for some specific orbits. In order to find these orbits, we look at equilibrium points of the system, their stability, and possible trajectories between them. A travelling pulse must have the same limits as x goes to $\pm\infty$, which can be represented by a homoclinic orbit (the orbit begins and ends at the same equilibrium) in the phase plane. On the other hand, a travelling front, must have different limits at $\pm\infty$, and hence, we must look for heteroclinic orbits (an orbit with different starting and ending equilibria).

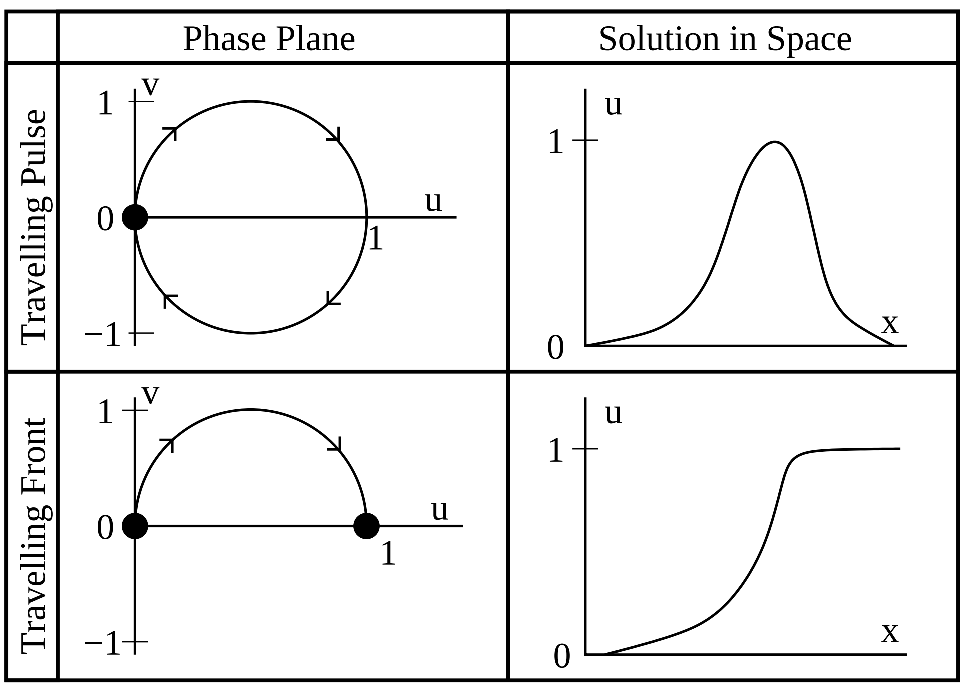

Figure 1.4: Orbits in the phase plane and their solution counterparts.

As an example, we consider a particular case of (1.1.1) with a cubic reaction term [7],

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)(u(x,t)-1)(u(x,t)-\alpha) \tag{1.1.3}$$

considering $\alpha \in (0,1)$ as a parameter.

Looking for a travelling wave, we apply the change of coordinates from (1.1.2),

$$cu' = u'' - u(u-1)(u-\alpha) \tag{1.1.4}$$

Next, we consider the reduced system

$$\begin{aligned} u_\xi &= v \\ v_\xi &= cv + u(u-1)(u-\alpha) \end{aligned} \tag{1.1.5}$$

The equilibrium points of this system are found when $(u_\xi, v_\xi) = (0,0)$, which forces $v = 0$ and $u = 0, 1$ or α. To find the stability of each of those equilibrium points, we consider the Jacobian of the system,

$$J(u,v) = \begin{pmatrix} 0 & 1 \\ 3u^2 - 2\alpha u - 2u + \alpha & c \end{pmatrix}$$

$$J(0,0) = \begin{pmatrix} 0 & 1 \\ \alpha & c \end{pmatrix}, J(\alpha,0) = \begin{pmatrix} 0 & 1 \\ \alpha^2 - \alpha & c \end{pmatrix}, J(1,0) = \begin{pmatrix} 0 & 1 \\ 1-\alpha & c \end{pmatrix}$$

The trace of all three matrices is c. The determinant of $J(0,0)$ and $J(1,0)$ is negative, while the determinant of $J(\alpha,0)$ is positive. (Since $\alpha \in (0,1)$). We can therefore conclude that $(0,0)$ and $(1,0)$ are saddle nodes, while $(\alpha,0)$ is a node or a focus, whose stability depends on the sign of c. We will build a heteroclinic connection between $(0,0)$ and $(1,0)$, leaving by the unstable manifold of $(0,0)$ and arriving onto the stable manifold of $(1,0)$, since they are both saddle nodes. As an initial guess, we consider quadratic polynomials passing by both of these points, namely

$$v = Au(u-1)$$

By substituting this into (1.1.4), and noting that $u'' = v_\xi = v_u u_\xi = v_u v$, we get

$$\begin{aligned} cAu(u-1) &= (A(u-1)+Au)\cdot Au(u-1) - u(u-1)(u-\alpha) \\ 0 &= A^2(2u-1) - cA - (u-\alpha) \\ 0 &= u(2A^2-1) + (-A^2 - cA + \alpha) \end{aligned}$$

For this to be true for all u, we require that

$$A = \pm\frac{1}{\sqrt{2}}$$

and get that the relation between the wave speed and the parameter α is given by

$$c = \frac{1}{\sqrt{2}}(1-2\alpha) \tag{1.1.6}$$

In order to compute the solution explicitly, we select $A = -\frac{1}{\sqrt{2}}$. The curve $v = Au(u-1)$ in the phase plane translates into the differential equation

$$\frac{du}{d\xi} = -\frac{1}{\sqrt{2}}u(u-1)$$

which we can solve using seperation of variables.

$$\begin{aligned} \int \frac{du}{u(u-1)} &= -\int \frac{d\xi}{\sqrt{2}} \\ -\int \frac{du}{u} + \int \frac{du}{u-1} &= -\int \frac{d\xi}{\sqrt{2}} \\ -\ln|u| + \ln|u-1| &= -\frac{\xi}{\sqrt{2}} + k \\ \ln\left|\frac{u-1}{u}\right| &= -\frac{\xi}{\sqrt{2}} + k \\ \left|\frac{u-1}{u}\right| &= e^{-\frac{\xi}{\sqrt{2}}+k} \end{aligned}$$

Since we want u to look like a wavefront, we need $0 < u(\xi) < 1$; or equivalently, $\frac{u-1}{u} < 0$. For simplicity, make the change of variable $\eta = -\sqrt{2}k$.

$$\begin{aligned} -\frac{u-1}{u} &= e^{-\frac{(\xi+\eta)}{\sqrt{2}}} \\ 1 &= ue^{-\frac{(\xi+\eta)}{\sqrt{2}}} + u \\ u &= \frac{1}{1+e^{-\frac{(\xi+\eta)}{\sqrt{2}}}} \end{aligned}$$

Equivalently, we can write

$$u = \frac{1}{2}\left(1 + \tanh\left(\frac{\xi+\eta}{2\sqrt{2}}\right)\right)$$

and we have just obtained the profile of our travelling wave. We can check that for any real η,

$$u(x,t) = \frac{1}{2}\left(1 + \tanh\left(\frac{x+ct+\eta}{2\sqrt{2}}\right)\right) \tag{1.1.7}$$

is indeed a solution to (1.1.3) with the relationship between c and α given in (1.1.6).

1.2 Symmetry

Here, we take a short break from the example to discuss an important property of some dynamical systems: symmetry. Given a symmetry group Γ, we say that the dynamical system

$$\frac{dx}{dt} = F(x) \tag{1.2.1}$$

exhibits symmetry if for every $\gamma \in \Gamma$

$$F(\gamma x) = \gamma F(x) \tag{1.2.2}$$

It follows from (1.2.2) that if $x(t)$ is a solution of (1.2.1) and $\gamma \in \Gamma$, then $\gamma x(t)$ is also a solution of (1.2.1). Furthermore, if for every $\gamma \in \Gamma$ the time orbits of $x(t)$ and $\gamma x(t)$ are equal as sets, we say that $R_e = \{x(t)|t \in \mathbb{R}\}$ is a relative equilibrium for the dynamical system. This concept of symmetry is explored in great detail in [4, 5].

We notice that in the example above, we found a family of solutions to our problem parametrized by ξ. In fact, for reaction-diffusion problems of the type (1.1.1), every solution will be associated to a translationally invariant family of such solutions. To prove this claim, assume that we have a solution $u(x,t)$ to $u_t = Du_{xx} + f(u)$, and consider $v(x,t) = u(x-\xi,t)$. By making the change of variable, $z = x - \xi$, we notice that

$$v(x,t)_t = u(z,t)_t,\ v(x,t)_{xx} = u(z,t)_{zz},\ f(v(x,t)) = f(u(z,t))$$

and since $u(z,t)$ satisfies $u_t = Du_{zz} + f(u)$, the claim holds for $v(x,t)$.

If u is a travelling wave, then the orbits of every member of the translation invariant family will be the same, shifted in time. Hence, $R_e = \{u(x-\xi)|\xi \in \mathbb{R}\}$ is a relative equilibrium. The elements of the relative equilibrium are translations of the wave profile $u(x)$, which are all continuous bounded functions and therefore elements of $\mathcal{C}^0_{\text{unif}}(\mathbb{R},\mathbb{R})$. As a consequence, solutions starting on the relative equilibrium will remain on it at all time (See Figure (1.5)).

1.3 Stability

In order for travelling waves obtained in our model to be meaningful, it is important that initial conditions close to our relative equilibirum converge to it as time progresses forward. In order words, we inquire about the stability of our relative equilibrium. To do so, we must consider the linearization of $u_t = Du_{xx} + f(u)$ at our relative equilibrium. Suppose that $u(x + ct)$ is a travelling wave solution. Consider the nearby solution

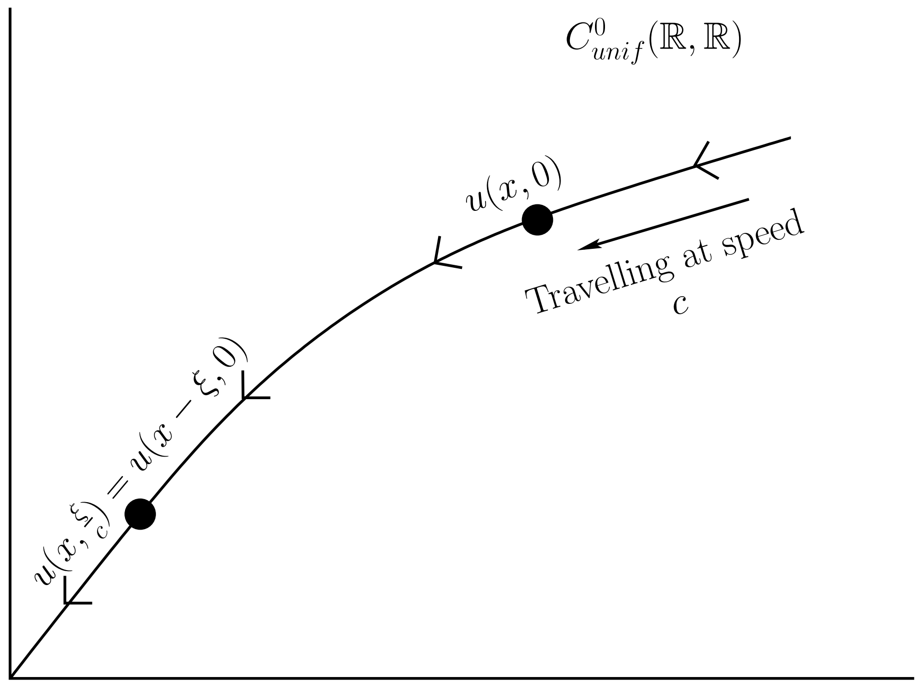

Figure 1.5: Relative Equilibrium

$$v(x,t) = u(x+ct) + e^{\lambda t}\eta(x+ct) \tag{1.3.1}$$

We see that

$$\begin{aligned} v_t &= cu'(x+ct) + \lambda e^{\lambda t}\eta(x+ct) + ce^{\lambda t}\eta'(x+ct) \\ v_{xx} &= u''(x+ct) + e^{\lambda t}\eta''(x+ct) \\ f(v) &= f(u) + f'(u)e^{\lambda t}\eta(x+ct) + (\text{Higher Order Terms}) \end{aligned}$$

For $v(x,t)$ to be a solution, we require that

$$\begin{aligned} & cu'(x+ct) + \lambda e^{\lambda t}\eta(x+ct) + ce^{\lambda t}\eta'(x+ct) \\ = \;& Du''(x+ct) + De^{\lambda t}\eta''(x+ct) + f(u) + f'(u)e^{\lambda t}\eta(x+ct) + (H.O.T.) \end{aligned}$$

Since $u(x+ct)$ is a solution, it satisfies (1.1.2), and these u terms cancel out. Next, we neglect the higher order terms since this is a linearization. Lastly, we can divide through by $e^{\lambda t}$. This brings us to the eigenvalue equation

$$\lambda\eta(\xi) = D\eta''(\xi) + f'(u(\xi))\eta(\xi) - c\eta'(\xi) \tag{1.3.2}$$

We notice that for values of λ with real part greater than 0, $v(x,t)$ is going to diverge from the relative equilibrium as $t \to \infty$. On the other hand, solutions corresponding to eigenvalues with negative real part are going to converge towards the relative equilibrium. First, we notice that the eigenvalue 0 is always present, corresponding to the eigenvector $\eta(\xi) = u'(\xi)$. Indeed, differentiating (1.1.2) with respect to ξ gives us the identity:

$$0 = Du'''(\xi) + f'(u(\xi))u'(\xi) - cu''(\xi)$$

The zero eigenvalue appears from the translation invariance of our solutions. If we assume that the rest of the spectrum is in the left hand plane bounded away from the imaginary axis, then any solution starting near the relative equilibrium will tend towards it as $t \to \infty$, in which case we say that our relative equilibrium is stable [6, 12]. On the other hand, if part of the spectrum is in the right hand plane, some solutions nearby will not converge to the relative equilibrium, in which case, we say that the relative equilibrium is unstable.

1.4 Approaching the Problem

As real life values always have a certain degree of variability/uncertainty, we are only able to observe travelling waves if they are stable. Along the same train of thought, natural systems are often faced with outside perturbations, some of which may break the symmetry of our equilibria. Therefore, it is of interest to study perturbed systems. In particular, much attention has been given in recent years to the phenomenon of wave blocking. Wave blocking occurs when a perturbation stops the progression of a travelling wave, forcing it into a steady state.

In [11], Lewis explores the effects of a “gap”-type perturbation on travelling waves in scalar reaction-diffusion equations. He shows that the speed of a travelling wave is intimately related to wave blocking. In particular, we learn that for a given speed c, there is a threshold value for the strength of the perturbation (length of the gap in his case) above which the wave will always get blocked. The analysis done in [11] uses comparison arguments and sub/super-solutions in order to obtain these results. Unfortunately, these techniques have no equivalent in higher dimensions, which makes them very hard to generalize to multi-component systems.

In [10, 8], the effect of Euclidean symmetry-breaking perturbations on spiral waves is analysed using techniques of center manifold reduction. In this thesis, we will apply those geometrical ideas to travelling waves. Considering the problem of blocking as a two-parameter problem (i.e. considering the speed of the wave as a distinct parameter) and reducing the PDE to a system of ODEs has been explored in [13]. In this sense, our approach will be similar, but our result will be more general. Indeed, [13] analyses only a particuliar exemple, and the analysis is hard to generalize.

We believe that this two-parameter approach is the key to understanding the organising center of wave blocking. Our results give us a linear relation between the speed of the travelling wave and the first point of blocking, near the organising center. This provides a deep insight on the origin of the block as opposed to specific values of blocking for individually chosen values of travelling speed, which is done in [11] (See Figure (1.6)).

This thesis is organised as follows: In sections 2.1-2.2, we set up our equation, define spaces and expose some key properties of the problem we are considering. In sections 2.3-2.5, we consider the linearization operator L and its spectrum. This allows us to find a decomposition of our space in section 2.6, which, combined with the center manifold theorem used in section 2.7, gives us a parametrization of our system. This parametrization is presented in section 2.8 and used to analyse the perturbed system in section 2.9. At this point, we have all the tools in hand to show

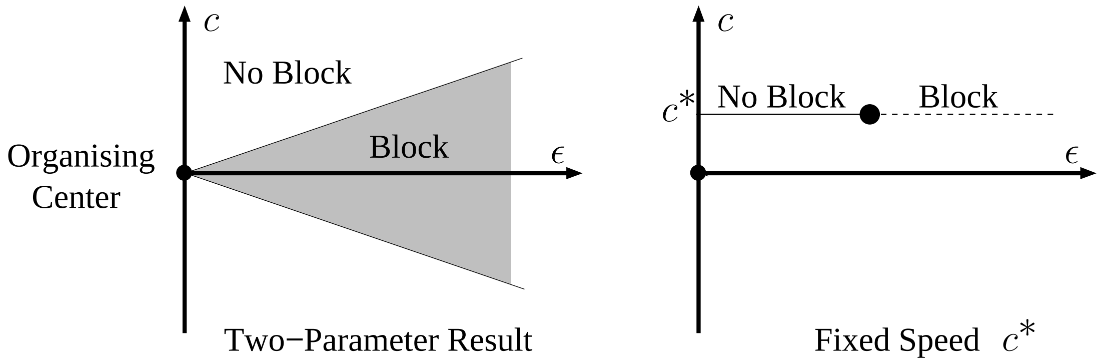

Figure 1.6: Advantage of our two-parameter approach

our main result, which is done in section 2.10. To support our result, we compare it with multiple numerical simulations in Chapter 3.

Chapter 2

Results

2.1 Setup

In our attempt to understand the effects of perturbations on wave-blocking, we consider the prototype partial differential equation:

$$u_t = \mathcal{A}u + u(1-u)(u-\alpha) + \epsilon g(u, x, \epsilon) \tag{2.1.1}$$

where $\mathcal{A}u := u_{xx}$, $0 < \epsilon \ll 1$ is a small parameter, and $g(u, x, \epsilon)$ is the shape of the perturbation. Furthermore, we will write $\mathcal{F}(u, \alpha) = u(1-u)(u-\alpha)$.

The advantage of considering this particular system is that we know an explicit travelling wave solution to the unperturbed system, namely (1.1.7). The profile of this wave will be referred to as

$$u^*(\alpha)(\eta) = u^*(\eta) = \frac{1}{2}(1 + \tanh(\frac{\eta}{2\sqrt{2}})). \tag{2.1.2}$$

Note that in this specific case, the wave profile is independent of α. However, it is important to note that all the ideas presented in this thesis are valid for a system of evolution equations much more general than (2.1.1) [9]. The reason for concentrating on the specific system (2.1.1) is for ease of exposition and for the fact that all computations can be done explicitly.

Next, it is important to define/choose the space with which we will be working. On both $\mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$ and $\mathcal{L}^2(\mathbb{R},\mathbb{R})$, the equation

$$u_t = \mathcal{A}u + \mathcal{F}(u,\alpha) = u_{xx} + u(1-u)(u-\alpha)$$

generates a local semiflow for any $\alpha \in \mathbb{R}$, which makes solving this equation a well-posed problem [6]. Furthermore, on these spaces the action of the group Γ (defined below, which consists of translations on the real line) is smooth, and the spaces satisfy the hypotheses of the center manifold reduction of [12]. Choosing to work in $\mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$ rather than $\mathcal{L}^2(\mathbb{R},\mathbb{R})$ allows us to consider travelling fronts, as they are uniformly continuous, but not square integrable. Hence, we will be considering the Banach space of bounded real uniformly continuous functions $Y := \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$.

In order to better visualize the symmetry of our problem, we can formally define $\Gamma = SE(1)$ to be the group of rigid translations of the real line. Let $\mathcal{T} : \Gamma \to GL(Y)$ be a faithful strongly continuous and isometric representation of Γ. For $a \in \Gamma$, we will write $\mathcal{T}_a$ to denote $\mathcal{T}(a)$. This notation in terms of group simply means that $\mathcal{T}_a v(x) = v(x+a)$ for any $v \in Y$.

Before proceeding with the analysis, we need to make some remarks about our system and add in a hypothesis.

Remark 2.1.3 (a) The infinitesimal generator of $\mathcal{T}$ is

$$\xi u(x) = \lim_{h\to 0} \frac{\mathcal{T}_h u(x) - Iu(x)}{h} = \lim_{h\to 0} \frac{u(x+h)-u(x)}{h} = u'(x)$$

(b) The domain of the diffusion operator $\mathcal{A} = \frac{\partial^2}{\partial x^2}$ is $Y_1 := \mathcal{C}^2_{unif}(\mathbb{R},\mathbb{R})$ and that of the infinitesimal generator of $\mathcal{T}$, $\xi = \frac{\partial}{\partial x}$, is $Y_2 := \mathcal{C}^1_{unif}(\mathbb{R},\mathbb{R})$. Both of these sets are dense in Y and invariant under translation. Also, we can pick $Y_3 := \mathcal{C}^3_{unif}(\mathbb{R},\mathbb{R}) \subseteq Y_1 \cap Y_2$ such that $\frac{\partial^2}{\partial x^2}(Y_3) \subseteq Y_2$, $\frac{\partial}{\partial x}(Y_3) \subseteq Y_1$, and $\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial x} = \frac{\partial}{\partial x}\frac{\partial^2}{\partial x^2}$ on Y_3, i.e. $\mathcal{A}\xi = \xi\mathcal{A}$.

(c) For any $u \in Y_3$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}(\mathcal{T}_a u)(x) &= \frac{\partial^2}{\partial x^2}(u(x+a)) \\ &= u''(x+a) \\ &= \mathcal{T}_a\left(\frac{\partial^2}{\partial x^2}u(x)\right) \quad i.e.\ \mathcal{A}\mathcal{T}_a = \mathcal{T}_a\mathcal{A}.\end{aligned}$$

(d) Since $\mathcal{F} = u(1-u)(u-\alpha)$ depends only, and polynomially on u and α, we have that for any $u \in Y_3, \alpha \in \mathbb{R}$,

$$\begin{aligned}(\mathcal{F}(u,\alpha))_x &= \mathcal{F}_u(u,\alpha)u'(x) \\ (\mathcal{F}(u,\alpha))_{xx} &= \mathcal{F}_{uu}(u,\alpha)u'(x)^2 + \mathcal{F}_u(u,\alpha)u''(x) \\ (\mathcal{F}(u,\alpha))_{xxx} &= \mathcal{F}_{uuu}(u,\alpha)u'(x)^3 + 3\mathcal{F}_{uu}(u,\alpha)u''(x)u'(x) + \mathcal{F}_u(u,\alpha)u'''(x)\end{aligned}$$

are all continuous and bounded, since u is $\mathcal{C}^3$ and bounded. This means that $\mathcal{F}(u,\alpha)$ is at least $\mathcal{C}^2$ and bounded, and therefore $\mathcal{F}(u,\alpha) \in Y_2$, $\forall u \in Y_3, \forall \alpha \in \mathbb{R}$. Also, the function $\mathcal{F}(u,\alpha)$ is $\mathcal{T}$-equivariant, i.e.

$$\begin{aligned}\mathcal{F}(\mathcal{T}_a u,\alpha) &= \mathcal{F}(u(x+a),\alpha) \\ &= \mathcal{F}(u,\alpha)(x+a) \\ &= \mathcal{T}_a\mathcal{F}(u,\alpha) \qquad ,\forall u \in Y, \alpha \in \mathbb{R}\end{aligned} \tag{2.1.4}$$

This implies also that for any $u, v \in Y, \alpha \in \mathbb{R}$

$$\frac{\partial}{\partial u}\left(\mathcal{F}(\mathcal{T}_a u,\alpha)\right)\mathcal{T}_a v = \mathcal{T}_a\left(\frac{\partial}{\partial u}\left(\mathcal{F}(u,\alpha)\right)v\right) \tag{2.1.5}$$

$$\xi\mathcal{F}(u,\alpha) = \frac{\partial}{\partial x}\mathcal{F}(u,\alpha) = \frac{\partial}{\partial u}\mathcal{F}(u,\alpha)\xi u \tag{2.1.6}$$

and

$$\frac{\partial}{\partial \alpha}\mathcal{F}(\mathcal{T}_a u,\alpha) = \mathcal{T}_a\frac{\partial}{\partial \alpha}\mathcal{F}(u,\alpha) \tag{2.1.7}$$

Hypothesis 2.1.8 The bounded function $g(u,x,\epsilon) \in \mathcal{C}^\infty$ is not Γ -equivariant, i.e

$$g(\mathcal{T}_a u, x, \epsilon) = \mathcal{T}_a g(u,x,\epsilon) \iff a = 0$$

As a consequence, equation (2.1.1) displays translational symmetry only if $\epsilon = 0$. If $\epsilon \neq 0$, we say that the perturbation is symmetry-breaking.

In order to analyse (2.1.1), we suspend c, the speed parameter, and consider the system

$$\begin{aligned} u_t &= \mathcal{A}u + u(1-u)(u-\alpha(c)) + \epsilon g(u,x,\epsilon) \\ c_t &= 0 \end{aligned} \tag{2.1.9}$$

with the associated norm

$$\left\| \begin{pmatrix} u \\ c \end{pmatrix} \right\| = ||u|| + |c|$$

where we take as norm on Y

$$||u|| = \sup_{x\in\mathbb{R}} \{|u(x)|\}$$

We can also define a scalar product on Y as

$$\langle u(x), v(x) \rangle = \int_{-\infty}^{\infty} u(x)v(x)\,dx$$

which is valid for functions u and v such that the integral converges. Furthermore, we generalize the group action $\mathcal{T}_a$ as follows:

$$\mathcal{T}_a \begin{pmatrix} u \\ c \end{pmatrix} = \begin{pmatrix} \mathcal{T}_a u \\ c \end{pmatrix}$$

In the same way, we can generalize the scalar product to $Y \times \mathbb{R}$ as follows:

$$\left\langle \begin{pmatrix} u(x) \\ a \end{pmatrix}, \begin{pmatrix} v(x) \\ b \end{pmatrix} \right\rangle = \int_{-\infty}^{\infty} u(x)v(x)\,dx + ab$$

2.2 Travelling Speed

With these definitions in hand, we begin our analysis by revisiting our unperturbed system, letting $\epsilon = 0$ in (2.1.9),

$$\begin{aligned} u_t &= \mathcal{A}u + u(1-u)(u-\alpha) \\ c_t &= 0 \end{aligned} \tag{2.2.1}$$

As we have calculated before, for a given α, a relative equilibrium of (2.2.1) is given by the orbit of

$$\begin{pmatrix} u(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{c(\alpha)t}u^* \\ c(\alpha) \end{pmatrix}$$

Substituting this parametrization of the relative equilibrium into (2.2.1) and applying the translation invariance of $\mathcal{F}(u^*, \alpha) = u^*(1-u^*)(u^*-\alpha)$ (2.1.4), gives us the identity

$$\mathcal{T}_{c(\alpha)t}c(\alpha)\xi u^* = \mathcal{T}_{c(\alpha)t}\mathcal{A}u^* + \mathcal{T}_{c(\alpha)t}\left(u^*(1-u^*)(u^*-\alpha)\right)$$

Since Γ is a group, every element is invertible, and hence we can cancel out the $\mathcal{T}_{c(\alpha)t}$. We then take the norm on both sides. We obtain an expression for the drift speed of the relative equilibrium.

$$|c(\alpha)|\,||\xi u^*|| = ||\mathcal{A}u^* + \mathcal{F}(u^*, \alpha)||$$

We then divide through by $||\xi u^*|| \neq 0$ since u^* is not constant.

$$|c(\alpha)| = \frac{||\mathcal{A}u^* + \mathcal{F}(u^*, \alpha)||}{||\xi u^*||}$$

We compute this explicitly, using the algebraic equation (2.1.2) for the wave profile u^*.

$$\xi u^* = \frac{du^*}{dx} \quad = \quad \frac{\partial}{\partial x}\left(\frac{1}{2}\left(1 + \tanh\left(\frac{x}{2\sqrt{2}}\right)\right)\right)$$

$$= \frac{\sqrt{2}}{8}\left(1-\tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right)$$

$$\begin{aligned}
\mathcal{A}u^* = \frac{d^2u^*}{dx^2} &= \frac{\partial^2}{\partial x^2}\left(\frac{1}{2}\left(1+\tanh\left(\frac{x}{2\sqrt{2}}\right)\right)\right) \\
&= -\frac{1}{8}\frac{\sinh\left(\frac{\sqrt{2}}{4}x\right)}{\cosh^3\left(\frac{\sqrt{2}}{4}x\right)} \\
||\mathcal{A}u^* + \mathcal{F}(u^*,\alpha)|| &= \sup_{x\in\mathbb{R}}\left\{\left|\frac{d^2u^*}{dx^2} + u^*(1-u^*)(u^*-\alpha)\right|\right\} \\
&= \left|\frac{1}{8} - \frac{\alpha}{4}\right| \\
&= \frac{1}{8}\,|2\alpha - 1| \\
||\xi u^*|| &= \sup_{x\in\mathbb{R}}\left\{\left|\frac{du^*}{dx}\right|\right\} \\
&= \frac{\sqrt{2}}{8} \\
|c(\alpha)| &= \frac{||\mathcal{A}u^* + \mathcal{F}(u^*,\alpha)||}{||\xi u^*||} \\
&= \frac{1}{\sqrt{2}}\,|2\alpha - 1|
\end{aligned}$$

This means that the drift speed along the relative equilibrium calculated here concurs with (1.1.6), the result obtained by phase-plane analysis.

Remark 2.2.2 Recall that the profile (1.1.7) is a solution to (1.1.1) for any $\alpha \in (0,1)$. Therefore, we consider the function $u^*(\alpha) : (0,1) \to Y_3$ which associates the same wave profile to every α, $u^*(\alpha) = u^* = \frac{1}{2}(1+\tanh(\frac{x}{2\sqrt{2}}))$.

(a) For every $\alpha \in (0,1)$, we have $\mathcal{F}(u^*(\alpha),\alpha) \in Y_3$, $\xi u^*(\alpha) \neq 0$ and both $\xi u^*(\alpha)$ and $\mathcal{A}u^*(\alpha)$ are smooth functions of α (trivially) with derivative $\xi u^*_\alpha(\alpha) = \mathcal{A}u^*_\alpha(\alpha) = 0$. Furthermore, we note that $\mathcal{T}_a u^*(\alpha) = u^*(\alpha)$ if and only if $a = 0$, since u^* does not have any translation symmetry.

(b) For all $\alpha \in (0, 1)$,

$$\begin{pmatrix} u(t) \\ c(t) \end{pmatrix} = \begin{pmatrix} \mathcal{T}_{c(\alpha)t} u^*(\alpha) \\ c(\alpha) \end{pmatrix}$$

is a relative equilibrium of the PDE system, where the drift speed $c(\alpha)$ is the smooth function defined by

$$c(\alpha)\xi u^* = \mathcal{A}u^* + u^*(1 - u^*)(u^* - \alpha) \tag{2.2.3}$$

(c) The function $c(\alpha) = \frac{1}{\sqrt{2}}(1 - 2\alpha)$ is monotone on $(0, 1)$, and $c(\frac{1}{2}) = 0$.

By virtue of the above remark, we forget about the parameter α, and consider c instead, via the function $\alpha(c)$. As a result, equation (2.2.3) becomes

$$c\xi u^* = \mathcal{A}u^* + u^*(1 - u^*)\left(u^* - \frac{1 - \sqrt{2}c}{2}\right) \tag{2.2.4}$$

or in the general form

$$c\xi u^*(c) = \mathcal{A}u^*(c) + \mathcal{F}(u^*(c), c) \tag{2.2.5}$$

Since in this case $u^*(c)$ is the same for every c, we will drop the argument and simply write u^*.

2.3 The Operator L

When $c = 0$, applying ξ to (2.2.5) yields

$$\xi\mathcal{A}u^* + \xi\mathcal{F}(u^*, 0) = 0$$

Applying property (2.1.6), and the fact that ξ and $\frac{\partial^2}{\partial x^2}$ commute, from remark (2.1.3), we get the identity

$$\mathcal{A}(\xi u^*) + \mathcal{F}_u(u^*, 0)\xi u^* = 0 \tag{2.3.1}$$

Also, by differentiating (2.2.5) with respect to c and evaluating at $c = 0$, we obtain

$$\mathcal{A}u_c^* + \mathcal{F}_u(u^*, 0)u_c^* + \mathcal{F}_c(u^*, 0) = \xi u^*. \tag{2.3.2}$$

Inspired by the above identities, we consider the densely defined operator $L : Y \times \mathbb{R} \to Y \times \mathbb{R}$

$$L \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \mathcal{A}\phi + \mathcal{F}_u(u^*, 0)\phi + \mathcal{F}_c(u^*, 0)\omega \\ 0 \end{pmatrix}$$

which is such that

$$L \begin{pmatrix} u_c^* \\ 1 \end{pmatrix} = L \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{F}_c(u^*, 0) \\ 0 \end{pmatrix} = \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} \tag{2.3.3}$$

and

$$L \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.3.4}$$

2.4 The Eigenvalue 0 of L

Equations (2.3.3) and (2.3.4) tell us that the operator L has an eigenvalue of 0 with at least one generalized eigenvector. We will denote the eigenvector by $\phi_1 = (\xi u^*, 0)^T$ and the generalized eigenvector by $\phi_2 = (u_c^*, 1)^T = (0, 1)^T$. Also, we let E be the generalized eigenspace associated with the 0 eigenvalue of L. Our goal now is to show that $E = \text{span}\{\phi_1, \phi_2\}$.

First, we consider the kernel of L, i.e. the set of all $\begin{pmatrix} \phi \\ \omega \end{pmatrix} \in \mathcal{C}^0_{unif}(\mathbb{R}, \mathbb{R}) \times \mathbb{R}$ such that

$$L \begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \mathcal{A}\phi + \mathcal{F}_u(u^*, 0)\phi + \mathcal{F}_c(u^*, 0)\omega \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$
\begin{aligned}
&\Leftrightarrow \quad \mathcal{A}\phi + \mathcal{F}_u(u^*,0)\phi + \mathcal{F}_c(u^*,0)\omega = 0 \\
&\Leftrightarrow \quad \phi_{xx} + \left(-3u^{*2} + 3u^* - \frac{1}{2}\right)\phi + \xi u^*\omega = 0 \\
&\Leftrightarrow \quad \phi_{xx} + f(x)\phi = -\xi u^*\omega
\end{aligned}
$$

where $f(x) = \left(-3u^{*2}(x) + 3u^*(x) - \frac{1}{2}\right)$.

Proposition 2.4.1 $\ker(L) = \text{span}\{\phi_1\}$.

Proof: By the argument above, the problem of finding the elements in the kernel of L is reduced to finding pairs (ϕ, ω), with ϕ a uniformly continuous and bounded function and $\omega \in \mathbb{R}$, which satisfy the ordinary differential equation $\phi_{xx} + f(x)\phi = -\xi u^*\omega$. In order to do so, we first find solutions of the homogeneous equation, and then use variation of parameters to find the general solution. Since this is a second-order ODE, it has two linearly independent solutions. They are found to be:

$$
\begin{aligned}
u_1(x) &= \xi u^*(x) = \frac{\sqrt{2}}{8}\left(1 - \tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right) \\
u_2(x) &= \frac{1}{\cosh^2\left(\frac{\sqrt{2}}{4}x\right)}\left[3\ln\left(\frac{1}{\left(\cosh\left(\frac{\sqrt{2}}{4}x\right) + \sinh\left(\frac{\sqrt{2}}{4}x\right)\right)^2}\right)\right. \\
&- \left. 4\sinh\left(\frac{\sqrt{2}}{4}x\right)\cosh^3\left(\frac{\sqrt{2}}{4}x\right) - 6\sinh\left(\frac{\sqrt{2}}{4}x\right)\cosh\left(\frac{\sqrt{2}}{4}x\right)\right]
\end{aligned}
$$

which are graphically represented in Figure 2.1, and Figure 2.2 respectively.

Next, we proceed with the variation of parameters to solve the non-homogeneous problem. We suppose that there exists a solution of the form $\psi(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$. In this case, theory tells us that

$$
y_1'(x) = -\frac{(-\xi u^*\omega)u_2(x)}{W[u_1,u_2](x)}, \; y_2'(x) = \frac{(-\xi u^*\omega)u_1(x)}{W[u_1,u_2](x)}
$$

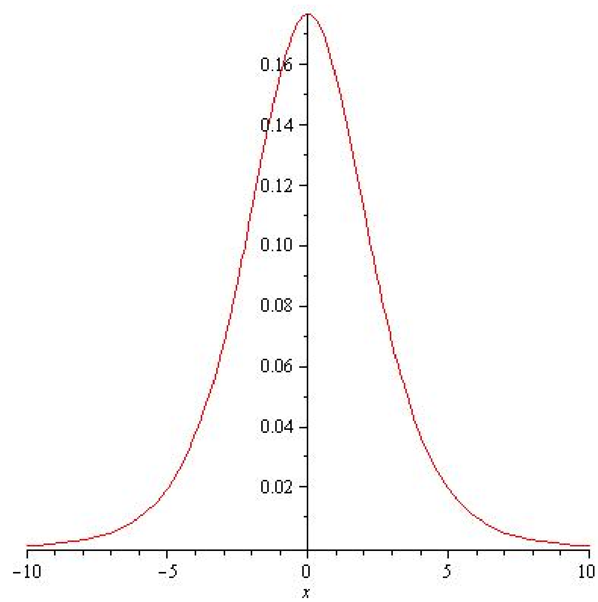

Figure 2.1: $u_1(x)$

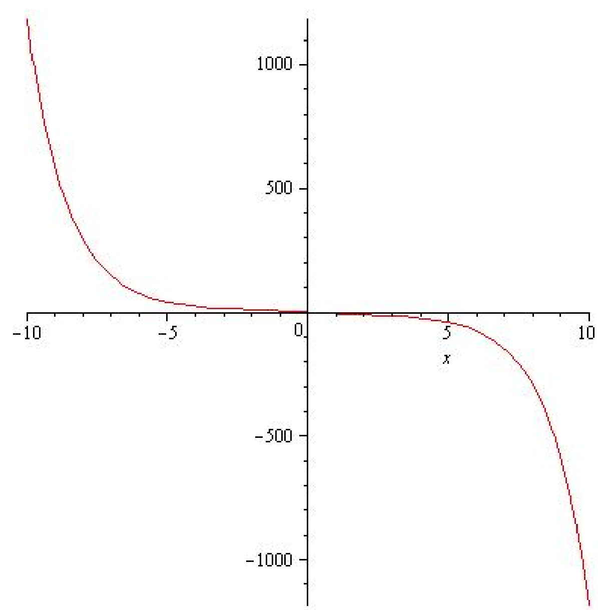

Figure 2.2: $u_2(x)$

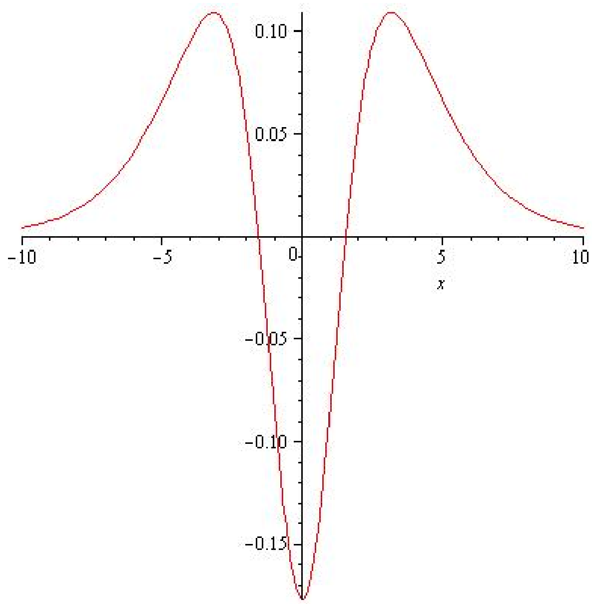

Figure 2.3: $y_1(x)u_1(x)$

We compute $W[u_1, u_2](x) = -1$. Which means that, if $\omega \neq 0$,

$$
\begin{aligned}
y_1(x) &= \int y_1'(x)\,dx \\
&= \frac{\omega}{2}\frac{1}{\left(e^{\frac{\sqrt{2}}{2}x}+1\right)^3}\left[-\frac{\sqrt{2}}{2}x - 6\sqrt{2}xe^{\frac{\sqrt{2}}{2}x} + \sqrt{2}xe^{\frac{3\sqrt{2}}{2}x}\right. \\
&+ \left. 3\sqrt{2}xe^{\sqrt{2}x} + 3\sqrt{2}xe^{\frac{\sqrt{2}}{2}x} + \sqrt{2}x - 8e^{\sqrt{2}x} - 8e^{\frac{\sqrt{2}}{2}x}\right] \\
y_2(x) &= \int y_2'(x)\,dx \\
&= -\frac{\omega}{48}\sqrt{2}\tanh\left(\frac{\sqrt{2}}{4}x\right)\left(-3+\tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right)
\end{aligned}
$$

We conclude that ψ is the sum of the functions graphed in Figure 2.3 and Figure 2.4 (for $\omega = 1$).

It is verified that $y_2(x)u_2(x)$ is unbounded, while $y_1(x)u_1(x)$ goes to 0 as $x \to$

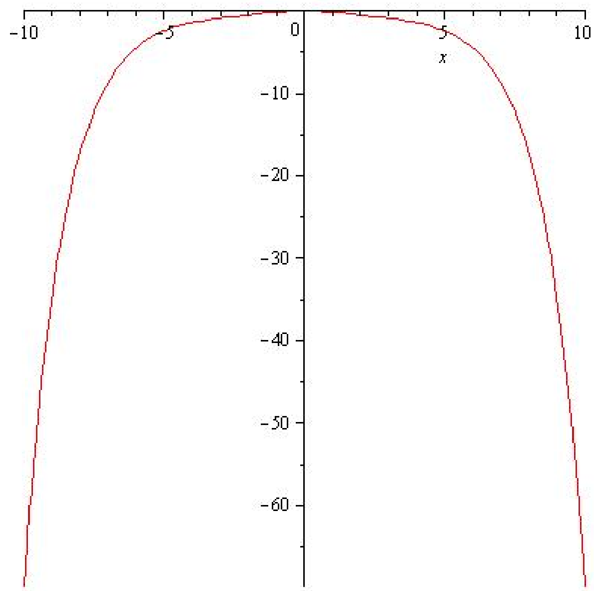

Figure 2.4: $y_2(x)u_2(x)$

$\pm\infty$. Hence, $\psi(x) = y_1(x)u_1(x) + y_2(x)u_2(x)$ is necessarely unbounded. We recall that we are looking for solutions $\phi \in \mathcal{C}^0_{unif}(\mathbb{R}, \mathbb{R})$. In particular, our solutions have to be bounded. Therefore, for any non-zero ω, there are no admissible solutions.

We are left to consider the case of $\omega = 0$, where the general solution becomes $Au_1(x) + Bu_2(x)$. Again, this solution is unbounded if and only if B is non-zero. Hence, the only remaining solutions are the multiples of $(u_1(x), 0) = (\xi u^*, 0)$.

∎

By the result above, the only eigenvector in E is $\phi_1 = \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix}$. It remains to check for generalized eigenvectors of L associated with the eigenvalue 0. We notice that

$$L\phi_2 = L \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \mathcal{F}_c(u^*, 0) \\ 0 \end{pmatrix} = \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix}$$

Hence, ϕ_2 is a generalized eigenvector of L associated with the eigenvalue 0. Furthermore, for any $\phi_3 \in \mathcal{C}^0_{unif}(\mathbb{R}, \mathbb{R}) \times \mathbb{R}$, $L\phi_3$ is of the form $\begin{pmatrix} * \\ 0 \end{pmatrix}$, so we conclude that ϕ_2 cannot have a generalized eigenvector.

We can therefore conclude that the generalized eigenspace associated to the eigenvalue 0 of L is $E = \text{span}\{\phi_1, \phi_2\}$.

2.5 The Spectrum of L

Now that we understand the eigenspace associated to the eigenvalue 0 of L, we must consider the rest of its spectrum. The spectrum of L is the set of complex numbers λ such that the operator $(L - \lambda I)^{-1}$ does not exist, is unbounded or not densely defined. The eigenvalues of L correspond to isolated values of λ and form the discrete spectrum of L. On the other hand, the spectrum of L can also contain other objects, such as curves of the form $\lambda(x)$. These elements are part of the continuous spectrum of L. We say that a point $\lambda \in \mathbb{C}$ is regular if it is not in the spectrum. In order to ensure the stability of our relative equilibrium, we need to show that the rest of the spectrum of L is bounded away from the imaginary axis, in the left-half plane. To do so, we first consider the relation between the following two operators:

$$\mathcal{L}\phi = \phi'' + G(x)\phi \tag{2.5.1}$$

$$L_{gen}\begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} \mathcal{L}\phi + \omega \cdot \xi u^* \\ 0 \end{pmatrix} \tag{2.5.2}$$

where $G(x)$ is some arbitrary function of x.

Proposition 2.5.3 *The set $\zeta(\mathcal{L})$, of regular points of $\mathcal{L}$, is contained in the set of regular points of L_{gen}, i.e. $\zeta(\mathcal{L}) \subseteq \zeta(L_{gen})$. Hence, the spectrum of L_{gen} is contained in the spectrum of $\mathcal{L}$, i.e. $\sigma(L_{gen}) \subseteq \sigma(\mathcal{L})$.*

Proof:

Let $\lambda \in \zeta(\mathcal{L})$ (it follows that $\lambda \neq 0$). Then, $(\mathcal{L} - \lambda I)^{-1}$ exists, is densily defined and bounded. In particular, the equation $(\mathcal{L} - \lambda I)\phi = h$, for some function h, has the unique solution $\phi = (\mathcal{L} - \lambda I)^{-1}h$. Next, we consider the equation :

$$\begin{aligned}
(L_{gen} - \lambda I)\begin{pmatrix} \phi \\ \omega \end{pmatrix} &= L_{gen}\begin{pmatrix} \phi \\ \omega \end{pmatrix} - \begin{pmatrix} \lambda\phi \\ \lambda\omega \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L}\phi + \omega \cdot \xi u^* - \lambda\phi \\ \lambda\omega \end{pmatrix} \\
&= \begin{pmatrix} (\mathcal{L} - \lambda I)\phi + \omega \cdot \xi u^* \\ \lambda\omega \end{pmatrix}
\end{aligned}$$

Hence, the equation

$$(L_{gen} - \lambda I)\begin{pmatrix} \phi \\ \omega \end{pmatrix} = \begin{pmatrix} h \\ c \end{pmatrix}$$

is equivalent to

$$\begin{pmatrix} (\mathcal{L} - \lambda I)\phi + \omega \cdot \xi u^* \\ \lambda\omega \end{pmatrix} = \begin{pmatrix} h \\ c \end{pmatrix}$$

which is in turn equivalent to the pair of equations

$$\lambda\omega = c$$

$$(\mathcal{L} - \lambda I)\phi + \frac{c}{\lambda} \cdot \xi u^* = h$$

We deduce from the first equation that

$$\omega = \frac{c}{\lambda}$$

Substituting this value in the latter equation yields

$$\begin{aligned}(\mathcal{L}-\lambda I)\phi+\frac{c}{\lambda}\cdot\xi u^{*} &= h\\ (\mathcal{L}-\lambda I)\phi &= h-\frac{c}{\lambda}\cdot\xi u^{*}\\ \phi &= (\mathcal{L}-\lambda I)^{-1}\cdot h-\frac{c}{\lambda}(\mathcal{L}-\lambda I)^{-1}\cdot\xi u^{*}\end{aligned}$$

We can therefore conclude that $(L_{gen}-\lambda I)^{-1}$ is the operator such that

$$(L_{gen}-\lambda I)^{-1}\begin{pmatrix} h\\ c\end{pmatrix}=\begin{pmatrix}(\mathcal{L}-\lambda I)^{-1}\cdot h-\frac{c}{\lambda}(\mathcal{L}-\lambda I)^{-1}\cdot\xi u^{*}\\ \frac{c}{\lambda}\end{pmatrix}$$

Furthermore, we can see that this operator inherits the properties of $(\mathcal{L}-\lambda I)^{-1}$ and so $\zeta(\mathcal{L})\subseteq\zeta(L_{gen})$.

∎

For the operator $\mathcal{L}$ with $G(x)=\mathcal{F}_u(u^*,0)$, it is shown in [6] that the spectrum of $\mathcal{L}$ is bounded away from the imaginary axis, in the left plane, except for the isolated eigenvalue at 0. In fact, [6] proves that the continuous spectrum is bounded by $Re(\lambda)\leq-\frac{1}{2}$ and that all non-zero eigenvalues of $\mathcal{L}$ are strictly negative. In this case, $L_{gen}=L$ and applying the result above tells us that the spectrum of L is also bounded away from the imaginary axis, in the left half plane, except for the eigenvalue at 0.

2.6 L-Invariant Splitting

Next, we wish to build an L-invariant splitting of the form :

$$Y\times\mathbb{R}=E\oplus W=span\{\phi_1,\phi_2\}\oplus W \tag{2.6.1}$$

To do so, consider the operator $\mathcal{L}$ from (2.5.1), with $G(x)=\mathcal{F}_u(u^*,0)$, i.e.

$$\mathcal{L}\phi = \phi_{xx}+\mathcal{F}_u(u^*,0)\phi$$

$$= \phi_{xx} + f(x)\phi$$

where $f(x) = -3u^{*2}(x) + 3u^*(x) - \frac{1}{2}$.

Our goal here is to write $\mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R}) = \ker(\mathcal{L}) \oplus \text{range}(\mathcal{L})$. Since for all $\phi \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$, $\mathcal{L}\phi$ corresponds to the first component of $L(\phi, 0)^T$, we get from proposition 2.4.1 that $\ker(\mathcal{L}) = \{\beta \cdot \xi u^*; \beta \in \mathbb{R}\}$. Hence, it remains to show that range($\mathcal{L}$) spans the orthogonal complement of $\ker(\mathcal{L})$ in $\mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$ and that the sum is direct.

Proposition 2.6.2 $\ker(\mathcal{L}) \cap \text{range}(\mathcal{L}) = \{0\}$

Proof: Let $\phi \in \ker(\mathcal{L}) \cap \text{range}(\mathcal{L})$. Then, since $\phi \in \ker(\mathcal{L})$, proposition 2.4.1 implies that $\phi = \beta \cdot \xi u^*$ for some $\beta \in \mathbb{R}$. Also, since $\phi \in \text{range}(\mathcal{L})$, $\exists \psi \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$ such that $\mathcal{L}\psi = \psi_{xx} + f(x)\psi = \phi$. The proof of proposition 2.4.1 tells us that the only way $\psi_{xx} + f(x)\psi = \beta \cdot \xi u^*$ has a bounded solution is if $\beta = 0$. From this, we conclude that $\ker(\mathcal{L}) \cap \text{range}(\mathcal{L}) = \{0\}$.

∎

Proposition 2.6.3 $\mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R}) = \ker(\mathcal{L}) + \text{range}(\mathcal{L})$

Proof: Let $\phi \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$. We can write

$$\phi(x) = \langle \phi(x), \xi u^* \rangle \xi u^* + (\phi(x) - \langle \phi(x), \xi u^* \rangle \xi u^*)$$

where we recall that

$$\langle \phi(x), \xi u^* \rangle = \int_{-\infty}^{\infty} \phi(x) \xi u^* \, dx$$

and that ξu^* is integrable. Therefore, $\langle \phi(x), \xi u^* \rangle$ is well defined $\forall \phi \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$. Relabelling $h(x) = (\phi(x) - \langle \phi(x), \xi u^* \rangle \xi u^*)$, we get that

$$\phi(x) = \langle \phi(x), \xi u^* \rangle \xi u^* + h(x)$$

where $h(x) \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$ is orthogonal to ξu^*, and $\langle \phi(x), \xi u^* \rangle \xi u^* \in \ker(\mathcal{L})$.

Hence, we need to show that $h(x) \in \text{range}(\mathcal{L})$. In other words, we need to prove the existence of a function $\psi \in \mathcal{C}^0_{unif}(\mathbb{R}, \mathbb{R})$ such that

$$\psi_{xx} + f(x)\psi = h(x)$$

We have found, in the proof of proposition 2.4.1, that the solutions of the homogeneous problem are $u_1(x)$ and $u_2(x)$. To find solutions to the non-homogeneous problem, we use variation of parameters ($\psi = y_1(x)u_1(x) + y_2(x)u_2(x)$). Recalling that $W[u_1, u_2](x) = -1$,

$$y_1'(x) = h(x)u_2(x) \ , \ y_2'(x) = -h(x)u_1(x)$$

We can now integrate

$$y_1(x) = \int_0^x h(z)u_2(z)\, dz + K_1$$

and let $K_1 = 0$ as a particular choice. In which case, since $h(x)$ is bounded (say $|h(x)| \leq \kappa$), we obtain

$$\begin{aligned} |y_1(x)| &= \left| \int_0^x h(z)u_2(z)\, dz \right| \\ &\leq \int_0^x |h(z)||u_2(z)|\, dz\,^1 \\ &\leq \kappa \int_0^x |u_2(z)|\, dz \end{aligned}$$

Therefore,

$$|y_1(x)u_1(x)| \leq \kappa \int_0^x |u_2(z)|\, dz \cdot |\xi u^*(x)|$$

Transforming $u_2(z)$ into exponentials, we can explicitly compute the integral to be

$$\int_0^x |u_2(z)|\, dz = \frac{\left(\sqrt{2}e^{-\frac{3\sqrt{2}}{2}x} + (6x - \sqrt{2})e^{-\frac{\sqrt{2}}{2}x} - (6x + \sqrt{2})e^{-\sqrt{2}x} + \sqrt{2}\right) e^{\frac{\sqrt{2}}{2}x}}{1 + e^{-\frac{\sqrt{2}}{2}x}}$$

for $x \geq 0$, adding a negative sign for $x \leq 0$.

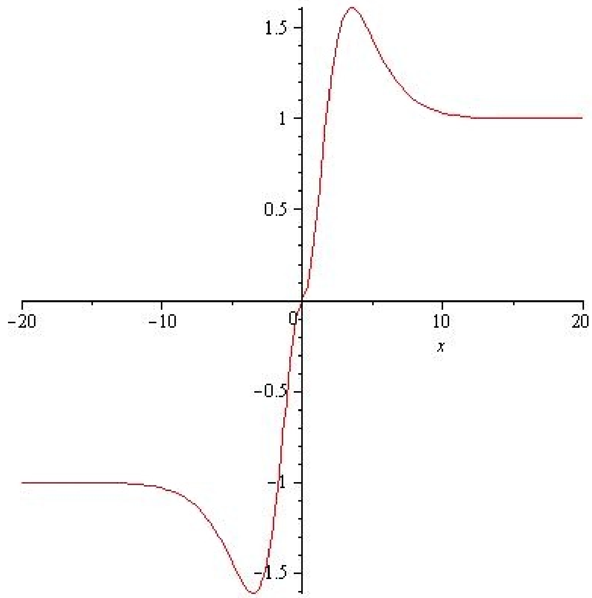

Figure 2.5: $\int_0^x |u_2(z)|\,dz \cdot |\xi u^*(x)|$

We compute that the limits as x goes to $\pm\infty$ of $\int_0^x |u_2(z)|\,dz \cdot |\xi u^*(x)|$ are ± 1 respectively. The graph of that function is shown in Figure 2.5.

We can therefore conclude that $y_1(x)u_1(x)$ is bounded. Also, $y_1'(x) = h(x)u_2(x)$ and $u_1'(x)$ are continuous functions, so $y_1(x)u_1(x)$ is continuously differentiable, and hence uniformly continuous. This tells us that $y_1(x)u_1(x) \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$.

We proceed with the integration of the second term,

$$y_2(x) = \int_{-\infty}^{x} -h(z)u_1(z)\,dz + K_2$$

letting $K_2 = 0$ for simplicity.

By the orthogonality property of $h(z)$ with $u_1(z) = \xi u^*$, we have that

$$\lim_{x\to\pm\infty} y_2(x) = 0$$

[1]Note that if $x < 0$, then the integral becomes $\int_x^0 |h(z)||u_2(z)|\,dz$

Since $h(x)$ and $u_1(x)$ are continuous, $y_2(x)$ is $\mathcal{C}^1$. We also have that $u_2(x)$ is $\mathcal{C}^1$, and

$$\lim_{x \to \pm\infty} u_2(x) = \mp\infty$$

The term which we want to bound is $y_2(x)u_2(x)$. In order to use theorem (A.0.1), it remains to show that

$$\frac{y_2'(x) \cdot u_2(x)^2}{u_2'(x)}$$

exists and is bounded for all $|x| > K$ for some K. Since

$$\frac{y_2'(x) \cdot u_2(x)^2}{u_2'(x)} = \frac{-h(x)u_1(x) \cdot u_2(x)^2}{u_2'(x)} = -h(x) \cdot \frac{u_1(x)u_2(x)^2}{u_2'(x)}$$

and that $-h(x)$ is bounded by hypothesis, it is sufficient to show that

$$\frac{u_1(x)u_2(x)^2}{u_2'(x)}$$

is bounded for $|x|$ large enough. This function is plotted in Figure 2.6.

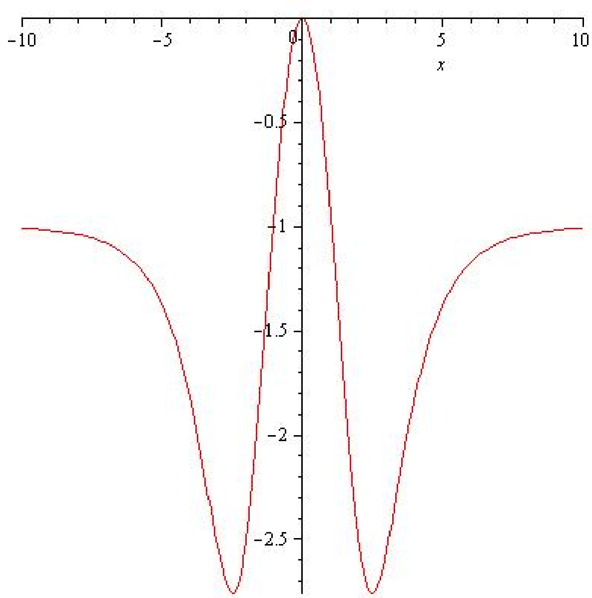

Figure 2.6: $\frac{u_1(x)u_2(x)^2}{u_2'(x)}$

We can compute explicitly the limits at $\pm\infty$ to be -1, confirming boundedness. Applying theorem (A.0.1), we conclude that $y_2(x)u_2(x)$ is bounded for all $x \in \mathbb{R}$.

Since both $y_2(x)$ and $u_2(x)$ are $\mathcal{C}^1$, their product is also $\mathcal{C}^1$, and therefore uniformly continuous. This tells us that $y_2(x)u_2(x) \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$.

We can conclude that $\psi = y_1(x)u_1(x) + y_2(x)u_2(x) \in \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R})$ is a solution to the non-homogeneous ODE, which implies that $h(x) \in \text{range}(\mathcal{L})$. ∎

We now have $\mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R}) = \ker(\mathcal{L}) \oplus \text{range}(\mathcal{L})$, and hence

$$\begin{aligned} \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R}) \times \mathbb{R} &= \begin{pmatrix} \ker(\mathcal{L}) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} \text{range}(\mathcal{L}) \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ &= E \oplus \begin{pmatrix} \text{range}(\mathcal{L}) \\ 0 \end{pmatrix} \end{aligned}$$

We note that this splitting is indeed L-invariant since $L\phi_2 = \phi_1$, $L\phi_1 = (0,0)^T$ and $L(a,0)^T$ is of the form $(b,0)^T$ with $b \neq \xi u^*$ [See the proof of proposition 2.4.1], for any $a \in \mathcal{C}^0_{unif}(\mathbb{R}.\mathbb{R})$. Furthermore, from the proof of proposition 2.6.3 we have that $L(\text{range}(\mathcal{L}),0)^T = (\text{range}(\mathcal{L}),0)^T$.

For this L-invariant splitting, we define $P : Y \times \mathbb{R} \to E$ to be the standard projection onto E.

2.7 Center Manifold Reduction

Setting up for a center manifold reduction (See Appendix B), we need to prove some facts about our setup:

Proposition 2.7.1 *The following function is $\mathcal{C}^\infty$.*

$$\begin{aligned} f : \mathbb{R} &\to \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R}) \\ a &\mapsto \mathcal{T}_a u^*(x) \end{aligned}$$

Proof: Recall that $\mathcal{T}_a u^*(x) = u^*(x+a)$. Therefore,

$$\begin{aligned} f'(a) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{u^*(x+a+h) - u^*(x+a)}{h} \\ &= u^*_x(x+a) \end{aligned}$$

Hence, f is as smooth as u^* is. Since $u^*(x+a) \in C^\infty$ for every $a \in \mathbb{R}$, then we conclude that f is C^∞. ∎

Proposition 2.7.2 *$\forall K > 0, \exists \delta > 0$ such that $||\mathcal{T}_a u^*(x) - u^*(x)|| \geq \delta$, whenever $|a| \geq K$.*

Proof: Let $K > 0$, and pick $a \in \mathbb{R}$ such that $|a| \geq K$.

$$\begin{aligned} ||\mathcal{T}_a u^*(x) - u^*(x)|| &= \sup_{x\in\mathbb{R}}\{|u^*(x+a) - u^*(x)|\} \\ &= \sup_{x\in\mathbb{R}} \left\{ \left| \frac{1}{2}\tanh\left(\frac{\sqrt{2}}{4}(x+a)\right) - \frac{1}{2}\tanh\left(\frac{\sqrt{2}}{4}x\right) \right| \right\} \\ &= \sup_{x\in\mathbb{R}}\{|M(x)|\} \end{aligned}$$

To find the supremum of this function, we must find the critical points, hence we compute its derivative.

$$\frac{d}{dx}M(x) = \frac{\sqrt{2}}{8}\left(1 - \tanh^2\left(\frac{\sqrt{2}}{4}(x+a)\right)\right) - \frac{\sqrt{2}}{8}\left(1 - \tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right)$$

First, we notice that $M'(x)$ exists for all $x \in \mathbb{R}$. Then, setting $M'(x) = 0$ and observing that $\tanh(x)$ is an odd, injective function, we get that

$$\frac{\sqrt{2}}{4}(x+a) = \pm\frac{\sqrt{2}}{4}x$$

Taking the + sign tells us that $M'(x) = 0 \ \forall x \in \mathbb{R} \iff a = 0$. We are interested in the case where we take the $-$ sign, which yields that $x = -\frac{a}{2}$ is the only extremum for a non-zero a. Since

$$\lim_{x\to\pm\infty} M(x) = 0, \ M(-\frac{a}{2}) = \tanh\left(\frac{\sqrt{2}}{8}a\right)$$

we conclude that $x = -\frac{a}{2}$ is the global maximum of $M(x)$ if $a > 0$ (Global minimum if $a < 0$).

Since $\tanh(x)$ is a strictly increasing function, then for any nonzero x, $|\tanh(x)| > 0$. Furthermore, using the fact that it is and odd function, we note that for all $|a| > K, |\tanh(a)| > \tanh(K)$. Hence,

$$\begin{aligned} ||\mathcal{T}_a u^*(x) - u^*(x)|| &= \sup_{x\in\mathbb{R}}\{|M(x)|\} \\ &= \left|\tanh\left(\frac{\sqrt{2}}{8}a\right)\right| \\ &> \tanh\left(\frac{\sqrt{2}}{8}K\right) \\ &> 0 \end{aligned}$$

and picking $\delta = \left|\tanh\left(\frac{\sqrt{2}}{8}K\right)\right|$ gives us the desired result. ∎

Proposition 2.7.3 *For any $v \in E = \operatorname{span}\{\phi_1, \phi_2\}$, the following function is $\mathcal{C}^\infty$.*

$$\begin{aligned} h : \mathbb{R} &\to \mathcal{C}^0_{unif}(\mathbb{R},\mathbb{R}) \times \mathbb{R} \\ a &\mapsto \mathcal{T}_a v \end{aligned}$$

Proof: Let $v \in E$. Then, $v = s\phi_1 + t\phi_2 = \begin{pmatrix} s \cdot \xi u^* \\ t \end{pmatrix}$. This implies that

$$\mathcal{T}_a v = \begin{pmatrix} s \cdot \mathcal{T}_a \xi u^* \\ t \end{pmatrix} = \begin{pmatrix} s \cdot u_x^*(x+a) \\ t \end{pmatrix}$$

Again, $u^*(x+a)$ is $\mathcal{C}^\infty$ $\forall a \in \mathbb{R}$, and we conclude that h is also $\mathcal{C}^\infty$ smooth. ∎

Proposition 2.7.4 *The projections $\mathcal{T}_a P \mathcal{T}_{-a}$ are $\mathcal{C}^\infty$ in a in the operator norm.*

Proof: First, we construct a formula for the projection for a given a.

Let $\begin{pmatrix} v(x) \\ c \end{pmatrix} \in \mathcal{C}^0_{unif}(\mathbb{R}.\mathbb{R}) \times \mathbb{R}$. Then,

$$P\begin{pmatrix} v(x) \\ c \end{pmatrix} = \frac{\left\langle \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix}, \begin{pmatrix} v(x) \\ c \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix}, \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} \right\rangle} \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} + \frac{\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} v(x) \\ c \end{pmatrix} \right\rangle}{\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

since $\phi_1 = \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix}$, $\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and P is the projection on span$\{\phi_1, \phi_2\}$. We note that

$$\left\langle \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix}, \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} \right\rangle = \int_{-\infty}^{\infty} (\xi u^*)^2 \, dx = \frac{\sqrt{2}}{12}$$

and also that

$$\left\langle \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle = 1$$

Hence,

$$P\begin{pmatrix} v(x) \\ c \end{pmatrix} = \frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(x)\xi u^* \, dx \cdot \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}$$

Therefore,

$$\begin{aligned} P\mathcal{T}_{-a}\begin{pmatrix} v(x) \\ c \end{pmatrix} &= P\begin{pmatrix} v(x-a) \\ c \end{pmatrix} \\ &= \left(\frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(x-a)\xi u^*(x) \, dx\right) \cdot \begin{pmatrix} \xi u^*(x) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix} \end{aligned}$$

We can make the change of variable $z = x - a$ in order to shift the explicit dependence on a from v to ξu^*, namely,

$$\left(\frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(x-a)\xi u^*(x) \, dx\right) = \left(\frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(z)\xi u^*(z+a) \, dz\right)$$

Finally, we get a formula for $\mathcal{T}_a P\mathcal{T}_{-a}$:

$$\mathcal{T}_a P\mathcal{T}_{-a}\begin{pmatrix} v(x) \\ c \end{pmatrix} = \left(\frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(z)\xi u^*(z+a) \, dz\right) \cdot \begin{pmatrix} \xi u^*(x+a) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ c \end{pmatrix}$$

Differentiating the resulting expression with respect to a, we get

$$\begin{aligned} \frac{d}{da}\mathcal{T}_a P\mathcal{T}_{-a}\begin{pmatrix} v(x) \\ c \end{pmatrix} &= \frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(z)u^*_{xx}(z+a) \, dz \cdot \begin{pmatrix} \xi u^*(x+a) \\ 0 \end{pmatrix} \\ &+ \frac{12}{\sqrt{2}}\int_{-\infty}^{\infty} v(z)\xi u^*(z+a) \, dz \cdot \begin{pmatrix} u^*_{xx}(x+a) \\ 0 \end{pmatrix} \end{aligned}$$

Since u^* is $\mathcal{C}^\infty$, we can compute further derivatives in a. It follows that the operator $\mathcal{T}_a P\mathcal{T}_{-a}$ is also $\mathcal{C}^\infty$ smooth. ∎

The fact that the spectrum of L is bounded away from the imaginary axis, in the left half plane, except for the eigenvalue at 0, which has a two-dimensional generalized eigenspace, and theorems 2.7.1, 2.7.2, 2.7.3, and 2.7.4 allow us to apply the center manifold theorem of Sandstede, Scheel and Wulff [12], which can be found in Appendix B. We obtain that the set

$$\mathcal{S} = \left\{ \begin{pmatrix} \mathcal{T}_a u^* \\ c \end{pmatrix} ; a \in \Gamma, c \in \mathbb{R} \right\}$$

is a normally hyperbolic locally exponentially attracting $\mathcal{C}^n$ smooth invariant center manifold for (2.2.1), for any $n \in \mathbb{N}$. The dynamics of (2.2.1) on our center manifold are governed by the system :

$$\begin{aligned} a_t &= c \\ c_t &= 0 \end{aligned} \tag{2.7.5}$$

We note here that the condition $g \in \mathcal{C}^\infty$ on the perturbation term of equation (2.1.1) can be relaxed to $g \in \mathcal{C}^{k+2}, k \geq 1$, at the cost of losing some smoothness of our center manifold, which would become $\mathcal{C}^{k+1}$ smooth.

In particular, the relative equilibrum corresponding to $c = 0$, which we name Θ, lies on the center manifold $\mathcal{S}$ (See Figure 2.7).

$$\Theta = \left\{ \begin{pmatrix} \mathcal{T}_a u^* \\ 0 \end{pmatrix} ; a \in \Gamma \right\}$$

2.8 Parametrization

As was shown in [12], we can parametrize a neighborhood of Θ as follows: for any $U \in \mathcal{C}^0_{unif}(\mathbb{R}, \mathbb{R}) \times \mathbb{R}$ close enough to Θ,

$$U = U_1 + U_2 + v$$

for some $U_1 \in \Theta$, U_2 of the form $c\phi_2$, $c \in \mathbb{R}$, and $v \in \mathcal{T}_a(W)$. (Note that ϕ_2 and W are the same as in the decomposition (2.6.1), and that the translation $a = a(U)$ is the same for U_1 and v.) A visual interpretation of this orthogonal decomposition can be seen in Figure 2.8. This allows us to decompose $U = (u, c)^t$ as

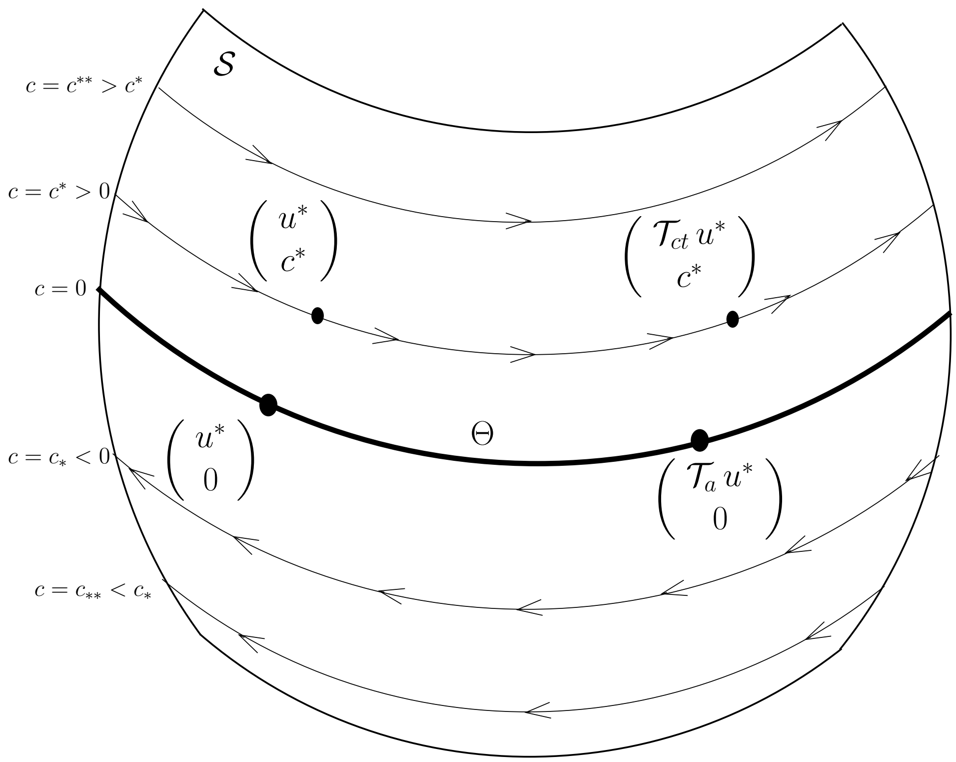

Figure 2.7: Relative equilibria on the center manifold

$$\begin{pmatrix} u \\ c \end{pmatrix} = \mathcal{T}_a \left[\begin{pmatrix} u^* \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} w \\ 0 \end{pmatrix} \right] \tag{2.8.1}$$

For c close to 0, this gives us a representation of our center manifold $\mathcal{S}$. Indeed, if $(u, c)^t \in \mathcal{S}$, then $u = \mathcal{T}_a u^*$, and

$$\begin{pmatrix} \mathcal{T}_a u^* \\ c \end{pmatrix} = \mathcal{T}_a \left[\begin{pmatrix} u^* \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} w \\ 0 \end{pmatrix} \right] \tag{2.8.2}$$

which means that $w = 0$. (In the general case, when the shape of the wave depends on the speed c, w would be non-zero).

Substituting this decomposition into $u_t = \mathcal{A}u + \mathcal{F}(u, c)$ allows us to obtain a compatibility condition. Hence, we proceed with the substitution, and then apply the translational invariance of $\mathcal{F}$:

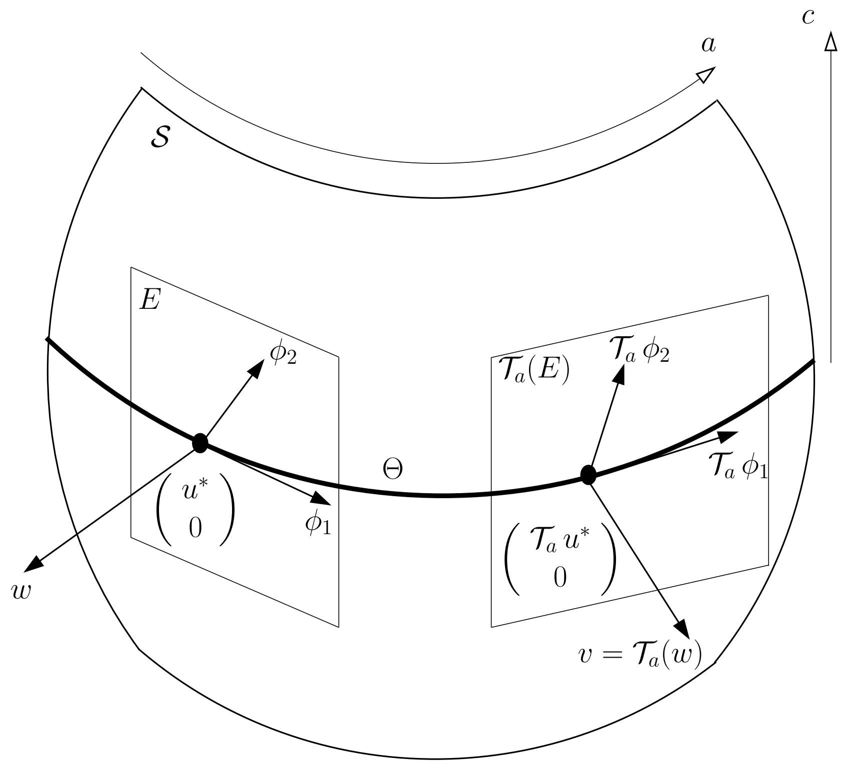

Figure 2.8: Decomposition of a neighborhood of Θ

$$\dot{a}(t)\mathcal{T}_{a(t)}\begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} = \mathcal{T}_{a(t)}\begin{pmatrix} \mathcal{A}u^* \\ 0 \end{pmatrix} + \mathcal{T}_{a(t)}\begin{pmatrix} \mathcal{F}(u^*, c) \\ 0 \end{pmatrix}$$

Next, we use the definition of L which tells us that

$$L\begin{pmatrix} 0 \\ c \end{pmatrix} = \mathcal{F}_c(u^*, 0)\begin{pmatrix} c \\ 0 \end{pmatrix} \tag{2.8.3}$$

Also, since u^* is a steady state solution of the PDE,

$$0 = \begin{pmatrix} \mathcal{A}u^* \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{F}(u^*, 0) \\ 0 \end{pmatrix} \tag{2.8.4}$$

Applying $\mathcal{T}_{-a(t)}$ and using the above properties, we obtain

$$\dot{a}(t)\begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} = L\begin{pmatrix} 0 \\ c \end{pmatrix} + \begin{pmatrix} \mathcal{F}(u^*, c) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{F}(u^*, 0) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{F}_c(u^*, 0)c \\ 0 \end{pmatrix}$$

We write $H = \mathcal{F}(u^*, c) - \mathcal{F}(u^*, 0) - \mathcal{F}_c(u^*, 0)c$. Furthermore, $L(0, c)^t = c(\xi u^*, 0)^t$, and since $(u, c)^t$ lies on $\mathcal{S}$, we can additionnally use the fact that $\dot{a}(t) = c$.

$$c \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} = c \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} + \begin{pmatrix} H \\ 0 \end{pmatrix} \tag{2.8.5}$$

This gives us our compatibility condition

$$\begin{pmatrix} H \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{2.8.6}$$

2.9 Analysis of the Perturbed System

By using similar arguments to those of [10], the spectral projections of L allow us to conclude that the set $\mathcal{S}$ will persist in the perturbed system (2.1.1) for values of ϵ small enough. It will remain a locally exponentially attracting invariant manifold $\mathcal{S}_\epsilon \cong \mathcal{S}$, but the dynamics on $\mathcal{S}_\epsilon$ may differ from those on $\mathcal{S}$. In particular, the equilibrium points on $\mathcal{S}_\epsilon$, near Θ, will correspond to blocking points for our travelling wave.

We are now ready to consider the perturbed system (2.1.1) for small $\epsilon > 0$. Let $\begin{pmatrix} u \\ c \end{pmatrix}$ be a solution of (2.1.1) on S_ϵ, near Θ. We will use the decomposition (2.8.2), adding a term proportional to ϵ to take into account the effects of the perturbation.

$$\begin{pmatrix} u \\ c \end{pmatrix} = \mathcal{T}_a \left[\begin{pmatrix} u^* \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \epsilon \begin{pmatrix} E_1 \\ 0 \end{pmatrix} \right] \tag{2.9.1}$$

Since $g(u, x, \epsilon)$ is assumed to be bounded and break symmetry, then $E(a(t), c, \epsilon) = (E_1(a(t), c, \epsilon), 0)^t \in W$ is bounded. Substituting this parametrization into (2.1.1) and applying the translational invariance of $\mathcal{F}$, we get

$$\dot{a}(t)\mathcal{T}_{a(t)} \left(\begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} + O(\epsilon) \right) \quad = \quad \mathcal{T}_{a(t)} \left(\begin{pmatrix} \mathcal{A} u^* \\ 0 \end{pmatrix} + \begin{pmatrix} \epsilon \mathcal{A} E_1 \\ 0 \end{pmatrix} \right)$$

$$
\begin{aligned}
&+ \quad \mathcal{T}_{a(t)} \begin{pmatrix} \mathcal{F}\left(u^* + \epsilon E_1, c\right) \\ 0 \end{pmatrix} \\
&+ \quad \epsilon \begin{pmatrix} g(\mathcal{T}_{a(t)}(u^* + \epsilon E_1), x, \epsilon) \\ 0 \end{pmatrix}
\end{aligned}
$$

We now apply $\mathcal{T}_{-a(t)}$ on both sides of the equation and use identities (2.8.3) and (2.8.4) as well as the fact that

$$
L \begin{pmatrix} \epsilon E_1 \\ 0 \end{pmatrix} = \begin{pmatrix} \epsilon \mathcal{A} E_1 \\ 0 \end{pmatrix} + \mathcal{F}_u(u^*, 0) \begin{pmatrix} \epsilon E_1 \\ 0 \end{pmatrix} \tag{2.9.2}
$$

which yields

$$
\begin{aligned}
\dot{a}(t) \left(\begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} + O(\epsilon) \right) \quad &= \quad L \begin{pmatrix} 0 \\ c \end{pmatrix} + \epsilon L \begin{pmatrix} E_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathcal{F}\left(u^* + \epsilon E_1, c\right) \\ 0 \end{pmatrix} \\
&- \quad \begin{pmatrix} \mathcal{F}(u^*, 0) \\ 0 \end{pmatrix} - \begin{pmatrix} \mathcal{F}_c(u^*, 0) c \\ 0 \end{pmatrix} - \epsilon \begin{pmatrix} \mathcal{F}_u(u^*, 0) E_1 \\ 0 \end{pmatrix} \\
&+ \quad \epsilon \begin{pmatrix} \mathcal{T}_{-a(t)} g(\mathcal{T}_{a(t)}(u^* + \epsilon E_1), x, \epsilon) \\ 0 \end{pmatrix}
\end{aligned}
$$

Next, we use the fact that as $(\epsilon, c) \to (0, 0)$, the following differences go to 0.

$$
\frac{1}{\epsilon} \left(\mathcal{F}\left(u^* + \epsilon E_1, c\right) - \mathcal{F}\left(u^*, c\right) \right) - \mathcal{F}_u\left(u^*, c\right) E_1 \to 0
$$

$$
\mathcal{F}_u\left(u^*, c\right) E_1 - \mathcal{F}_u\left(u^*, 0\right) E_1 \to 0
$$

$$
\mathcal{T}_{-a(t)} g\left(\mathcal{T}_{a(t)}\left(u^* + \epsilon E_1\right), x, \epsilon\right) - \mathcal{T}_{-a(t)} g\left(\mathcal{T}_{a(t)} u^*, x, 0\right) \to 0
$$

This allows us to define

$$
\begin{aligned}
q(a, c, \epsilon) \quad &= \quad \frac{1}{\epsilon}(\mathcal{F}(u^* + \epsilon E_1, c) - \mathcal{F}(u^*, c)) - \mathcal{F}_u(u^*, c) E_1 + \mathcal{F}_u(u^*, c) E_1 - \mathcal{F}_u(u^*, 0) E_1 \\
&+ \quad \mathcal{T}_{-a(t)} g(\mathcal{T}_{a(t)}(u^* + \epsilon E_1), x, \epsilon) - \mathcal{T}_{-a(t)} g(\mathcal{T}_{a(t)} u^*, x, 0)
\end{aligned}
$$

where $\lim_{(c,\epsilon)\to(0,0)} q(a, c, \epsilon) = 0$.

Adding and substracting $(\epsilon q(a,c,\epsilon),0)^t$ from the equation, we notice the appearance of $(H,0)^t$. We then apply projection P to the resulting equation. We get

$$\begin{aligned}\dot{a}(t)\left(P\begin{pmatrix}\xi u^*\\ 0\end{pmatrix}+O(\epsilon)\right) &= PL\begin{pmatrix}0\\ c\end{pmatrix}+\epsilon PL\begin{pmatrix}E_1\\ 0\end{pmatrix}+P\begin{pmatrix}H\\ 0\end{pmatrix}\\ &+ \epsilon P\begin{pmatrix}\mathcal{T}_{-a(t)}g(\mathcal{T}_{a(t)}u^*,x,0)\\ 0\end{pmatrix}+\epsilon P\begin{pmatrix}q\\ 0\end{pmatrix}\end{aligned}$$

Remark 2.9.3 For any $y\in\mathcal{C}^0_{\text{unif}}(\mathbb{R},\mathbb{R})$, $PL\begin{pmatrix}y\\ 0\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$.

Proof: We can write $\begin{pmatrix}y\\ 0\end{pmatrix}=a\phi_1+\omega$ with $\omega\in W, a\in\mathbb{R}$ and,

$$\begin{aligned}L(a\phi_1+\omega) &= aL(\phi_1)+L(\omega)\\ &= L(\omega)\in W\end{aligned}$$

∎

To simplify further, we use remark (2.9.3), the compatibility condition (2.8.6) and the facts that $L(0,c)^t=c(\xi u^*,0)^t$, and $P(\xi u^*,0)=(\xi u^*,0)=\phi_1$.

$$\dot{a}(t)\left(\phi_1+O(\epsilon)\right)=c\phi_1+\epsilon P\begin{pmatrix}\mathcal{T}_{-a(t)}g(\mathcal{T}_{a(t)}u^*,x,0)\\ 0\end{pmatrix}+\epsilon P\begin{pmatrix}q\\ 0\end{pmatrix}$$

At this point, we project the resulting equation on ϕ_1 to obtain

$$\begin{aligned}\dot{a}(t)\left(\langle\phi_1,\phi_1\rangle(1+O(\epsilon))\right) &= c\langle\phi_1,\phi_1\rangle+\epsilon\left\langle\begin{pmatrix}\mathcal{T}_{-a(t)}g(\mathcal{T}_{a(t)}u^*,x,0)\\ 0\end{pmatrix},\phi_1\right\rangle\\ &+\epsilon\left\langle\begin{pmatrix}q\\ 0\end{pmatrix},\phi_1\right\rangle.\end{aligned}$$

Solving for $\dot{a}(t)$, we get

$$\dot{a}(t) = \frac{c\langle\phi_1,\phi_1\rangle + \epsilon\left\langle \begin{pmatrix} \mathcal{T}_{-a(t)}g(\mathcal{T}_{a(t)}u^*,x,0) \\ 0 \end{pmatrix}, \phi_1 \right\rangle + \epsilon\left\langle \begin{pmatrix} q \\ 0 \end{pmatrix}, \phi_1 \right\rangle}{\langle\phi_1,\phi_1\rangle(1+O(\epsilon))}.$$

We notice that

$$\frac{1}{1+O(\epsilon)} = 1 - O(\epsilon) + H.O.T.$$

We then neglect every term of order ϵ^2 or $c\epsilon$ since we assume ϵ, c to be small. This brings us to the key equation

$$\begin{aligned} \dot{a}(t) &= c + \epsilon\frac{\left\langle \begin{pmatrix} \mathcal{T}_{-a(t)}g(\mathcal{T}_{a(t)}u^*,x,0) \\ 0 \end{pmatrix}, \phi_1 \right\rangle}{\langle\phi_1,\phi_1\rangle} + \epsilon q^*(a,c,\epsilon) \qquad (2.9.4) \\ &= c - \epsilon r(a) + \epsilon q^*(a,c,\epsilon) \end{aligned}$$

where

$$r(a) = -\frac{\left\langle \begin{pmatrix} \mathcal{T}_{-a(t)}g(\mathcal{T}_{a(t)}u^*,x,0) \\ 0 \end{pmatrix}, \phi_1 \right\rangle}{\langle\phi_1,\phi_1\rangle} \qquad (2.9.5)$$

and $\lim_{(c,\epsilon)\to(0,0)} q^*(a,c,\epsilon) = 0$.

Proposition 2.9.6 *For any (ϵ, c) sufficiently close to $(0,0)$, there exists a curve of equilibria, $\Theta_\epsilon \subset \mathcal{S}_\epsilon$, such that on this curve, $\dot{a}(t) = 0$ for the perturbed system. This curve is the graph of a function $c_\epsilon(a) = \epsilon r(a) + \epsilon\sigma(a,\epsilon)$, where $\lim_{\epsilon\to 0}\sigma(a,\epsilon) = 0$.*

Proof: We take $c_\epsilon(a)$ to be the function defined implicitly by $c = \epsilon r(a) - \epsilon q^*(a,c,\epsilon)$, which exists by the implicit function theorem. ∎

2.10 The Wedge of Wave-Blocking

Consider a travelling wave with speed $c = c^*$ (c^* near 0) in the perturbed system. Its flow is governed by $(a(t), c^*)$. We conclude from proposition 2.9.6 that if the pair (c^*, ϵ) satisfies

$$\inf_{a\in\mathbb{R}}\{r(a) + \sigma(a,\epsilon)\} < \frac{c^*}{\epsilon} < \sup_{a\in\mathbb{R}}\{r(a) + \sigma(a,\epsilon)\} \tag{2.10.1}$$

then at some point between $a = -\infty$ and $a = \infty$, $\dot{a} = 0$. This means that at some point, the wave will reach an equilibrium point. In other words, our wave will be blocked.

On the other hand, if

$$\frac{c^*}{\epsilon} < \inf_{a\in\mathbb{R}}\{r(a) + \sigma(a,\epsilon)\} \quad \text{or} \quad \frac{c^*}{\epsilon} > \sup_{a\in\mathbb{R}}\{r(a) + \sigma(a,\epsilon)\}$$

then $\dot{a}$ will be either strictly negative or strictly positive, in which case the wave will progress indefinitely, and no block will occur. We summarize the results in the following theorem:

Theorem 2.10.2 *Consider the differential equation* (2.1.1) *satisfying hypothesis* (2.1.8). *For all $\epsilon \geq 0$ close enough to 0, the semi-flow generated by the differential equation admits a two-dimensional asymptotically stable $\mathcal{C}^n$-smooth invariant manifold $\mathcal{S}_\epsilon$ for any $n \in \mathbb{N}$. In the case where $\epsilon = 0$, $\mathcal{S}_0 = \mathcal{S}$, and solutions on $\mathcal{S}$ correspond to relative equilibria. These are waves propagating at constant speed. On the other hand, when $\epsilon \neq 0$, solutions on $\mathcal{S}_\epsilon$ correspond to perturbed relative equilibria with non-constant drift speed. In this case, wave-blocking occurs if (c^*, ϵ) belongs to a wedge in parameter space, characterized by inequality* (2.10.1).

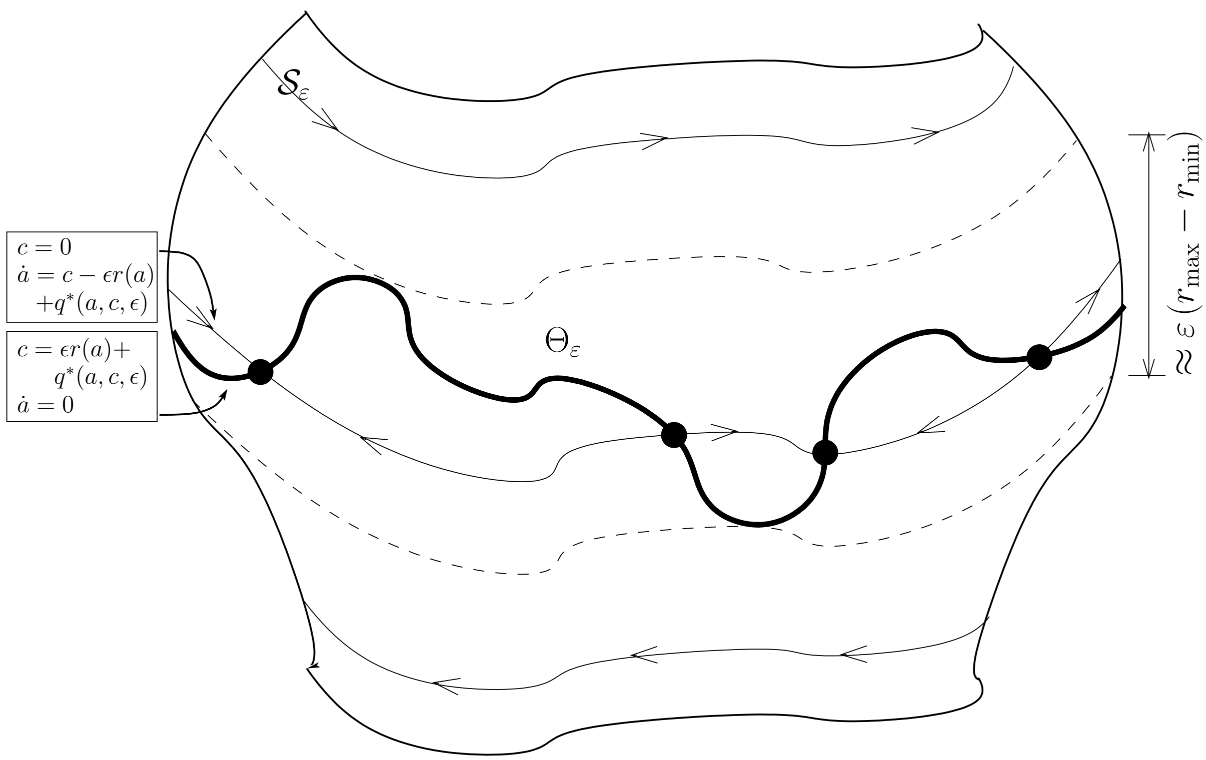

Figure 2.9: Perturbed equilibrium on $\mathcal{S}_\epsilon$. The flow is indicated for two of the curves by $\dot{a}$. On the curve $c = 0$, the flow is non-constant. The curve Θ_ϵ ($\dot{a} = 0$) corresponds to the set of blocking points.

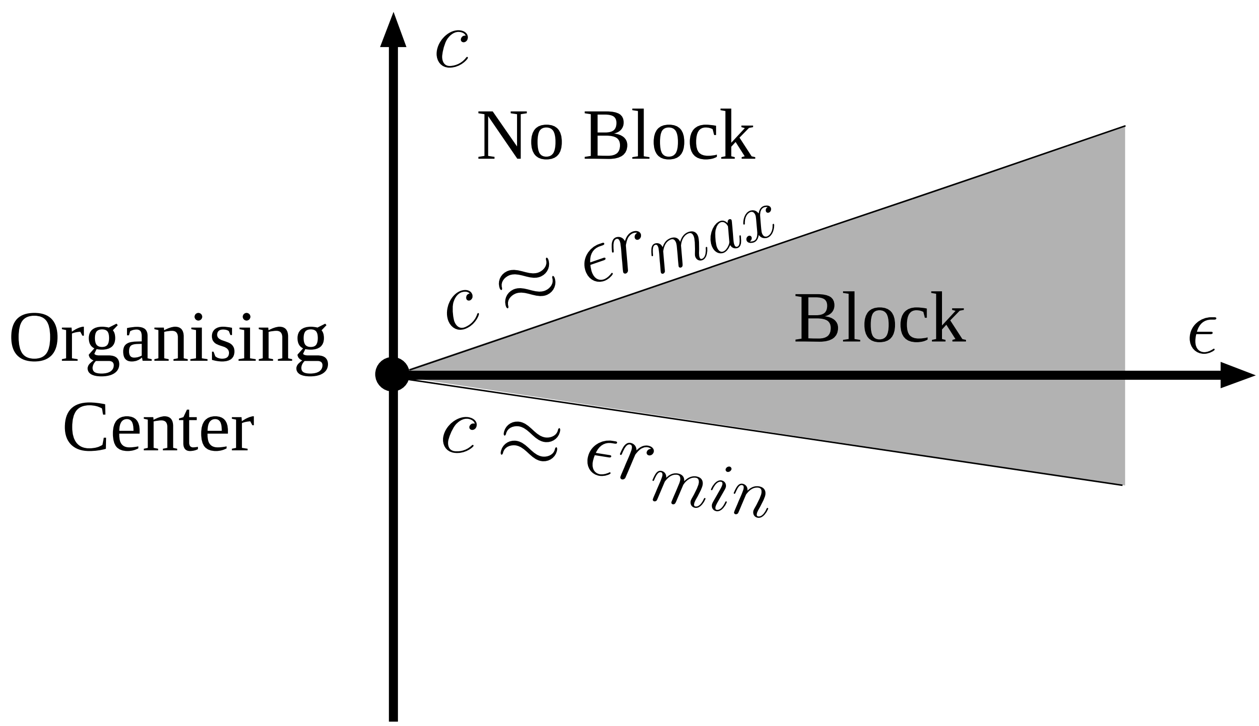

Figure 2.10: Wedge in (ϵ, c) space where wave blocking occurs.

Chapter 3

Numerical Examples

3.1 Applying our Result

In this chapter, we show the implications of theorem 2.10.2 using specific examples of bounded perturbations. This allows us to compute explicitely the wedge near the organising center where wave-blocking will occur. We will then compare the theoretical result of theorem 2.10.2 with numerically computed values of wave-blocking for our examples.

As pointed out in inequality (2.10.1), the terms which define the wedge of wave blocking are $\sup_{a\in\mathbb{R}}\{r(a)+\sigma(a,\epsilon)\}$ and $\inf_{a\in\mathbb{R}}\{r(a)+\sigma(a,\epsilon)\}$. On the other hand, close enough to $(c,\epsilon)=(0,0)$, the term $\sigma(a,\epsilon)$ becomes negligeable. Hence, the slope of the curves of initial block will be approximated by $\sup_{a\in\mathbb{R}}\{r(a)\}$ and $\inf_{a\in\mathbb{R}}\{r(a)\}$.

We begin by computing $\langle\phi_1,\phi_1\rangle$.

$$\begin{aligned}\langle\phi_1,\phi_1\rangle &= \langle(\xi u^*,0)^t,(\xi u^*,0)^t\rangle\\ &= \int_{-\infty}^{\infty}\left(\frac{\sqrt{2}}{8}\left(1-\tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right)\right)^2 dx\\ &= \frac{\sqrt{2}}{12}\end{aligned}$$

To proceed further, we need to specify the form of the perturbation g. Note that all the figures are placed at the end of the chapter.

3.2 Example 1: Smooth Example with Explicit Calculations

Example 3.2.1 Consider the perturbation $g(x) = \xi u^* = \frac{\sqrt{2}}{8}\left(1 - \tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right)$ (See Figure 2.1). We verify that $g(x+a) \neq g(x)$ except when $a = 0$, and that $g(x) \in \mathcal{C}^\infty$ is bounded. Hence, $g(x)$ satisfies hypothesis (2.1.8). To finish computing $r(a)$, we need to find the following scalar product:

$$\left\langle \begin{pmatrix} \mathcal{T}_{-a} g(\mathcal{T}_a u^*, x, 0) \\ 0 \end{pmatrix}, \begin{pmatrix} \xi u^* \\ 0 \end{pmatrix} \right\rangle = \langle g(x-a), \xi u^* \rangle$$

$$= \int_{-\infty}^{\infty} \frac{\sqrt{2}}{8}\left(1 - \tanh^2\left(\frac{\sqrt{2}}{4}x\right)\right)\frac{\sqrt{2}}{8}\left(1 - \tanh^2\left(\frac{\sqrt{2}}{4}(x-a)\right)\right) dx$$

$$= \frac{\sqrt{2}e^{\frac{\sqrt{2}a}{2}}\left(-4e^{\frac{\sqrt{2}a}{2}} + 4 + \sqrt{2}a + \sqrt{2}ae^{\frac{\sqrt{2}a}{2}}\right)}{4\left(-1 + 3e^{\frac{\sqrt{2}a}{2}} - 3e^{\sqrt{2}a} + e^{\frac{3\sqrt{2}a}{2}}\right)}$$

We find that this function reaches its maximum value of $\frac{\sqrt{2}}{12}$ when $a = 0$, and that it tends to zero as $a \to \pm\infty$. From this, we conclude that $\sup_{a\in\mathbb{R}}\{r(a)\} = \frac{\sqrt{2}}{12}/\frac{\sqrt{2}}{12} = 1$ and $\inf_{a\in\mathbb{R}}\{r(a)\} = 0$. Hence, our theoretical approach suggests that wave blocking will occur inside a wedge defined by $0 < \frac{c}{\epsilon} < 1$.

We compare the slope of our blocking wedge with numerically obtained values from simulations. The simulations used an explicit method to solve problem (2.1.1) with a translation of u^* as initial condition. As such, the wave was considered blocked if it could not progress past the perturbation. Initial blocking values of ϵ were obtained by binary search with fixed decimal precision, for some values of c taken at a regular interval, starting at 0. The results are graphed in Figure 3.2.

3.3 Example 2 : Smooth Numerical Example with Blocking Wedge

Example 3.3.1 Consider the perturbation $g(x) = \frac{x}{(1+x^4)}$. Again, we verify that $g(x)$ breaks the translationnal symmetry, and that $g(x) \in \mathcal{C}^\infty$ is bounded. Hence, this perturbation also satisfies hypothesis (2.1.8). A graphical representation of this perturbation is shown in Figure 3.3.

This time, the scalar product $\langle g(x-a), \xi u^* \rangle$ cannot be calculated analytically. Therefore, we approximate $\sup_{a\in\mathbb{R}} \langle g(x-a), \xi u^* \rangle$ and $\inf_{a\in\mathbb{R}} \langle g(x-a), \xi u^* \rangle$ by calculating $\int_{-\infty}^{\infty} g(x-a)\xi u^* \, dx$ numerically for different values of a and then taking the maximum/minimum. Since u^* and $g(x)$ are both bounded and localised, the terms in the integral become negligeable outside [-100,100], which makes this computation feasible. By the same reasoning, the value of a which maximises this scalar product must be in this interval. We therefore pick values of a at a fixed interval inside [-100,100], and the continuity of the integral with respect to a guarantees that this will yield a good approximation of the real maximum/minimum.

We have found that $\sup_{a\in\mathbb{R}}\{r(a)\} \approx 0.5712$ and $\inf_{a\in\mathbb{R}}\{r(a)\} \approx -0.5712$. Hence, our theoretical approach suggests that wave blocking will occur inside a wedge defined by $c = 0.5712\epsilon$ and $c = -0.5712\epsilon$. We compare this result with numerical simulations like in the previous example. Some screenshots of a typical numerical simulation are presented in Figure 3.4. The results are graphed in Figure 3.5.

3.4 Example 3: Triangular Perturbation with Explicit Calculations

Example 3.4.1 The last perturbation we will look at is the triangle of base $2h$, $h \in \mathbb{R}$, and height 1, parametrized by the following equation:

$$g(x) = \begin{cases} 1 + \frac{x}{h} & \text{if } -h \leq x \leq 0; \\ 1 - \frac{x}{h} & \text{if } 0 \leq x \leq h; \\ 0 & \text{otherwise.} \end{cases}$$

This perturbation breaks translational symmetry and is bounded. On the other hand, it is not differentiable at $x = 0, \pm h$. Hence, it satisfies hypothesis (2.1.8) almost everywhere. We show this example for two reasons. First, we wish to motivate the generality of our result. Secondly, this example is simple enough that we can compute $r(a)$ analytically for a family of perturbations parametrized by $h \in \mathbb{R}$. We note that $\sup_{a\in\mathbb{R}}\{\langle g(x-a), \xi u^*(x)\rangle\} = \sup_{a\in\mathbb{R}}\{\langle g(x), \xi u^*(x-a)\rangle\}$. Hence, to simplify the calculations, we will evaluate $\langle g(x), \xi u^*(x-a)\rangle$, using the exponential form of ξu^*.

$$\begin{aligned}
&\langle g(x), \xi u^*(x-a)\rangle \\
= \; & \int_{-h}^{0} (1 + \frac{x}{h}) \xi u^*(x-a) dx + \int_{0}^{h} (1 - \frac{x}{h}) \xi u^*(x-a) dx \\
= \; & \int_{-h}^{h} \xi u^*(x-a) dx + \frac{1}{h} \int_{-h}^{0} x \xi u^*(x-a) dx - \frac{1}{h} \int_{0}^{h} x \xi u^*(x-a) dx \\
= \; & u(x-a)|_{-h}^{h} + \frac{1}{h} \left[-\frac{4}{\sqrt{2}} \ln\left(1 + e^{-\frac{a}{\sqrt{2}}}\right) + h \left(1 - \frac{1}{1 + e^{\frac{-h-a}{\sqrt{2}}}}\right) \right. \\
& \left. - h \left(1 - \frac{1}{1 + e^{\frac{h-a}{\sqrt{2}}}}\right) + \frac{2}{\sqrt{2}} \ln\left(1 + e^{\frac{-h-a}{\sqrt{2}}}\right) + \frac{2}{\sqrt{2}} \ln\left(1 + e^{\frac{h-a}{\sqrt{2}}}\right) \right] \\
= \; & \left(\frac{1}{1 + e^{-\frac{h-a}{\sqrt{2}}}} - \frac{1}{1 + e^{\frac{h+a}{\sqrt{2}}}} \right) + \left(\frac{1}{1 + e^{\frac{h-a}{\sqrt{2}}}} - \frac{1}{1 + e^{\frac{-h-a}{\sqrt{2}}}} \right) \\
& + \frac{1}{h} \left[-\frac{4}{\sqrt{2}} \ln\left(1 + e^{\frac{-a}{\sqrt{2}}}\right) + \frac{2}{\sqrt{2}} \ln\left(1 + e^{\frac{-h-a}{\sqrt{2}}}\right) + \frac{2}{\sqrt{2}} \ln\left(1 + e^{\frac{h-a}{\sqrt{2}}}\right) \right]
\end{aligned}$$

The maximum of $\langle g(x), \xi u^*(x-a)\rangle$ is obtained for $a=0$, in which case we get :

$$\sup_{a\in\mathbb{R}}\{\langle g(x), \xi u^*(x-a)\rangle\} = \frac{1}{h}\left[-\frac{4}{\sqrt{2}}\ln 2 + \frac{2}{\sqrt{2}}\ln\left(1+e^{-\frac{h}{\sqrt{2}}}\right) + \frac{2}{\sqrt{2}}\ln\left(1+e^{\frac{h}{\sqrt{2}}}\right)\right]$$

We also note that the infimum is 0.

In order to compare with numerical simulations, we chose the value of $h=10$, in which case, we get that the line of initial block has approximately the equation $c = 6,824\epsilon$.

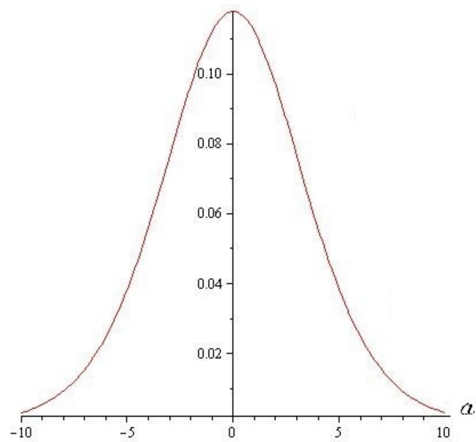

Figure 3.1: $\langle g(x-a), \xi u^* \rangle$

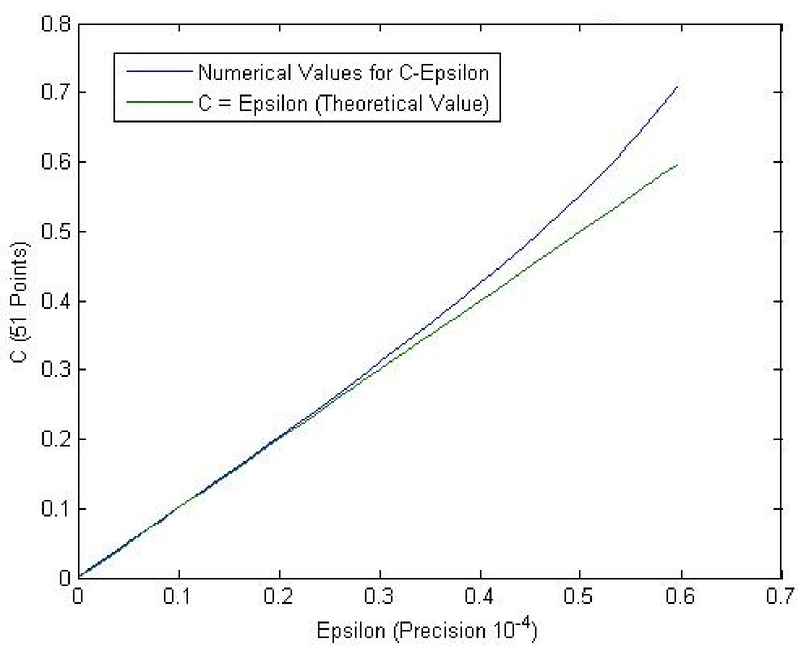

Figure 3.2: Comparison between the blocking wedge (green) and the numerically calculated initial values of wave blocking (blue) for the perturbation $g(x) = \xi u^*$. The graph is built from 51 points, and the initial blocking values of ϵ are calculated with a precision of 10^{-4}.

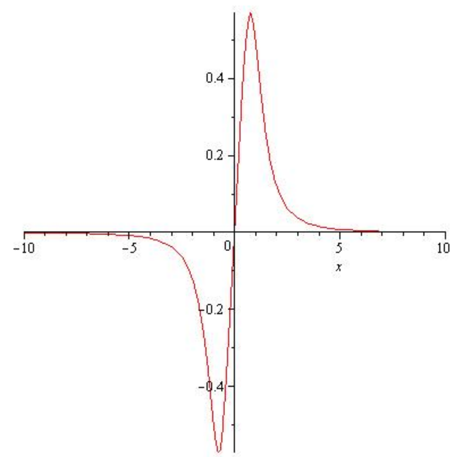

Figure 3.3: $g(x) = \frac{x}{(1+x^4)}$

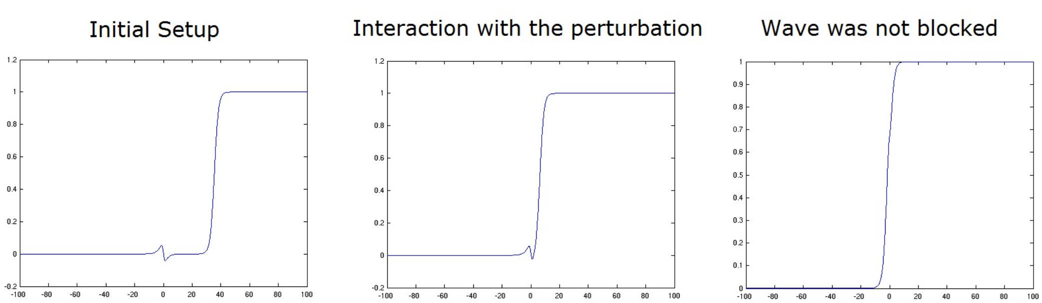

Figure 3.4: Three screenshots of a simulation taken at different times.

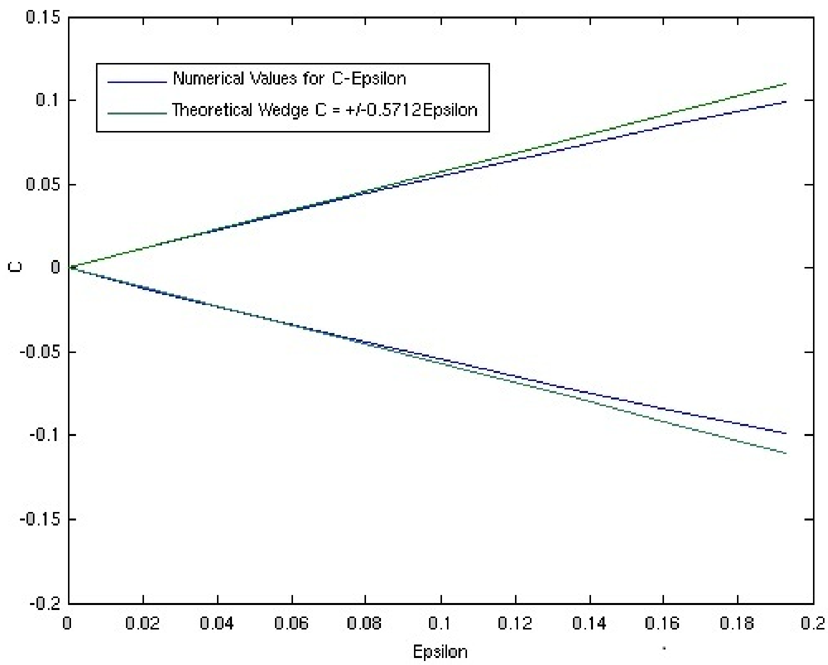

Figure 3.5: Comparison between the blocking wedge (green) and the numerically calculated initial values of wave blocking (blue) for the rational perturbation $g(x) = \frac{x}{(1+x^4)}$.

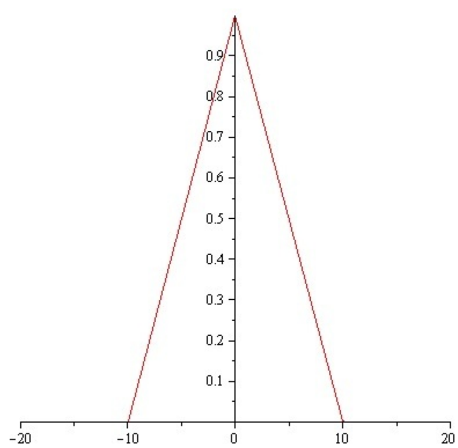

Figure 3.6: $g(x)$, a triangular perturbation. In the graph shown here, $h = 10$.

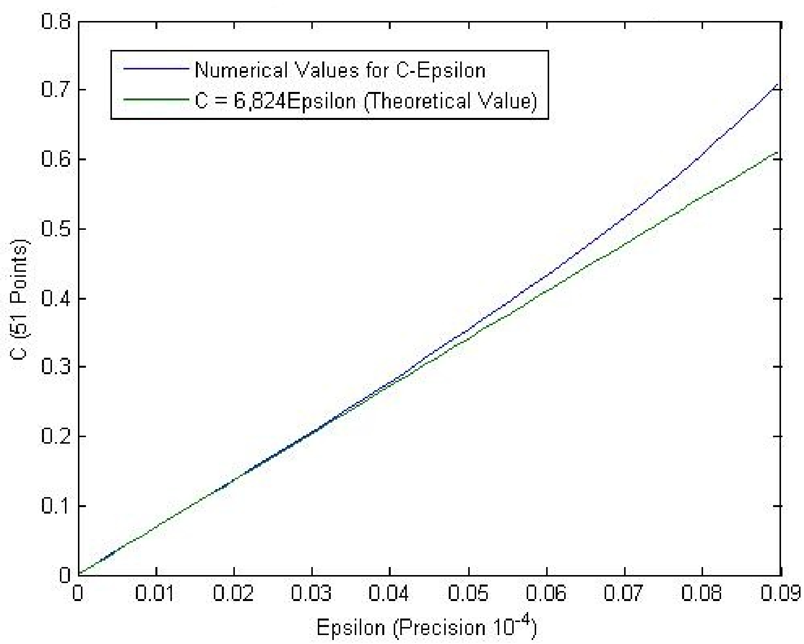

Figure 3.7: Comparison between the blocking wedge (green) and the numerically calculated initial values of wave blocking (blue) for the triangle-shaped perturbation with $h = 10$.

Chapter 4

Conclusion

4.1 Summary

In this thesis, we have analyzed the phenomenon of wave-blocking, focusing on the interplay between the speed of the travelling wave and the size of the perturbation. We believe that understanding the organizing center (small wave speed and small perturbation) is a great leap towards understanding wave blocking. As such, we have demonstrated a technique that allows us to find a wedge originating from the organizing center in (ϵ, c) parameter space. This wedge defines when the wave will be blocked. We used the bistable equation (1.1.3) as an example of a reaction-diffusion equation satisfying our hypotheses in order to demonstrate our technique. This choice allowed us to do most calculations explicitly, which is not possible in general. In turn, this allowed us to compare our theory with numerical simulations, supporting our result.

Our technique is based on exploiting the translational symmetry of our reaction-diffusion equation. First off, we formulated a hypothesis on the symmetry of the reaction term $\mathcal{F}$, while asking that g be a perturbation that breaks this symmetry. We also assumed that $\mathcal{F}$ and g are both smooth and bounded functions. In order to guarantee that solutions will persist after being lightly perturbed, we verified a spectral condition on the linearization of our system which ensures stability. We then applied the center manifold theorem of [12] to reduce the dynamics of our system to a two-dimensionnal manifold, near our equilibrium. Using the dynamics on this manifold, as well as some carefully chosen projections, we derived an expression for the wedge of wave-blocking near the relative equilibrium. Our main result is summarized in theorem 2.10.2.

We have compared the blocking wedge predicted from this theory with values of blocking obtained through numerical simulations for various perturbations. In every case, near the organizing center, both approaches yield almost the same values, which confirms the theory. As the value of ϵ becomes bigger, the non-linear terms start to influence the blocking point, and the linear approximation begins to lose precision, while remaining a decent approximation.

4.2 Generalization

While our result is given for a particular equation (the bistable equation), our technique is general in nature. In particular, symmetry conditions, spectral stability and the center manifold theorem of [12] are all tools that are available in multiple dimensions. Furthermore, our result is a continuous wedge in parameter space, which is better than having to compute punctual values of blocking for a wide array of perturbation sizes. On the other hand, our result is focused around the organising center, and gives us very little information as to what happens elsewhere.

The above advantages distinguish our result from that of [11], which uses comparison theorems for scalar reaction-diffusion PDEs to find punctual values of blocking for specific values of ϵ. Indeed, comparison theorems do not have analogues in higher dimensions which makes this technique difficult to generalize.

A generalization of the result exposed here can be found in [9], where we consider the following setup:

Let X be a Banach space. Consider the differential equation:

$$\frac{du}{dt} = \mathcal{A}u + \mathcal{F}(u, c) + \epsilon\mathcal{G}(u, \epsilon)$$

where c and ϵ are real parameters, ϵ small, $\mathcal{A} : X \to X$ is such that $-\mathcal{A}$ is sectorial, the functions $\mathcal{F} : X^\alpha \times \mathbb{R} \to X$ and $\mathcal{G} : X^\alpha \times \mathbb{R} \to X$ are $\mathcal{C}^{k+2}$ smooth for some $k \geq 1$ and some $\alpha \in [0, 1)$, and $\mathcal{G}$ is unifomly bounded on $X \times [0, \epsilon_0]$ for some $\epsilon_0 > 0$ (see [6] for notation). This setup includes not only reaction-diffusion PDEs, but also integro-differential equations and delay reaction-diffusion PDEs. Using ideas similar to those in this thesis, we can arrive to a theorem similar to 2.10.2 for this general setup.

Appendix A

Modified l'Hospital's Rule

Theorem A.0.1 *Let $u(x) : \mathbb{R} \to \mathbb{R}$ and $v(x) : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$ functions such that*

$$\lim_{x \to \pm\infty} u(x) = 0 \ , \ \lim_{x \to +\infty} v(x) = \pm\infty \ , \ \lim_{x \to -\infty} v(x) = \pm\infty$$

and

$$\frac{u'(x) \cdot v(x)^2}{v'(x)}$$

exists and is bounded for all $|x| > K$ for some K.

Then, $u(x) \cdot v(x)$ is bounded for all $x \in \mathbb{R}$.

Proof: Define $f(y) = u(\frac{1}{y})$ and $g(y) = v(\frac{1}{y})^{-1}$. We notice that

$$\lim_{y \to 0^+} f(y) = 0, \ \lim_{y \to 0^+} g(y) = 0$$

To complete the definitions of $f(y)$ and $g(y)$, we impose $f(0) = g(0) = 0$, thus making the functions continuous. Let $b \in \mathbb{R}$ such that $\frac{1}{b} > K$. Define $H(y) = f(y)g(b) - g(y)f(b)$. We notice that $H(0) = H(b) = 0$ and that $H(y)$ is continuous. We can then apply Rolle's theorem, which states that there exists a $c \in (0, b)$ such that $H'(c) = 0$.

$$H'(c) = f'(c)g(b) - g'(c)f(b) = 0$$

We can rearrange the above equation to get:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)}{g(b)}$$

In order to get an expression in terms of $u(x)$ and $v(x)$, we need the identities

$$\begin{aligned}
f'(y) &= \frac{d}{dy}u\left(\frac{1}{y}\right) = u'\left(\frac{1}{y}\right) \cdot \left(\frac{-1}{y^2}\right) = u'(x) \cdot (-x^2) \\
g'(y) &= \frac{d}{dy}v\left(\frac{1}{y}\right)^{-1} = -v\left(\frac{1}{y}\right)^{-2} v'\left(\frac{1}{y}\right) \cdot \left(\frac{-1}{y}\right) = \frac{-v'(x)(-x^{-2})}{v(x)^2} \\
\frac{f(b)}{g(b)} &= u\left(\frac{1}{b}\right) v\left(\frac{1}{b}\right)
\end{aligned}$$

By substituting these identities in the above equation, we get that

$$u\left(\frac{1}{b}\right) v\left(\frac{1}{b}\right) = \frac{-u'\left(\frac{1}{c}\right) \cdot v\left(\frac{1}{c}\right)^2}{v'\left(\frac{1}{c}\right)}$$

Since $c \in (0, b)$, then $\frac{1}{c} > \frac{1}{b} > K$. Using the second hypothesis, we get that the right hand side of the equation is bounded for such a value of c. Hence, for $x = \frac{1}{b}$, $u(x)v(x)$ is bounded. Since this is true for every $x > K$, then $u(x)v(x)$ is bounded as $x \to \infty$. (A symmetric argument can be used to show that $u(x)v(x)$ is also bounded for $x \to -\infty$). Finally, the fact that $u(x)v(x)$ is a continuous function generalizes boundedness to all $x \in \mathbb{R}$.

∎

Appendix B

Center Manifold Theorem

In this section, we cite the center manifold theorem from [12].

Consider a semilinear differential equation

$$u_t = -Au + F(u), \tag{B.0.1}$$

on some Banach space X. We assume that A is sectorial and F is a C^{k+2}-function from $Y = X^\alpha$ to X for some $k \geq 1$ and $\alpha \in [0, 1)$, see Henry [6] for the notation. The norms for vectors and operators on Y are denoted $|\cdot|$ and $||\cdot||$, respectively. The local semiflow on Y associated with (B.0.1) is denoted by $\Phi_t(u)$. Let G be a finite-dimensional but possibly non-compact Lie group, and $\rho : G \to GL(Y)$, $g \mapsto \rho_g$ be a representation of G in the space of bounded invertible operators. We assume that there exists a constant K such that $||\rho_g|| \leq K$ for all $g \in G$. After introducing an equivalent norm on Y, we may assume that $||\rho_g|| = 1$ for all g. We suppose that $\Phi_t(u)$ is G-equivariant, that is, $\Phi_t(\rho_g u) = \rho_g \Phi_t(u)$ for $t \geq 0$, $g \in G$, and $u \in Y$.

Throughout, we fix a point u_* and denote its group orbit and the isotropy group by Gu_* and H, respectively, that is, we set $Gu_* = \{\rho_g u_*; g \in G\}$ and $H = \{g \in G; \rho_g u_* = u_*\}$. Suppose that the element u_* chosen is a relative equilibrium of (B.0.1):

Hypothesis B.0.2 Let $u_* \in Y$ and assume that there exists an element $\xi_* \in \mathbf{alg}(G)$ in the Lie algebra of G such that

$$\Phi_t(u_*) = \rho_{g_*(t)} u_*,$$

where $g_*(t) = \exp(\xi_* t) \in G$ is the one-parameter family generate by ξ_*.

Hypothesis B.0.3 Assume that $\{\lambda \in \mathbb{C}; |\lambda| \geq 1\}$ is a spectral set for the linearization

$$\rho_{\exp(-\xi_*)} D\Phi_1(u_*) \in L(Y)$$

with associated projection $P_* \in L(Y)$ such that the generalized eigenspace $E_*^{cu} = R(P_*)$ is finite-dimensional.

Hypothesis B.0.4 (i) $\rho_g u_*$ is C^{k+2} in $g \in G$.

(ii) For any $\epsilon > 0$ there exists a $\delta > 0$ such that $|\rho_g u_* - u_*| \geq \delta$ for all $g \in G$ satisfying $\text{dist}(g,H) \geq \epsilon$.

(iii) $\rho_g v$ is C^{k+1} in $g \in G$ for any point v in E_*^{cu}.

(iv) The projections $\rho_g P_* \rho_{g^{-1}}$ are C^{k+1} in $g \in G$ in the operator norm.

Theorem B.0.5 *Assume that Hypotheses (B.0.2), (B.0.3), and (B.0.4) are obeyed. Under these conditions, there exists a G-invariant manifold $M_*^{cu} \subset Y$ which is locally invariant under Φ_t for any $t \geq 0$. The manifold M_*^{cu} and the action of G on M_*^{cu} are of class C^{k+1}. Furthermore, M_*^{cu} is locally exponentially attracting and contains all solutions which stay close to the group orbit of u_* for all backward times.*

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