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Linear and Nonlinear Deflection Analysis of Thick Rectangular Plates Using Finite Differences

By

Nasr-Eddine Bencharif

A thesis presented to the University of Ottawa in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Civil Engineering

DEPARTMENT OF CIVIL ENGINEERING UNIVERSITY OF OTTAWA

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Thanks are also due to my parents and my family for their understanding, patience and encouragement.
Abstract

Variational methods are widely used for the solution of complex differential equations in mechanics for which exact solutions are not possible. The finite difference method, although well known as an efficient numerical method was applied in the past only for the solution of linear and nonlinear thin plates.

In the present study, the suitability of the method for the solution of nonlinear deflection of thick plates is studied for the first time. While there is major differences between small deflection and large deflection plate theories, the former can be treated as a particular case of the latter, when the centre deflection of the plate is less than or equal to 0.2-0.25 of the thickness of the plate. The finite difference method as applied here is a modified finite difference approach to the ordinary finite difference method generally used for the solution of thin plate problems. In this thesis thin plates are treated as a particular case of the corresponding thick plate when the boundary conditions of the plates are taken into account.

The method is first applied to investigate the deflection behaviour of square clamped and simply supported square isotropic thick plates. After the validity of the method is established, it is then extended to the solution of rectangular thick plates of various aspect ratios and thicknesses.

Generally, beginning with the use of a limited number of mesh sizes for a given plate aspect ratio and boundary conditions, a general solution of the problem including the investigation of accuracy and convergence was
extended to rectangular thick plates by providing more detailed functions satisfying the rectangular mesh sizes generated automatically by the programme.

Whenever possible results of the present method are compared with the existing solutions in the technical literature obtained by much more laborious methods and close agreements are found. Significant amounts of results presented herein are not currently available in the technical literature for various plate aspect ratios and Poisson's ratios. The submatrices involved in the formation of the finite difference equations from the governing differential equations forming the general system are generated directly by the computer programme. The subroutine SOLINV from the second directed method as developed and illustrated in Chapter V takes care of the inversion of the general matrix. The subroutine developed by the author has been proven to be more efficient than the former methods known for the computation of linear simultaneous equations [61].

Simplicity in formulation and quick convergence are the obvious advantages of the finite difference formulation developed here for small and large deflection analysis of thick plate in comparison with other numerical methods requiring extensive computer facilities.
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$x, y, z$  rectangular co-ordinate system.
$\phi, \chi, \psi$  dimensionless rectangular co-ordinate system.
$u, v, w$  displacements of the nodes in $x, y,$ and $z$ direction respectively.
$U, V, W$  dimensionless displacements of the nodes in $\phi, \chi,$ and $\psi$ direction respectively.
$a$  longitudinal dimension of the plate.
$b$  transverse dimension of the plate.
$h$  thickness of the plate.
$\frac{a}{b}$  plate aspect ratio.
$\frac{h}{a}$  plate thickness aspect ratio.
$N_x$  axial forces parallel to the $x$-axis per unit length of a section of plate perpendicular to $y$-axis.
$N_y$  axial forces parallel to the $y$-axis per unit length of a section of plate perpendicular to $x$-axis.
$Q_x, Q_y$  shearing forces parallel to the $z$-axis per unit length of a section of plate perpendicular to $x$- and $y$-axes respectively.
$N_{xy}$  the in-plane shearing intensities per unit length.
$M_{xx}, M_{yy}$  bending moment intensities per unit length.
$M_{xy}$  twisting moment intensity per unit length.
$q = q(x, y)$  transverse external force per unit area.
$Q = \frac{q a^4}{E h}$  dimensionless transverse external force per unit area.
$\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$  normal stress components in $x, y, z$ directions respectively.
$\sigma_{xy}, \sigma_{xz}, \sigma_{yz}$  shearing stress components in rectangular co-ordinates.
\[ \varepsilon_x, \varepsilon_y, \varepsilon_z \]  
unit elongation in x-, y-, and z- directions.

\[ \varepsilon_{xy}, \varepsilon_{xz}, \varepsilon_{yz} \]  
shearing strain components in rectangular co-ordinates.

\[ E \]  
Modulus of elasticity in tension and compression.

\[ G \]  
Modulus of elasticity in shear.

\[ \nu \]  
Poisson’s ratio 0.316.

\[ z = \pm h/2 \]  
lower and upper face of a plate, respectively.

\[ D \]  
flexural rigidity of an isotropic plate.

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \]  
Laplacian operator in rectangular co-ordinates.

\[ \gamma \]  
Relaxation factor

\[ \alpha = \frac{W}{h} \]  
Dimensionless value for the central deflection.

\[ \bar{\alpha} = \frac{\alpha}{Q} \]  
Dimensionless factor for the linear central deflection.

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  
Laplacian operator in space co-ordinates.

\[ [AA] \]  
General matrix.

\[ [FF] \]  
Nodal loading.

\[ [FOR] \]  
Force vector.

\[ [DIS] \]  
Linear vector displacement.

\[ [DISN] \]  
Nonlinear vector displacement.
Chapter 1

Introduction

1.1 General

Plates are used as components of large scale structures in both mechanical and civil engineering structures. They are plane surface structures bounded either by straight or curved lines. Plates may have free, simply-supported, or fixed boundary conditions, including elastic supports or, in some cases, point supports. The static loads carried by plates are predominantly perpendicular to the plate surface. Plates are classified either by their geometrical forms or by their physical characteristics. Thick plates are a type of plates where the thickness is considerable and the approximate theories of thin plates are no longer applicable. In such cases, the thick plate theory must be applied. This theory considers the problem of
thick plates as a three-dimensional problem of elasticity. The stress analysis becomes, consequently, much more involved and up to now, the problem of thick plates is solved only for a few simple particular cases. The difficulty of obtaining solutions for thick plates is a direct result of the complexities of the partial differential equations governing thick plate behaviour. Furthermore, the boundary conditions for thick plates also have to be satisfied in three directions and hence, three functions defining all three displacements are required. The thick plate theory finds important application in the static analysis of heavy floor slabs and the dynamic analysis of heavy floors carrying rotating machinery in industrial buildings. An analysis is often required to determine the deformation of the thick plates, under the type of loading the plate is designed to carry.

The fundamental problems in mechanics are generally governed either by deriving the differential equations or by minimum energy principles. The governing differential equations and the equations derived by minimum energy principles are usually rather complex and do not lend themselves to exact solutions. The general equations defining the large deformation theory for thick plate are very complex and their exact solutions are not available. Confronted with this problem engineers and researchers have to resort to numerical methods to effect a solution. Variational methods were often exploited by engineers and scientists as a very effective tool for solving applied mechanics problems. These methods gained popularity in recent years due to the development of high speed digital computers. Finite differences methods were also used extensively for solving such complex differential equations. This method can be considered as a means of obtaining
the approximate solutions of differential equations, where the exact solutions are not available. According to the method, the problem is solved at chosen appropriate points where the system of equations is defined. In applying the finite difference method, a mesh size must first be chosen. The mesh size covers the entire domain of the plate with n points yielding n simultaneous equations in each direction. Convergence is easily investigated by increasing the mesh size covering the plate domain. For this study, the finite difference method is applied to the three dimensional problem of thick plates where the governing differential equations of equilibrium of such plates are expressed in terms of the displacements u, v, w and the rotations \( \omega_x, \omega_y \), for the non-linear analysis following the co-ordinates axes x, y and z respectively. The first objective of this thesis is to study the applicability of the finite difference method for the non-linear analysis of thick plates.

As a supporting objective of the present study, a few somewhat innovative methods for the solution of simultaneous equations are presented. Equations-solving methods can be generally categorized as either direct techniques, which yield answers in a finite, predictable number of operations, or as iterative techniques, which yield answers that become increasingly more accurate as the number of iterations becomes large. Until fairly recently, it was commonly accepted that direct methods were most suitable for small sets of equations with dense coefficient matrices, while iterative methods were best suited for large sets of equations involving sparse coefficient matrices. The current viewpoint is somewhat different. Iterative techniques are still preferred for very large sets of equations with sparse
but not banded coefficient matrices. However, it has been found that direct methods are also highly suitable for quite large sets of equations having banded coefficient matrices. These banded matrices usually result either from the application of finite difference or of finite element methods to partial differential equations. New innovative methods for solving simultaneous equations involving both the direct method and the iteration technique will be presented as one of the objectives of the thesis.

1.2 Object and Scope

The main object of this thesis is to solve the problem of large deflection non-linear analysis of thick plates using the finite difference method. To study the applicability of the method, the problem of the bending of a square clamped thick plate is first studied using a rather crude mesh size. Once the validity of the method is established, the same method is extended to analyse the corresponding problems of rectangular clamped and simply-supported thick plates. The method can be further applied to determine the buckling and vibration of thick plates by the introduction of corresponding appropriate elasticity equations.

After the large systems of simultaneous equations have been obtained from finite differences, it became quite obvious that some new mathematical subroutines must be developed to handle efficiently the solution of these equations. As a result, some rather innovative efficient subroutines have since been developed [61] to handle the solutions of simultaneous equations
resulting from the finite difference modelling of large deflection of thick plate structures.

1.3 Present Investigation

In the analysis of thin plates subjected to lateral loading, in particular, when the central deflection is equal to or greater than 25% of the thickness of the plate, the Kirchhoff theory which neglects the membrane effect cannot yield satisfactory results. Hence, for the finite deflection analysis of plates, the Kirchhoff theory is replaced by the refined Von-Karman theory, where the involvement of the membrane forces in the equations contributes a significant relief to the plate deflection. However, the solutions of the Von-Karman equations are extremely tedious and complicated. Linear analysis of thick plates is much more complicated than the corresponding analysis of thin plates. For thick plate analysis three equations of equilibrium are considered compared to the Kirchhoff equation where only the lateral deflection is considered. The degree of difficulty increases even higher when we consider the nonlinear terms in the three equations of equilibrium. It has since been noted that not only the equations are difficult and complicated to handle; moreover, their solution requires new subroutines in order to deal with large systems of simultaneous equations resulting from the finite difference modelling. Since no results are available in the technical literature on large deflection of thick isotropic rectangular plates and the existing results are meagre even for small deflection of thick
square plates, convergence of the problem becomes important to yield confidence to the solution. In this investigation only deflections are considered, and the stresses are not computed because they do not, in general, yield sufficiently accurate results. In order to yield both bending and membrane stresses that are of acceptable accuracy, the finite difference mesh would have to be so refined resulting in enormous additional requirements for computer storage and time far beyond the present capacity allocation by the present computer facilities at the University of Ottawa. The discussion of the difficulties in generating stresses from displacement functions for large deflections of thin plates was well documented by Wang [16] and Yang [16-a]. Chapter II will be devoted to a brief review of the large deflection of thin isotropic rectangular plates. Since one of the objectives of this thesis is to study the linear and nonlinear analysis of thick isotropic rectangular plates and the applicability of the finite difference method as a numerical solution for such plates, existing literature related to the solution of small deflection of thick plates by different numerical methods is also reviewed in this chapter. The second or supporting objective of the thesis is to develop innovative efficient subroutines to solve systems of simultaneous equations. For this part, a brief review for the most popular method is also presented in chapter II. The main analysis of thick isotropic rectangular plates for small and large deflections, formulation of the general governing equations of equilibrium, stress-strain relationship, compatibility equations, dimensionless displacements are carried out in chapter III. In chapter IV, the finite difference expressions required to analyse the deflection problem of clamped and simply-supported isotropic rectangular thick
plates are formulated. The formulation of finite difference in the case of thick plates is not an easy task. For example, in the case of thin plate theory, the problem can be reduced to one governing equation in terms of the vertical displacement \( w \), whereas for thick plates the governing differential equations require the inclusion of two additional components \( u \) and \( v \), thus increasing significantly the complexity of the problem. Finite difference modelling is developed for the first time to solve the problem of small and large deflection of thick rectangular plates. In the technical literature, a small amount of data is available for small deflection of square thick plates using various numerical methods. In thick plate theory, the requirement to satisfy the boundary conditions deserves careful consideration especially for boundary conditions applied at extreme surfaces of the plate. For this reason, the thick plate is divided into layers with appropriate boundary expressions for each layer. The modified finite difference method presented here may be taken as a general method applicable to both thin and thick plate structures.

The formulation of the finite difference expression for each point is presented in detail, showing how the partial derivatives are replaced by finite difference expressions, and how the resulting equations are reduced in matrix form to yield the solution of a particular problem. Chapter V is devoted to the innovative method for the solution of systems of simultaneous equations, using iterative methods and direct methods, numerical examples are included in the appendix to demonstrate the applicability and advantages of the methods in comparison with the conventional methods of analysis. Results for small and large deflection analysis of thick rectan-
gular plates obtained are summarized in chapter VI for both the clamped and simply supported boundary conditions. For small deflection analysis of thick plates, results are compared, whenever possible with the solutions previously obtained by other researchers.
Chapter 2

PREVIOUS STUDIES

The classical theory of plates has been well established and widely applied since Thomson and Tait resolved the controversy over the boundary conditions of Poisson’s theory in the latter half of the nineteenth century. Due to approximations inherent in their derivations, classical theories restrict themselves in their application to plates having thickness much less than their lateral dimensions. As a result, these theories cannot be applied with acceptable accuracy to thick plates or problems where local effects predominate such as stress concentration problems. Thus, there are cases where a much more refined theory is required. This led to several significant developments in recent years in the field of analysis of thick plates. The fundamental non-linear large deflection equations for thin homogeneous plates were derived by Von-Karman in 1910. S. Way [1] solved these equations for a clamped rectangular thin plate using the Ritz method by assuming the
displacements $u$, $v$, and $w$ as polynomials functions. S. Levy [2] considered the case of the square simply supported thin plate using double trigonometric functions. Levy also solved [3] the problem of the square clamped plate using Fourier series functions. Then Levy and Greenman [4] solved the problem for the clamped rectangular plate with the plate ratio 1.5 using the same technique. A general procedure for solving these equations for multiple boundary conditions has been worked out at National Bureau of Standards by S. Levy [5] for the NACA. E. Reissner [6] proposed a new approach to analyse finite deflection of circular plates. Due to the deformation, the thin circular plate becomes a shell of revolution. A simplified non-linear equations for a flat plate with large deflections are derived by H.M. Berger [7]. He assumed that the strain energy due to second invariant of the middle-surface strains can be neglected. This method was used to solve thin circular and rectangular plates with different boundary conditions. E. Reissner [8] obtained an exact solution of the non-linear equations for finite deflections of thin elastic rectangular plates, with twisting moments $T$ and bending moments $M$ simultaneously applied to two opposite edges of the plate. Three methods were discussed by E.L. Bernstein [9] for the derivation of the equilibrium equations for large deflection of plates, namely, the free body method, the direct strain energy method, and the equilibrium equation method. In all the three methods, the rotation component $\omega_z$ was neglected. It was later observed that when thickness shear and thickness-stretch deformations were included, application of the three methods produced inconsistent sets of equilibrium equations, none of which is entirely satisfactory.
M. Balachandra and S. Gopalacharyulu [10] solved the problem of clamped thin rectangular plates with large deflections using a modified Fourier series and results were compared with Timoshenko [47] using polynomials. Numerical solutions are obtained by F.Bauer et al. [11] using an iterative method for the solution of the Von-Karman equations. Numerical solutions are obtained by an iterative method which employs an acceleration parameter θ. They are numerically evaluated by approximating the Dirichletlet problem using corresponding difference equations. The resulting system of linear algebraic equations which is of quasi-tridiagonal form, is solved by a factored method. Approximate solutions for the non-linear bending of thin rectangular plates were presented by K.T. Sandara Raja Iyengar and M.Martin Nagri [12] considering various boundary conditions. The solution was given by using a double series consisting of appropriate beam functions which satisfy the boundary conditions. The differential equations are satisfied by using the orthogonality properties of the series. A finite difference method was carried out by J.C. Brown and J.M Harvey [13] to compute the finite deflection of rectangular plates subjected to uniform lateral pressure and compressive edge loading. The results are in good agreement with the some numerical solutions and some experimental values. S.S. Tezcan [14] used the framework method to calculate large deflections of plates. For his analysis he used a typical Hrennikoff cell. Each cell of the framework contains six unidimensional bars. A solution of the large deflection equations for thin rectangular clamped plates was carried out by R. Hooke [15] using the perturbation method. Wang [16] used relaxation technique to solve a square simply supported thin plate. D.W. Murray and E.L. Wilson [17]
used the finite elements to determine large deflection of cantilever and simply supported thin plates using a triangular mesh. The results are in good agreement with Timoshenko [47], and Levy [2]. R. Hooke [18] determined a non-linear function which will fit the post-elastic deflection prediction of the plates for design purposes, and the technique matched some solutions with low percentage of error. Large deflection of elastic plates under patch loading was carried out by B. Aalami [19] using two numerical methods namely finite difference and relaxation. They yielded the same result for both deflection and stresses.

The shear deformation theory of Reissner [20] derived on the basis of a variational criteria has been one of the foremost in the treatment of thick plates. This theory has been applied to various rectangular plates by many investigators. Kromm [21] demonstrated the absence of corner reactions using a theory that takes into account transverse shear deformation. This point has also been discussed in detail by Marguerre and Woernle [22]. Green [23] has pointed out that the Reissner equations can be obtained directly from the three dimensional elasticity equations without recourse to variational considerations. Donnel [24] has given a three dimensional thick plate theory in which the solution is obtained in the form of infinite series with the first term representing classical thin plate theory results. Lee [25] has applied this method to simply supported rectangular plates. Friedrichs and Dressler [26] and Goldenveiser [27] have given approximate theories which could predict boundary layer effects, by the method of asymptotic integration of the governing equations of elasticity. Lur'e [28] has given a general power series method to solve the elasticity equations. Poniotovskii
[29] employed Legendre polynomial expansion in the thickness coordinate to derive a system of two dimensional equations. A three dimensional solution for rectangular plates and laminates has been developed by Srinivas et al. [30] in which the solution for displacements is taken in the form of a double trigonometric series satisfying the equations of equilibrium in terms of displacements. Srinivas and Rao [31] solved the small deflection problem of thick plates using a three dimensional analysis by expressing the displacement functions in terms of hyperbolic functions using the techniques of collocation, orthogonalization, and orthogonalization along with collocation at corners. The generalized Levy solution [32] has been taken by David W. Cooke to solve the problem of a statically loaded, rectangular plate which is simply supported on two opposite edges and has arbitrary boundary conditions on the remaining edges. Krashchina [33] presented a method based on the expressions obtained by Berdichevskii to solve the problem of bending of thick plates. Kishida [34] carried out a three-dimensional elastic stress analysis of thick plates, and solved the problem of axisymmetric bending of a clamped circular plate subjected to an annularly distributed load. An analysis of an infinite elastic thick plate subjected to loads symmetrical to the axis of revolution has been developed by Nakajima, Iamsopana and Kakuzen [35]. After they have represented the part of hyperbolic functions of the integrand of stress functions in terms of exponential functions, Maclaurin’s law was used to expand the denominator and each term was then integrated separately. The approximate stress components were calculated by solving the simultaneous linear algebraic equations with four unknowns by satisfying the boundary conditions at two points.
on the boundary surfaces. Kobayashi, Nagasawa, Ishikawa, and Hata [36] proposed an extension of Lover’s moderately thick plate theory to solve the problem of a three dimensional rectangular cantilever plate. Bending of thick plates under an arbitrary load has been carried out by Gruzdev [37] using the method proposed by Lur’e, who reduces the three dimensional plate problem to the two dimensional problem using the infinite order differential operators of the displacements and rotations of the middle plane of the plate. Deshmukh and Archer [38] proposed a new method which considerably reduces the amount of numerical computations for certain classes of plate problems as compared to the alternate methods, i.e., finite element or finite difference methods. The method proposed is based on Reissner’s theory where the total unknown component is equal to the sum of particular and complimentary solutions. Mixed finite difference scheme for the analysis of simply supported thick plates was proposed by Noor [39]. His analysis is based on the linear, three dimensional theory of orthotropic elasticity and a Fourier approach is used to reduce the governing equations to six first-order ordinary differential equations in the thickness coordinate and a finite difference method is carried out to form a system of linear equations from the previous differential equations. Ryabov and Rasskazov [40] examined the bending of thick nonuniform plates through their thickness. Their assumption is based on the distribution of the shear components in the cross sections and its change over the plate thickness in accordance with a square parabola law.

Finite element is the most rigorous method used to solve complicated problems during the last three decades. Here, Epstein and Huttermaier [41]
made use of the finite element concept for the numerical analysis of multilayered plates and thick plates. Guruswamy and Tang [42] have expressed a twenty four degree of freedom sector finite element for the static and dynamic analysis of thick circular plates. Sundara, Chandrashekhar and Sebastian [43] applied a method of initial functions (MIF) originally proposed by Vlassov for the analysis of thick rectangular plates. The unknowns are expanded in Maclaurin series in the thickness coordinate and hence the solution is obtained in terms of unknown initial functions on the reference plane. Ng and Bencharif [44] have programmed a solution using finite difference representation for the small deflection analysis of thick plates and the results are in good agreement with previous researchers. The applicability of the finite difference method and its ease application was well demonstrated more recently [45] by Ng and Bencharif with extensive results for the design and analysis of thick plates not previously available in the technical literature.

For the solution of systems of equations, it has been noted by Ralston [57] that sets of simultaneous linear equations can usually be put into one of two categories: either the coefficient matrix is dense (few zero elements) but the set is not large, or the matrix is sparse (many zero elements) and the set is large. The most popular methods of solving simultaneous equations with their algorithms can be found in [58] by R. W. Hornbeck, and Wang [59]. Recently, Bencharif and Ng [61] used the change of variable decomposition method to solve a system of simultaneous equations and the results show good improvement in comparison with the conventional meth-
ods Gauss-Jordan and LU decomposition methods in terms of time and space.
Chapter 3

Theoretical Derivations

3.1 General

In this chapter, formulation of the governing equilibrium and compatibility equations of the plate based on the theory of elasticity is presented. For the analysis of large deformations, it is obvious that the analysis of small deformations will be considered as a particular case. The same analogy is applied to thin plate analysis as a particular case of thick plate problem. Geometrical and material nonlinearity can arise in plate problems. In this study only geometrical nonlinearity is taken under consideration and homogeneous, isotropic material is assumed. Figure (3.1) illustrates the geometry and orientation of a plate in the cartesian coordinates system. The x and y axis delimit the longitudinal and transversal directions of the
surface of the plate respectively and the z axis the perpendicular direction to the surface of the plate.

![Diagram of coordinate system](image)

**Figure (3.1) Position of The Coordinate System.**

Internal forces and moments acting on the limits of plate element, as shown in Figure (3.2) and Figure (3.3), are related to the internal stresses by the equations:

\[
N_x = \int_{-h/2}^{h/2} \sigma_{xx} \, dz \quad (3.1)
\]

\[
N_y = \int_{-h/2}^{h/2} \sigma_{yy} \, dz \quad (3.2)
\]

\[
N_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} \, dz \quad (3.3)
\]

\[
Q_x = \int_{-h/2}^{h/2} \sigma_{xz} \, dz \quad (3.4)
\]

\[
Q_y = \int_{-h/2}^{h/2} \sigma_{yz} \, dz \quad (3.5)
\]

\[
M_z = \int_{-h/2}^{h/2} \sigma_{zz} \, dz \quad (3.6)
\]
\[
M_y = \int_{-h/2}^{h/2} \sigma_{yy} z dz \\
M_{xy} = \int_{-h/2}^{h/2} \sigma_{xy} z dz
\] (3.7) (3.8)

Figure (3.2) Equilibrium State of The Plate Element

Figure (3.3) Distribution of The Moments in The Plate Element
3.1.1 Stress and Strain functions

From elasticity [48], and [49] the general Hook’s law stress strain relationships are expressed by the following equations.

\[
\begin{align*}
\sigma_{xx} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)}(\epsilon_{yy} + \epsilon_{zz}) + \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}\epsilon_{xx} \quad (3.9) \\
\sigma_{yy} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)}(\epsilon_{xx} + \epsilon_{zz}) + \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}\epsilon_{yy} \quad (3.10) \\
\sigma_{zz} &= \frac{\nu E}{(1 + \nu)(1 - 2\nu)}(\epsilon_{xx} + \epsilon_{yy}) + \frac{E(1 - \nu)}{(1 + \nu)(1 - 2\nu)}\epsilon_{zz} \quad (3.11) \\
\sigma_{xy} &= \frac{E}{2(1 + \nu)}\epsilon_{xy} \quad (3.12) \\
\sigma_{yz} &= \frac{E}{2(1 + \nu)}\epsilon_{yz} \quad (3.13) \\
\sigma_{zx} &= \frac{E}{2(1 + \nu)}\epsilon_{zx} \quad (3.14)
\end{align*}
\]

In terms of stresses the six previous equations can be defined as:

\[
\begin{align*}
\epsilon_{xx} &= \frac{1}{E}[\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})] \quad (3.15) \\
\epsilon_{yy} &= \frac{1}{E}[\sigma_{yy} - \nu(\sigma_{zz} + \sigma_{xx})] \quad (3.16) \\
\epsilon_{zz} &= \frac{1}{E}[\sigma_{zz} - \nu(\sigma_{xx} + \sigma_{yy})] \quad (3.17) \\
\epsilon_{xy} &= \frac{2(1 + \nu)}{E}\sigma_{xy} \quad (3.18) \\
\epsilon_{yz} &= \frac{2(1 + \nu)}{E}\sigma_{yz} \quad (3.19) \\
\epsilon_{zx} &= \frac{2(1 + \nu)}{E}\sigma_{zx} \quad (3.20)
\end{align*}
\]
3.2 Nonlinear Analysis

3.2.1 Geometry of Deformations

Given the positions of the points of the body in its initial state before deformation and in its final state after deformation, the geometry of strain is determined by the change in the distance between two arbitrary infinitely near points of the body caused by its transition from the first state to the final state.

If we consider $x, y, z$ and $\xi, \eta, \zeta$ as the coordinates of the initial and final state respectively, and let the points of the body undergo displacements with components $u, v, w$, then the final state of an arbitrary point of the body will be given by the cartesian coordinates [51]:

![Diagram](image)

Figure (3.4) Deformation of a Line Element.
\[\xi = x + u(x, y, z) \quad (3.21)\]
\[\eta = y + v(x, y, z) \quad (3.22)\]
\[\zeta = z + w(x, y, z) \quad (3.23)\]

Where the functions \(u, v, w\) and their partial derivatives with respect to \(x, y, z\) are considered to be continuous.

From (3.21), (3.22), (3.23) above we note that the final state of the points can be determined either by the cartesian final coordinates or by the initial coordinates. The initial coordinates become curvilinear coordinates at the final state.

As a result of movement, the point \(M(x, y, z)\) is displaced to the position \(M^*\) having cartesian coordinates \(\xi, \eta, \zeta\) whereas the point \(N(x+dx, y+dy, z+dz)\) infinitely near \(M\) is displaced to the position \(N^*\) having coordinates \(\xi + d\xi, \eta + d\eta, \zeta + d\zeta\) as shown in Figure (3.4). The deformed vector \(M^*N^*\) (with projections \(d\xi, d\eta, d\zeta\)), determines the magnitude and direction of that line element of the body whose magnitude and direction just before deformation were given by the vector \(MN\) (with projections \(dx, dy, dz\)). By applying the vector rule and using the point \(O\) as the origin of the cartesian coordinates we get:

\[M^*N^* = O\tilde{N}^* - O\tilde{M}^* \quad (3.24)\]

Then the projection of the deformed vector will be expressed as shown:

\[d\xi = x + dx + u(x + dx, y + dy, z + dz) - x - u(x, y, z) \quad (3.25)\]
\[ d\eta = y + dy + v(x + dx, y + dy, z + dz) - y - v(x, y, z) \quad (3.26) \]
\[ d\zeta = z + dz + w(x + dx, y + dy, z + dz) - z - w(x, y, z) \quad (3.27) \]

Simplifying and then expanding the right sides in Taylor series about the point \( x, y, z \) and retaining only the first orders we get:

\[ d\xi = dx + u(x, y, z) + \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz - u(x, y, z) \quad (3.28) \]
\[ d\eta = dy + v(x, y, z) + \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz - v(x, y, z) \quad (3.29) \]
\[ d\zeta = dz + w(x, y, z) + \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + \frac{\partial w}{\partial z} dz - w(x, y, z) \quad (3.30) \]

After collecting and arranging the terms we get the final expressions;

\[ d\xi = (1 + \frac{\partial u}{\partial x}) dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz \quad (3.31) \]
\[ d\eta = \frac{\partial v}{\partial x} dx + (1 + \frac{\partial v}{\partial y}) dy + \frac{\partial v}{\partial z} dz \quad (3.32) \]
\[ d\zeta = \frac{\partial w}{\partial x} dx + \frac{\partial w}{\partial y} dy + (1 + \frac{\partial w}{\partial z}) dz \quad (3.33) \]

Any arbitrary line element of the body after deformation can be expressed by the equations (3.31), (3.32), (3.33) in terms of its projections before deformation.

If we use the following notation;

\[ e_{xx} = \frac{\partial u}{\partial x}, \quad e_{yy} = \frac{\partial v}{\partial y}, \quad e_{zz} = \frac{\partial w}{\partial z} \]
\[ e_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad e_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \quad e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \quad (3.34) \]

and

\[ 2\omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \quad 2\omega_y = \frac{\partial w}{\partial z} - \frac{\partial u}{\partial x}, \quad 2\omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad (3.35) \]
then equations (3.31), (3.32), (3.33) will be expressed in the form below:

\[ d\xi = (1 + e_{xx})dx + \left(\frac{1}{2}e_{xy} - \omega_z\right)dy + \left(\frac{1}{2}e_{xz} + \omega_y\right)dz \]  
(3.36)

\[ d\eta = \left(\frac{1}{2}e_{xy} + \omega_z\right)dx + (1 + e_{yy})dy + \left(\frac{1}{2}e_{yz} - \omega_x\right)dz \]  
(3.37)

\[ d\zeta = \left(\frac{1}{2}e_{xz} - \omega_y\right)dx + \left(\frac{1}{2}e_{yz} + \omega_x\right)dy + (1 + e_{zz})dz \]  
(3.38)

### 3.2.2 Strain Components

To compute the strain change between the two stages we need first to determine the square distances of the two vectors \( \overrightarrow{MN} \) and \( \overrightarrow{M'N'} \) before and after deformation respectively.

Before deformation we have:

\[ ds^2 = dx^2 + dy^2 + dz^2 \]  
(3.39)

and after deformation we get:

\[ ds_0^2 = d\xi^2 + d\eta^2 + d\zeta^2 \]  
(3.40)

Proceeding with the difference between these two distances will give us the change due to the deformation.

\[ ds_0^2 - ds^2 = d\xi^2 + d\eta^2 + d\zeta^2 - dx^2 - dy^2 - dz^2 \]  
(3.41)

By using equations (3.31), (3.32), (3.3) for \( d\xi, d\eta, d\zeta \) and substituting into (3.41) we find:

\[ ds^2 - ds_0^2 = (1 + \frac{\partial u}{\partial x})dx^2 + 2(1 + \frac{\partial u}{\partial x})\frac{\partial u}{\partial y}dx\,dy + 2(1 + \frac{\partial u}{\partial x})\frac{\partial u}{\partial z}dx\,dz \]
\[\begin{align*}
+(\frac{\partial u}{\partial y})^2 dy^2 + 2 \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} dy dz + (\frac{\partial u}{\partial z})^2 dz^2 \\
+(\frac{\partial v}{\partial x})^2 dx^2 + 2 \frac{\partial v}{\partial x} (1 + \frac{\partial v}{\partial y}) dx dy + 2 \frac{\partial v}{\partial x} \frac{\partial v}{\partial z} dx dz \\
+(1 + \frac{\partial v}{\partial y})^2 dy^2 + 2 (1 + \frac{\partial v}{\partial y}) \frac{\partial v}{\partial z} dy dz + (\frac{\partial v}{\partial z})^2 dz^2 \\
+(\frac{\partial w}{\partial x})^2 dx^2 + 2 \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} dx dy + 2 \frac{\partial w}{\partial x} (1 + \frac{\partial w}{\partial z}) dx dz \\
+(\frac{\partial w}{\partial y})^2 dy^2 + 2 \frac{\partial w}{\partial y} (1 + \frac{\partial w}{\partial z}) dy dz + (1 + \frac{\partial w}{\partial z})^2 dz^2 \\
-dx^2 - dy^2 - dz^2
\end{align*}\]

(3.42)

Simplifying the equation by arranging the terms as shown below;

\[\begin{align*}
\frac{ds^2}{ds^2} - \frac{ds^2}{ds^2} &= \left[1 + 2 \frac{\partial u}{\partial x} + (\frac{\partial u}{\partial x})^2 + (\frac{\partial v}{\partial x})^2 + (\frac{\partial w}{\partial x})^2 - 1\right] dx^2 + \\
&\left[(\frac{\partial u}{\partial y})^2 + 1 + 2 \frac{\partial u}{\partial y} + (\frac{\partial v}{\partial y})^2 + (\frac{\partial w}{\partial y})^2 - 1\right] dy^2 + \\
&\left[(\frac{\partial u}{\partial z})^2 + (\frac{\partial v}{\partial z})^2 + 1 + 2 \frac{\partial w}{\partial z} + (\frac{\partial w}{\partial z})^2 - 1\right] dz^2 + \\
&\left[2 (1 + \frac{\partial u}{\partial z}) \frac{\partial u}{\partial z} + \frac{2 \partial v}{\partial z} (1 + \frac{\partial v}{\partial z}) + 2 \frac{\partial w}{\partial z} \frac{\partial w}{\partial z}\right] dx dy + \\
&\left[2 (1 + \frac{\partial u}{\partial y}) \frac{\partial u}{\partial z} + 2 \frac{\partial v}{\partial y} \frac{\partial v}{\partial z} + 2 \frac{\partial w}{\partial y} \frac{\partial w}{\partial z}\right] dx dz + \\
&\left[2 \frac{\partial u}{\partial y} \frac{\partial u}{\partial z} + 2 (1 + \frac{\partial v}{\partial y}) \frac{\partial v}{\partial y} + 2 \frac{\partial w}{\partial y} \frac{\partial w}{\partial y}\right] dy dz
\end{align*}\]

(3.43)

we get:

\[\begin{align*}
\frac{ds^2}{ds^2} - \frac{ds^2}{ds^2} &= 2 (\varepsilon_{xx} dx^2 + \varepsilon_{yy} dy^2 + \varepsilon_{zz} dz^2 + \varepsilon_{xy} dx dy + \varepsilon_{xz} dx dz + \varepsilon_{yz} dy dz)
\end{align*}\]

(3.44)
Figure (3.5) Deformation of a Body Element.

Where:

\[ \epsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial v}{\partial x} \right)^2 + \left( \frac{\partial w}{\partial x} \right)^2 \right] \]  
(3.45)

\[ \epsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial y} \right)^2 + \left( \frac{\partial v}{\partial y} \right)^2 + \left( \frac{\partial w}{\partial y} \right)^2 \right] \]  
(3.46)

\[ \epsilon_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2} \left[ \left( \frac{\partial u}{\partial z} \right)^2 + \left( \frac{\partial v}{\partial z} \right)^2 + \left( \frac{\partial w}{\partial z} \right)^2 \right] \]  
(3.47)

\[ \epsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \]  
(3.48)

\[ \epsilon_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x} \frac{\partial u}{\partial z} + \frac{\partial v}{\partial x} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial z} \]  
(3.49)
\[ e_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial z} + \frac{\partial v}{\partial y} \frac{\partial w}{\partial z} + \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \]  

(3.50)

By using the notations given in equations (3.34), (3.35) we get the following interpretation.

\[ \varepsilon_{xx} = \varepsilon_{xx} + \frac{1}{2} \left[ e_{xx}^2 + \left( \frac{1}{2} e_{xy} + \omega_z \right)^2 + \left( \frac{1}{2} e_{xz} - \omega_y \right)^2 \right] \]  

(3.51)

\[ \varepsilon_{yy} = \varepsilon_{yy} + \frac{1}{2} \left[ e_{yy}^2 + \left( \frac{1}{2} e_{xy} - \omega_z \right)^2 + \left( \frac{1}{2} e_{yz} + \omega_x \right)^2 \right] \]  

(3.52)

\[ \varepsilon_{zz} = \varepsilon_{zz} + \frac{1}{2} \left[ e_{zz}^2 + \left( \frac{1}{2} e_{xz} + \omega_y \right)^2 + \left( \frac{1}{2} e_{yz} - \omega_x \right)^2 \right] \]  

(3.53)

\[ \varepsilon_{xy} = \varepsilon_{xy} + e_{xx} \left( \frac{1}{2} e_{xy} - \omega_z \right) + e_{yy} \left( \frac{1}{2} e_{xy} + \omega_z \right) + \left( \frac{1}{2} e_{xz} - \omega_y \right) \left( \frac{1}{2} e_{yz} + \omega_x \right) \]  

(3.54)

\[ \varepsilon_{xz} = \varepsilon_{xz} + e_{xx} \left( \frac{1}{2} e_{xz} + \omega_y \right) + e_{zz} \left( \frac{1}{2} e_{xz} - \omega_y \right) + \left( \frac{1}{2} e_{xy} + \omega_x \right) \left( \frac{1}{2} e_{yz} - \omega_x \right) \]  

(3.55)

\[ \varepsilon_{yz} = \varepsilon_{yz} + e_{yy} \left( \frac{1}{2} e_{yz} - \omega_x \right) + e_{zz} \left( \frac{1}{2} e_{yz} + \omega_x \right) + \left( \frac{1}{2} e_{xy} - \omega_y \right) \left( \frac{1}{2} e_{xz} + \omega_y \right) \]  

(3.56)

3.2.3 Equations of Equilibrium

From a deformed body, let us isolate an elementary rectangular parallelepiped whose edges are parallel to the x, y, z axis and equal to \( d\xi, d\eta, d\zeta \), respectively as shown in Figure (3.6).

Accordingly, the following surface forces act on them [51]:
Figure (3.6) Acting Forces on a Body Element.
\begin{align*}
F_{A_1 B_1 B} &= - \left[ \sigma_\xi(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \zeta} \, d\zeta \right] \, d\eta \, d\zeta \quad (3.57) \\
F_{D D_1 C_1 C} &= \left[ \sigma_\xi(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \xi} \, d\xi + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \zeta} \, d\zeta \right] \, d\eta \, d\zeta \quad (3.58) \\
F_{D D_1 A_1 A} &= - \left[ \sigma_\eta(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial \sigma_\eta}{\partial \xi} \, d\xi + \frac{1}{2} \frac{\partial \sigma_\eta}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial \sigma_\eta}{\partial \zeta} \, d\zeta \right] \, d\xi \, d\zeta \quad (3.59) \\
F_{B B_1 C_1 C} &= \left[ \sigma_\zeta(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial \sigma_\zeta}{\partial \xi} \, d\xi + \frac{1}{2} \frac{\partial \sigma_\zeta}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial \sigma_\zeta}{\partial \zeta} \, d\zeta \right] \, d\xi \, d\zeta \quad (3.60) \\
F_{A D C B} &= - \left[ \sigma_\xi(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \xi} \, d\xi + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial \sigma_\xi}{\partial \zeta} \, d\zeta \right] \, d\xi \, d\eta \quad (3.61) \\
F_{A_1 D_1 C_1 B_1} &= \left[ \sigma_\eta(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial \sigma_\eta}{\partial \xi} \, d\xi + \frac{1}{2} \frac{\partial \sigma_\eta}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial \sigma_\eta}{\partial \zeta} \, d\zeta \right] \, d\xi \, d\eta \quad (3.62)
\end{align*}

In addition to surface forces, there will be also body forces acting on the element. Their resultant is equal to:

\[ F_{body} = \left[ F(\xi, \eta, \zeta) + \frac{1}{2} \frac{\partial F}{\partial \xi} \, d\xi + \frac{1}{2} \frac{\partial F}{\partial \eta} \, d\eta + \frac{1}{2} \frac{\partial F}{\partial \zeta} \, d\zeta \right] \, d\xi \, d\eta \, d\zeta \approx F^*(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta \quad (3.63) \]

Where \( F^*(\xi, \eta, \zeta) \) is the specific body force at point \( M^*(\xi, \eta, \zeta) \).

Summing all surface forces and body forces and setting the result equal to zero, we arrive at the following relation after eliminating the common factors:

\[ \frac{\partial \sigma_{\xi}}{\partial \xi} + \frac{\partial \sigma_{\eta}}{\partial \eta} + \frac{\partial \sigma_{\zeta}}{\partial \zeta} + F^* = 0 \quad (3.64) \]

The vector equation (3.30) is equivalent to the three scalar equations

\[ \frac{\partial \sigma_{\xi \xi}}{\partial \xi} + \frac{\partial \sigma_{\eta \xi}}{\partial \eta} + \frac{\partial \sigma_{\zeta \xi}}{\partial \zeta} + F^* = 0 \quad (3.65) \]
\[ \frac{\partial \sigma_{\xi \eta}}{\partial \xi} + \frac{\partial \sigma_{\eta \eta}}{\partial \eta} + \frac{\partial \sigma_{\zeta \eta}}{\partial \zeta} + F^* = 0 \quad (3.66) \]
\[ \frac{\partial \sigma_{\xi \zeta}}{\partial \xi} + \frac{\partial \sigma_{\eta \zeta}}{\partial \eta} + \frac{\partial \sigma_{\zeta \zeta}}{\partial \zeta} + F^* = 0 \quad (3.67) \]
If we neglect the body forces and transform the governing equilibrium equations from differentiation with respect to \( \xi, \eta, \zeta \) to the differentiation in cartesian coordinates with respect to \( x, y, z \), we get:

\[
\begin{align*}
\frac{\partial}{\partial x} \left[ (1 + e_{xx})\sigma_{xx} + \left( \frac{1}{2} e_{xy} - \omega_z \right)\sigma_{xy} + \left( \frac{1}{2} e_{xz} + \omega_y \right)\sigma_{xz} \right] & + \\
\frac{\partial}{\partial y} \left[ (1 + e_{yy})\sigma_{yy} + \left( \frac{1}{2} e_{xy} - \omega_z \right)\sigma_{xy} + \left( \frac{1}{2} e_{yz} + \omega_x \right)\sigma_{yz} \right] & + \\
\frac{\partial}{\partial z} \left[ (1 + e_{zz})\sigma_{zz} + \left( \frac{1}{2} e_{xy} - \omega_z \right)\sigma_{xy} + \left( \frac{1}{2} e_{yz} + \omega_x \right)\sigma_{yz} \right] & = 0 \quad (3.68) \\
\frac{\partial}{\partial x} \left[ \frac{1}{2} e_{xy} + \omega_z \right]\sigma_{xx} + (1 + e_{yy})\sigma_{xy} + \left( \frac{1}{2} e_{yx} - \omega_x \right)\sigma_{xz} & + \\
\frac{\partial}{\partial y} \left[ \frac{1}{2} e_{xy} + \omega_z \right]\sigma_{yx} + (1 + e_{yy})\sigma_{yy} + \left( \frac{1}{2} e_{yx} - \omega_x \right)\sigma_{yz} & + \\
\frac{\partial}{\partial z} \left[ \frac{1}{2} e_{xy} + \omega_z \right]\sigma_{zx} + (1 + e_{yy})\sigma_{zy} + \left( \frac{1}{2} e_{yx} - \omega_x \right)\sigma_{zz} & = 0 \quad (3.69) \\
\frac{\partial}{\partial x} \left[ \frac{1}{2} e_{xz} - \omega_y \right]\sigma_{xx} + \left( \frac{1}{2} e_{yz} + \omega_x \right)\sigma_{xy} + (1 + e_{zz})\sigma_{xz} & + \\
\frac{\partial}{\partial y} \left[ \frac{1}{2} e_{xz} - \omega_y \right]\sigma_{yx} + \left( \frac{1}{2} e_{yz} + \omega_x \right)\sigma_{yy} + (1 + e_{zz})\sigma_{yz} & + \\
\frac{\partial}{\partial z} \left[ \frac{1}{2} e_{xz} - \omega_y \right]\sigma_{zx} + \left( \frac{1}{2} e_{yz} + \omega_x \right)\sigma_{zy} + (1 + e_{zz})\sigma_{zz} & = 0 \quad (3.70)
\end{align*}
\]

### 3.2.4 Compatibility Equations

In the six strain equations (3.45-50), only the three displacement unknowns \( u, v, w \) are included. Let us consider that the strains are known and we need to compute the displacement components. The equations may be regarded as a system of partial differential equations for the determination of the displacements \( u, v, w \) when the strain components \( \epsilon_{xx}, \epsilon_{yy}, \epsilon_{zz}, \epsilon_{xy}, \epsilon_{yz}, \epsilon_{xz} \) are expressed as functions of \( x, y, z \). Since there are six equations for three
unknown functions, we cannot expect in general that these equations will possess a solution if the strain components are arbitrarily prescribed. Thus, there must be some conditions to be imposed on the strain components in order that these six equations will give a set of single-valued continuous solutions for the three displacement components. The fact that the strain components cannot be prescribed arbitrarily can be seen from the following geometrical considerations: Imagine that an elastic body is subdivided into a number of small cubic elements before deformation. Now, suppose that each element is subjected to an arbitrary deformation. After the deformation, these elements become parallelepipeds, and it may happen that it is impossible to arrange the parallelepipeds to form a continuous body, the strain components for each element must satisfy certain relations. The determination of the six components of strain at each point is completely satisfied by the three functions $u, v, w$ defining the components of displacement. If the elongations and shears are very small compared to unity; and as a result of this simplification, we obtain the following six approximate equations:

\[
\begin{align*}
\frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} - \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} &= \left(\frac{\partial \varepsilon_{xx}}{\partial y}\right)^2 + \left(\frac{\partial \varepsilon_{yy}}{\partial x}\right)^2 + \frac{1}{4} \left(\frac{\partial \varepsilon_{yy}}{\partial x} + \frac{\partial \varepsilon_{xx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial z}\right)^2 \\
- \left(\frac{\partial \varepsilon_{xx}}{\partial x} - \frac{\partial \varepsilon_{yy}}{\partial z}\right) \left(\frac{\partial \varepsilon_{yx}}{\partial y} - \frac{\partial \varepsilon_{xy}}{\partial y}\right) &= \frac{\partial \varepsilon_{yy}}{\partial y} \left(\frac{\partial \varepsilon_{xy}}{\partial x} - \frac{\partial \varepsilon_{yx}}{\partial y}\right) - \frac{\partial \varepsilon_{xx}}{\partial x} \left(\frac{\partial \varepsilon_{xy}}{\partial y} - \frac{\partial \varepsilon_{yx}}{\partial z}\right) \\
\frac{\partial^2 \varepsilon_{yy}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} - \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} &= \left(\frac{\partial \varepsilon_{yy}}{\partial z}\right)^2 + \left(\frac{\partial \varepsilon_{zz}}{\partial y}\right)^2 + \frac{1}{4} \left(\frac{\partial \varepsilon_{zz}}{\partial y} + \frac{\partial \varepsilon_{yy}}{\partial z} - \frac{\partial \varepsilon_{yz}}{\partial x}\right)^2 \\
- \left(\frac{\partial \varepsilon_{yz}}{\partial y} - \frac{\partial \varepsilon_{zy}}{\partial x}\right) \left(\frac{\partial \varepsilon_{zz}}{\partial z} - \frac{\partial \varepsilon_{zz}}{\partial z}\right) &= \frac{\partial \varepsilon_{zz}}{\partial z} \left(\frac{\partial \varepsilon_{zy}}{\partial y} - \frac{\partial \varepsilon_{yz}}{\partial y}\right) - \frac{\partial \varepsilon_{yz}}{\partial y} \left(\frac{\partial \varepsilon_{zy}}{\partial z} - \frac{\partial \varepsilon_{yz}}{\partial y}\right) \\
\frac{\partial^2 \varepsilon_{zz}}{\partial x^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial z^2} - \frac{\partial^2 \varepsilon_{yx}}{\partial x \partial z} &= \left(\frac{\partial \varepsilon_{zz}}{\partial z}\right)^2 + \left(\frac{\partial \varepsilon_{zz}}{\partial x}\right)^2 + \frac{1}{4} \left(\frac{\partial \varepsilon_{zz}}{\partial x} + \frac{\partial \varepsilon_{yy}}{\partial z} - \frac{\partial \varepsilon_{yx}}{\partial y}\right)^2
\end{align*}
\]
\[- \left( \frac{\partial \varepsilon_{xz}}{\partial z} - \frac{\partial \varepsilon_{yz}}{\partial y} \right) \left( \frac{\partial \varepsilon_{xy}}{\partial x} - \frac{\partial \varepsilon_{yx}}{\partial y} \right) - \frac{\partial \varepsilon_{xx}}{\partial x} \left( \frac{\partial \varepsilon_{xz}}{\partial x} - \frac{\partial \varepsilon_{zx}}{\partial x} \right) - \frac{\partial \varepsilon_{zz}}{\partial z} \left( \frac{\partial \varepsilon_{zx}}{\partial x} - \frac{\partial \varepsilon_{xz}}{\partial x} \right) \right] (3.73)

\[\frac{\partial^2 \varepsilon_{xx}}{\partial y \partial z} - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_{yx}}{\partial y} \right) \frac{\partial \varepsilon_{yz}}{\partial y} - \frac{1}{2} \frac{\partial}{\partial x} \left( \frac{\partial \varepsilon_{yy}}{\partial z} + \frac{\partial \varepsilon_{zy}}{\partial z} \right) \frac{\partial \varepsilon_{yz}}{\partial z} = \frac{\partial \varepsilon_{xx}}{\partial y} \frac{\partial \varepsilon_{xx}}{\partial x} + \frac{1}{2} \frac{\partial \varepsilon_{yy}}{\partial z} \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{1}{2} \frac{\partial \varepsilon_{zy}}{\partial z} \frac{\partial \varepsilon_{yz}}{\partial x} \right] (3.74)

\[\frac{\partial^2 \varepsilon_{yy}}{\partial x \partial z} - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \varepsilon_{yx}}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial x} \right) \frac{\partial \varepsilon_{yz}}{\partial y} - \frac{1}{2} \frac{\partial}{\partial y} \left( \frac{\partial \varepsilon_{zy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial z} \right) \frac{\partial \varepsilon_{yz}}{\partial y} = \frac{\partial \varepsilon_{yy}}{\partial x} \frac{\partial \varepsilon_{yy}}{\partial x} + \frac{1}{2} \frac{\partial \varepsilon_{yx}}{\partial z} \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{1}{2} \frac{\partial \varepsilon_{zy}}{\partial z} \frac{\partial \varepsilon_{yz}}{\partial x} \right] (3.75)

\[\frac{\partial^2 \varepsilon_{zz}}{\partial x \partial y} - \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial \varepsilon_{xz}}{\partial x} + \frac{\partial \varepsilon_{zx}}{\partial x} \right) \frac{\partial \varepsilon_{yz}}{\partial y} - \frac{1}{2} \frac{\partial}{\partial z} \left( \frac{\partial \varepsilon_{zy}}{\partial z} + \frac{\partial \varepsilon_{yz}}{\partial z} \right) \frac{\partial \varepsilon_{yz}}{\partial y} = \frac{\partial \varepsilon_{zz}}{\partial x} \frac{\partial \varepsilon_{zz}}{\partial x} + \frac{1}{2} \frac{\partial \varepsilon_{xz}}{\partial y} \frac{\partial \varepsilon_{yz}}{\partial x} + \frac{1}{2} \frac{\partial \varepsilon_{zx}}{\partial y} \frac{\partial \varepsilon_{yz}}{\partial x} \right] (3.76)

### 3.2.5 Simplification for Small Rotations

If the angles of rotation are small compared to unity, then, the linear parameters $e_{ij}$ differ from the strain components only by quantities of the same order as the squares of the angles of rotation. This allows us to simplify the strain system (3.51-56) to:

\[\varepsilon_{xx} = e_{xx} + \frac{1}{2} \left[ + \omega_x^2 + \omega_y^2 \right] \] (3.77)

\[\varepsilon_{yy} = e_{yy} + \frac{1}{2} \left[ + \omega_x^2 + \omega_y^2 \right] \] (3.78)
\[ \varepsilon_{zz} = \varepsilon_{zz} + \frac{1}{2} \left( \omega^2_y + \omega^2_x \right) \]  
(3.79)

\[ \varepsilon_{xy} = \varepsilon_{xy} - \omega_y \omega_x \]  
(3.80)

\[ \varepsilon_{xz} = \varepsilon_{xz} - \omega_z \omega_x \]  
(3.81)

\[ \varepsilon_{yz} = \varepsilon_{yz} - \omega_z \omega_y \]  
(3.82)

and the equilibrium system of equations can be written as:

\[ \frac{\partial}{\partial x} [\sigma_{xx} - \omega_z \sigma_{xy} + \omega_y \sigma_{xz}] + \frac{\partial}{\partial y} [\sigma_{yx} - \omega_z \sigma_{yy} + \omega_y \sigma_{yz}] + \frac{\partial}{\partial z} [\sigma_{zx} - \omega_z \sigma_{zy} + \omega_y \sigma_{zz}] = 0 \]  
(3.83)

\[ \frac{\partial}{\partial x} [\omega_x \sigma_{xx} + \sigma_{xy} - \omega_z \sigma_{xz}] + \frac{\partial}{\partial y} [\omega_x \sigma_{yx} + \sigma_{yy} - \omega_z \sigma_{yz}] + \frac{\partial}{\partial z} [\omega_x \sigma_{zx} + \sigma_{zy} - \omega_z \sigma_{zz}] = 0 \]  
(3.84)

\[ \frac{\partial}{\partial x} [\omega_y \sigma_{xx} - \omega_z \sigma_{xy} - \sigma_{xz}] + \frac{\partial}{\partial y} [\omega_y \sigma_{yx} - \omega_z \sigma_{yy} - \sigma_{yz}] + \frac{\partial}{\partial z} [\omega_y \sigma_{zx} - \omega_z \sigma_{zy} - \sigma_{zz}] = 0 \]  
(3.85)

If we rewrite the equations of equilibrium by taking the non-linear terms to the right side and we neglect the rotation \( \omega_z \) by considering that the plate is rigid enough to undergo rotation about the z axis then we get the following equations of equilibrium. The notation used below stands for the iterative solution deployed to solve the non-linear equations of thick plates. First we solve the problem of small deflection then we start correcting the force vector in the right hand side for the large deflection analysis. In any iterative method the terms with the subscript \( i+1 \) are devoted to the next iteration where the terms with the subscript \( i \) are devoted to the present iteration. To ease our representation, the terms with the symbol \( o \) stands
for the achieved iteration.

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = -\frac{\partial}{\partial x} \left[ \omega_c^{\sigma_{xx}} \right] - \frac{\partial}{\partial y} \left[ \omega_c^{\sigma_{xy}} \right] - \frac{\partial}{\partial z} \left[ \omega_c^{\sigma_{xz}} \right]
\]  
(3.86)

\[
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = \frac{\partial}{\partial x} \left[ \omega_c^{\sigma_{xy}} \right] + \frac{\partial}{\partial y} \left[ \omega_c^{\sigma_{yy}} \right] + \frac{\partial}{\partial z} \left[ \omega_c^{\sigma_{yz}} \right]
\]  
(3.87)

\[
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial}{\partial x} \left[ \omega_c^{\sigma_{xz}} - \omega_c^{\sigma_{yz}} \right] + \frac{\partial}{\partial y} \left[ \omega_c^{\sigma_{yz}} - \omega_c^{\sigma_{yx}} \right] + \frac{\partial}{\partial z} \left[ \omega_c^{\sigma_{zz}} - \omega_c^{\sigma_{zy}} \right]
\]  
(3.88)

### 3.3 Linear Analysis

For the small deflection the equations of equilibrium can be derived easily by just dropping the the non-linear terms from the above or by considering the right hand side of the equations equal to zero, so we get:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0
\]  
(3.89)

\[
\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0
\]  
(3.90)

\[
\frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} = 0
\]  
(3.91)

#### 3.3.1 Formulation of the elasticity problem

If we are interested in finding only the stress components, we may reduce the system of equations to six equations with six unknown stress components. Since the displacement components are not required, the compati-
bility equations must be satisfied to ensure the existence of single valued displacements. By using the notation [48], and [49]

\[ \theta = \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \]  

(3.92)

the first three strain stress relations are

\[ \varepsilon_{xx} = \frac{1}{E} ((1 + \nu) \sigma_{xx} - \nu \theta) \]  

(3.93)

\[ \varepsilon_{yy} = \frac{1}{E} ((1 + \nu) \sigma_{yy} - \nu \theta) \]  

(3.94)

\[ \varepsilon_{zz} = \frac{1}{E} ((1 + \nu) \sigma_{zz} - \nu \theta) \]  

(3.95)

Now if we take:

\[ \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial y^2} = \frac{\partial^2 \varepsilon_{yz}}{\partial y \partial z} \]  

(3.96)

and deriving the equations (3.94), (3.95), (3.19) with respect to the differential equation (3.96) above we find

\[ (1 + \nu) \left( \frac{\partial^2 \sigma_{yy}}{\partial x^2} + \frac{\partial^2 \sigma_{zz}}{\partial y^2} \right) - \nu \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) = 2 (1 + \nu) \frac{\partial^2 \sigma_{yz}}{\partial y \partial z} \]  

(3.97)

Again, if we take

\[ \frac{\partial^2 \varepsilon_{xx}}{\partial z^2} + \frac{\partial^2 \varepsilon_{zz}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xx}}{\partial z \partial z} \]  

(3.98)

and deriving the equations (3.93), (3.95), (3.20) with respect to the second differential equation (3.98) we find

\[ (1 + \nu) \left( \frac{\partial^2 \sigma_{xx}}{\partial z^2} + \frac{\partial^2 \sigma_{zz}}{\partial x^2} \right) - \nu \left( \frac{\partial^2 \theta}{\partial z^2} + \frac{\partial^2 \theta}{\partial x^2} \right) = 2 (1 + \nu) \frac{\partial^2 \sigma_{xx}}{\partial z \partial z} \]  

(3.99)

Similarly, using

\[ \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{yy}}{\partial x^2} = \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} \]  

(3.100)
and deriving the equations (3.93), (3.94), (3.18) with respect to the third differential equation (3.100) we find:

\[(1 + \nu) \left( \frac{\partial^2 \sigma_{xx}}{\partial y^2} + \frac{\partial^2 \sigma_{xy}}{\partial x^2} \right) - \nu \left( \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} \right) = 2(1 + \nu) \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} \tag{3.101} \]

From the equilibrium system, we form three couple of equations; where each couple corresponds to a simplification of the above expressions respectively.

\[\frac{\partial \sigma_{yz}}{\partial y} = \frac{\partial \sigma_{zz}}{\partial z} - \frac{\partial \sigma_{xz}}{\partial x} - F_{zz} \tag{3.102} \]
\[\frac{\partial \sigma_{yz}}{\partial z} = \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial \sigma_{xy}}{\partial x} - F_{yy} \tag{3.103} \]
\[\frac{\partial \sigma_{zz}}{\partial z} = \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xy}}{\partial y} - F_{xx} \tag{3.104} \]
\[\frac{\partial \sigma_{xz}}{\partial x} = \frac{\partial \sigma_{zz}}{\partial z} - \frac{\partial \sigma_{yz}}{\partial y} - F_{zz} \tag{3.105} \]
\[\frac{\partial \sigma_{xy}}{\partial y} = \frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \sigma_{xz}}{\partial z} - F_{xx} \tag{3.106} \]
\[\frac{\partial \sigma_{xy}}{\partial x} = \frac{\partial \sigma_{yy}}{\partial y} - \frac{\partial \sigma_{yz}}{\partial z} - F_{yy} \tag{3.107} \]

Differentiating the first of these equations of each couple with respect to \(i\) and the second with respect to \(j\), and adding the equations of each differentiated couple together, i.e.,

\[\frac{\partial \sigma_{ij}}{\partial i \partial j}; \quad \frac{\partial \sigma_{ji}}{\partial j \partial i} \tag{3.108} \]

and then

\[\frac{\partial \sigma_{ij}}{\partial i \partial j} + \frac{\partial \sigma_{ji}}{\partial j \partial i} = 2 \frac{\partial \sigma_{ij}}{\partial i \partial j} \tag{3.109} \]

the first couple gives

\[2 \frac{\partial^2 \sigma_{yz}}{\partial y \partial z} = - \frac{\partial^2 \sigma_{zz}}{\partial z^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial}{\partial x} \left( \frac{\partial \sigma_{xx}}{\partial z} + \frac{\partial \sigma_{xy}}{\partial y} \right) - \frac{\partial F_{zz}}{\partial z} - \frac{\partial F_{yy}}{\partial y} \tag{3.110} \]
using the 1st relation from equilibrium equations

\[ \frac{\partial \sigma_{xx}^2}{\partial x^2} + \frac{\partial F_{xx}}{\partial x} = -\frac{\partial}{\partial x} \left( \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} \right) \]  
(3.111)

then

\[ 2 \frac{\partial^2 \sigma_{xz}}{\partial y \partial z} = \frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial^2 \sigma_{zz}}{\partial z^2} + \frac{\partial F_{xx}}{\partial x} - \frac{\partial F_{yy}}{\partial y} - \frac{\partial F_{zz}}{\partial z} \]  
(3.112)

the second couple gives

\[ 2 \frac{\partial^2 \sigma_{zz}}{\partial x \partial z} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial}{\partial y} \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} \right) - \frac{\partial F_{zz}}{\partial x} - \frac{\partial F_{xx}}{\partial x} \]  
(3.113)

using the 2nd relation from equilibrium equation

\[ \frac{\partial \sigma_{yy}^2}{\partial y^2} + \frac{\partial F_{yy}}{\partial y} = -\frac{\partial}{\partial y} \left( \frac{\partial \sigma_{yx}}{\partial x} + \frac{\sigma_{yz}}{\partial z} \right) \]  
(3.114)

then

\[ 2 \frac{\partial^2 \sigma_{yy}}{\partial x \partial z} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial^2 \sigma_{zz}}{\partial z^2} - \frac{\partial F_{yy}}{\partial y} - \frac{\partial F_{zz}}{\partial z} \]  
(3.115)

the third couple gives

\[ 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} - \frac{\partial}{\partial z} \left( \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial z} \right) - \frac{\partial F_{yz}}{\partial x} - \frac{\partial F_{xy}}{\partial y} \]  
(3.116)

using the 3rd relation from equilibrium equation

\[ \frac{\partial \sigma_{yz}^2}{\partial z^2} + \frac{\partial F_{yz}}{\partial z} = -\frac{\partial}{\partial z} \left( \frac{\partial \sigma_{zz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} \right) \]  
(3.117)

then

\[ 2 \frac{\partial^2 \sigma_{yz}}{\partial x \partial y} = -\frac{\partial^2 \sigma_{xx}}{\partial x^2} - \frac{\partial^2 \sigma_{yy}}{\partial y^2} + \frac{\partial^2 \sigma_{zz}}{\partial z^2} - \frac{\partial F_{yy}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \]  
(3.118)
using the notation below to simplify the writing

\[ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]  

(3.119)

substituting equation (3.118) in equation (3.101) and using the symbol \( \nabla^2 \)
we find:

\[ (1 + \nu) \left( \nabla^2 \theta - \nabla^2 \sigma_{xx} - \frac{\partial^2 \theta}{\partial x^2} \right) - \nu \left( \nabla^2 \theta - \frac{\partial^2 \theta}{\partial x^2} \right) = (1 + \nu) \left( \frac{\partial F_{xx}}{\partial x} - \frac{\partial F_{yy}}{\partial y} - \frac{\partial F_{zz}}{\partial z} \right) \]  

(3.120)

Two analogous equation can be obtained by substituting (3.116) in (3.99) and (3.115) in (3.97)

\[ (1 + \nu) \left( \nabla^2 \theta - \nabla^2 \sigma_{yy} - \frac{\partial^2 \theta}{\partial y^2} \right) - \nu \left( \nabla^2 \theta - \frac{\partial^2 \theta}{\partial y^2} \right) = (1 + \nu) \left( \frac{\partial F_{xx}}{\partial x} - \frac{\partial F_{yy}}{\partial y} - \frac{\partial F_{zz}}{\partial z} \right) \]  

(3.121)

and

\[ (1 + \nu) \left( \nabla^2 \theta - \nabla^2 \sigma_{zz} - \frac{\partial^2 \theta}{\partial z^2} \right) - \nu \left( \nabla^2 \theta - \frac{\partial^2 \theta}{\partial z^2} \right) = (1 + \nu) \left( \frac{\partial F_{xx}}{\partial x} - \frac{\partial F_{yy}}{\partial y} - \frac{\partial F_{zz}}{\partial z} \right) \]  

(3.122)

adding together all the three equations (3.120), (3.121), (3.122) we find

\[ (1 - \nu) \nabla^2 \theta = -(1 + \nu) \left( \frac{\partial F_{xx}}{\partial x} + \frac{\partial F_{yy}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \right) \]  

(3.123)

substituting this expression for \( \nabla^2 \theta \) in (3.120), (3.121) and (3.122) gives

\[ \nabla^2 \sigma_{xx} + \frac{1}{1 + \nu} \frac{\partial^2 \theta}{\partial x^2} = -\nu \left( \frac{\partial F_{xx}}{\partial x} + \frac{\partial F_{yy}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \right) - 2 \frac{\partial F_{xx}}{\partial x} \]  

(3.124)

\[ \nabla^2 \sigma_{yy} + \frac{1}{1 + \nu} \frac{\partial^2 \theta}{\partial y^2} = -\nu \left( \frac{\partial F_{xx}}{\partial x} + \frac{\partial F_{yy}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \right) - 2 \frac{\partial F_{yy}}{\partial y} \]  

(3.125)

\[ \nabla^2 \sigma_{zz} + \frac{1}{1 + \nu} \frac{\partial^2 \theta}{\partial z^2} = -\nu \left( \frac{\partial F_{xx}}{\partial x} + \frac{\partial F_{yy}}{\partial y} + \frac{\partial F_{zz}}{\partial z} \right) - 2 \frac{\partial F_{zz}}{\partial z} \]  

(3.126)

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In the same manner the remaining three conditions (3.97), (3.99), (3.101) can be transformed into equations of the following kinds,

\[ \nabla^2 \sigma_{yz} + \frac{1}{1 + \nu} \frac{\partial^2 \theta}{\partial y \partial z} = - \left( \frac{\partial F_{zz}}{\partial y} + \frac{\partial F_{yz}}{\partial z} \right) \quad (3.127) \]

\[ \nabla^2 \sigma_{xx} + \frac{1}{1 + \nu} \frac{\partial^2 \theta}{\partial x \partial z} = - \left( \frac{\partial F_{zz}}{\partial x} + \frac{\partial F_{zx}}{\partial z} \right) \quad (3.128) \]

\[ \nabla^2 \sigma_{xy} + \frac{1}{1 + \nu} \frac{\partial^2 \theta}{\partial x \partial y} = - \left( \frac{\partial F_{zz}}{\partial y} + \frac{\partial F_{yz}}{\partial x} \right) \quad (3.129) \]

If there are no body forces or if the body forces are constant, the (3.124), (3.125), (3.126) equations and (3.127), (3.128), (3.129) equations become

\[ (1 + \nu) \nabla^2 \sigma_{xx} + \frac{\partial^2 \theta}{\partial x^2} = 0 \quad (3.130) \]

\[ (1 + \nu) \nabla^2 \sigma_{yy} + \frac{\partial^2 \theta}{\partial y^2} = 0 \quad (3.131) \]

\[ (1 + \nu) \nabla^2 \sigma_{zz} + \frac{\partial^2 \theta}{\partial z^2} = 0 \quad (3.132) \]

\[ (1 + \nu) \nabla^2 \sigma_{yz} + \frac{\partial^2 \theta}{\partial y \partial z} = 0 \quad (3.133) \]

\[ (1 + \nu) \nabla^2 \sigma_{xz} + \frac{\partial^2 \theta}{\partial x \partial z} = 0 \quad (3.134) \]

\[ (1 + \nu) \nabla^2 \sigma_{xy} + \frac{\partial^2 \theta}{\partial x \partial y} = 0 \quad (3.135) \]

In addition to the equilibrium equation and the boundary conditions the stress components in an isotropic body must satisfy the six conditions of compatibility. Now the generalized Hook’s stress components (3.9-14), expressed in terms of strain components become,

\[ \sigma_{xx} = \lambda e + 2G\varepsilon_{xx} \quad (3.136) \]
\[
\sigma_{yy} = \lambda e + 2G\varepsilon_{yy} \quad (3.137) \\
\sigma_{xx} = \lambda e + 2G\varepsilon_{zz} \quad (3.138) \\
\sigma_{yx} = G\varepsilon_{xy} \quad (3.139) \\
\sigma_{yz} = G\varepsilon_{yz} \\
\sigma_{zx} = G\varepsilon_{zx} \quad (3.141)
\]

where, \( e = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \), \( \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \), \( G = \frac{E}{2(1+\nu)} \), Substituting the relation \( \sigma_{xx}, \sigma_{xy}, \sigma_{zz} \) into the first equation of equilibrium (3.89), (3.90), (3.91) we get
\[
\frac{\lambda}{\partial x} + G \left( 2\frac{\partial \varepsilon_{xx}}{\partial x} + \frac{\partial \varepsilon_{xy}}{\partial y} + \frac{\partial \varepsilon_{zx}}{\partial z} \right) = 0 \quad (3.142)
\]

and if we substitute for the strain component the expressions (3.34) we find that (3.141) can be written in the form
\[
(\lambda + G)\frac{\partial e}{\partial x} + G \nabla^2 u = 0 \quad (3.143)
\]

The other two equations can be transformed in a similar manner.

Thus the three equations of equilibrium, expressed in terms of displace-
mements are
\[
(\lambda + G)\frac{\partial e}{\partial x} + G \nabla^2 u = 0 \quad (3.144) \\
(\lambda + G)\frac{\partial e}{\partial y} + G \nabla^2 v = 0 \quad (3.145) \\
(\lambda + G)\frac{\partial e}{\partial z} + G \nabla^2 w = 0 \quad (3.146)
\]

The governing differential equation can be obtained by adding the three equations of equilibrium together
\[
(\lambda + G)\left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\} e + G \nabla^2 \{u, v, w\} = 0 \quad (3.147)
\]
and by rearranging and simplifying:

\[ \nabla^2 \{u, v, w\} + \frac{1}{1 - 2\nu} \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \quad \text{(3.148)} \]

Now, we can extend the equations of equilibrium in terms of displacement components to their explicit forms in terms of the derivatives in the case where no body forces exist. The first equation after the extension gives

\[ (\lambda + G) \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) + G \left( \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 w}{\partial y \partial z} \right) = 0 \quad \text{(3.149)} \]

Similarly, the other two equations can be derived and the general system of equations is transformed to the following:

\[ (\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial y^2} + G \frac{\partial^2 u}{\partial z^2} + (\lambda + G) \frac{\partial^2 v}{\partial x \partial y} + (\lambda + G) \frac{\partial^2 w}{\partial x \partial z} = 0 \quad \text{(3.150)} \]

\[ (\lambda + G) \frac{\partial^2 u}{\partial x \partial y} + G \frac{\partial^2 v}{\partial x^2} (\lambda + 2G) \frac{\partial^2 v}{\partial y^2} + G \frac{\partial^2 v}{\partial z^2} + (\lambda + G) \frac{\partial^2 w}{\partial y \partial z} = 0 \quad \text{(3.151)} \]

\[ (\lambda + G) \frac{\partial^2 u}{\partial x \partial z} + (\lambda + G) \frac{\partial^2 v}{\partial y \partial z} + G \frac{\partial^2 w}{\partial x^2} + G \frac{\partial^2 w}{\partial y^2} + (\lambda + 2G) \frac{\partial^2 w}{\partial z^2} = 0 \quad \text{(3.152)} \]

and in matrix form the system of equation of equilibrium can be represented by:

\[
\begin{bmatrix}
UU & UV & UW \\
VU &VV & VW \\
WU & WV & WW
\end{bmatrix}
\]

The explanation of these notations will be shown at chapter IV. From the matrix notations, it should be noted that if we only consider thin plate
theory, only one element WW, is needed. This element WW takes care of
the vertical deflection w. In thin plate theory all the other terms in the ma-
trix may be neglected and their contributions are considered insignificant.
However, in thick plate theory, all nine elements in the matrix are involved
and all their contributions must be considered in solving the problem. We
can see the increase in difficulty and complexity of the thick plate problem
from this matrix interpretation. If we note by N the number of nodes, the
dimension of the general matrix shown above is three times greater than
the dimension of the submatrice WW which is $N \times N$. Thus, to improve the
convergence and the accuracy of the numerical technique, we need to use
more nodes thus resulting in very large systems of simultaneous equations.

3.4 Dimensionless Representation

To consider the non-dimensional representation we need to interprete the
equations of equilibrium for large analysis in terms of the displacements $u,
v, w$ as was shown for the small deflection analysis. As mentioned above,
and by considering the iterative notation for the left hand side and the right
hand side we get:

$$(\lambda + 2G) \frac{\partial^2 u}{\partial x^2} + G \frac{\partial^2 u}{\partial y^2} + G \frac{\partial^2 u}{\partial z^2} + (\lambda + G) \frac{\partial^2 v}{\partial x \partial y} + (\lambda + G) \frac{\partial^2 w}{\partial x \partial z} =$$

$-G(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial z}) \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}) - G(\frac{\partial w}{\partial z} + \frac{\partial u}{\partial z}) \frac{1}{2}(\frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial x^2}) -$

$G(\frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 w}{\partial y^2}) \frac{1}{2}(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial z}) - G(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial z}) \frac{1}{2}(\frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 w}{\partial x \partial y})$
\[
\left( (\lambda + 2G) \frac{\partial^2 w}{\partial x^2} + \lambda \left( \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} \right) \right) \frac{1}{2} \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} - \\
\left( (\lambda + 2G) \frac{\partial w}{\partial z} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \frac{1}{2} \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial x \partial z}
\]

\[
(\lambda + G) \frac{\partial^2 u}{\partial x \partial y} + G \frac{\partial^2 v}{\partial x^2} (\lambda + 2G) \frac{\partial^2 v}{\partial y^2} + G \frac{\partial^2 v}{\partial z^2} + (\lambda + G) \frac{\partial^2 w}{\partial y \partial z} = \\
G \left( \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 u}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial z} \right) + G \left( \frac{\partial^2 w}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial z} \right) \frac{1}{2} \left( \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial z} \right) + \\
G \left( \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 u}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial u}{\partial y} - \frac{\partial v}{\partial z} \right) + G \left( \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 u}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 u}{\partial y \partial z} \right) + \\
\left( (\lambda + 2G) \frac{\partial^2 w}{\partial z^2} + \lambda \left( \frac{\partial^2 u}{\partial x \partial z} + \frac{\partial^2 v}{\partial y \partial z} \right) \right) \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \\
\left( (\lambda + 2G) \frac{\partial w}{\partial z} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \frac{1}{2} \left( \frac{\partial^2 w}{\partial y \partial z} - \frac{\partial^2 u}{\partial z^2} \right)
\]

\[
(\lambda + G) \frac{\partial^2 u}{\partial x \partial z} + (\lambda + G) \frac{\partial^2 v}{\partial y \partial z} + G \frac{\partial^2 w}{\partial x \partial z} + G \frac{\partial^2 w}{\partial y^2} + (\lambda + 2G) \frac{\partial^2 w}{\partial z^2} = \\
\left( (\lambda + 2G) \frac{\partial^2 u}{\partial x \partial z} + \lambda \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \right) \right) \frac{1}{2} \left( \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) + \\
\left( (\lambda + 2G) \frac{\partial u}{\partial z} + \lambda \left( \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right) \frac{1}{2} \left( \frac{\partial^2 u}{\partial x \partial z} - \frac{\partial^2 w}{\partial z^2} \right) - \\
G \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial u}{\partial z} \right) - G \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial z} \right) \frac{1}{2} \left( \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial z} \right) + \\
G \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial x \partial z} \right) \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial u}{\partial z} \right) + G \left( \frac{\partial^2 v}{\partial y \partial z} + \frac{\partial^2 u}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 u}{\partial y \partial z} \right) - \\
\left( (\lambda + 2G) \frac{\partial^2 v}{\partial y \partial z} + \lambda \left( \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 w}{\partial y \partial z} \right) \right) \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) + \\
\left( (\lambda + 2G) \frac{\partial v}{\partial z} + \lambda \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \right) \frac{1}{2} \left( \frac{\partial^2 v}{\partial y \partial z} - \frac{\partial^2 u}{\partial z^2} \right) + \\
G \left( \frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial x \partial z} \right) \frac{1}{2} \left( \frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} \right) + G \left( \frac{\partial^2 w}{\partial x \partial z} + \frac{\partial^2 u}{\partial x \partial z} \right) \frac{1}{2} \left( \frac{\partial^2 w}{\partial x \partial z} - \frac{\partial^2 u}{\partial x \partial z} \right) - \\
G \left( \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) - G \left( \frac{\partial^2 w}{\partial y \partial z} + \frac{\partial^2 v}{\partial y \partial z} \right) \frac{1}{2} \left( \frac{\partial^2 w}{\partial y \partial z} - \frac{\partial^2 v}{\partial y \partial z} \right)
\]

\[\text{(3.155)}\]
If we consider a system of dimensionless coordinate with $\phi$, $\chi$, $\psi$ where:

$$U = \frac{a u}{h^2}, \quad \phi = \frac{x}{a} \quad (3.156)$$

$$V = \frac{a v}{h^2}, \quad \chi = \frac{y}{b} \quad (3.157)$$

$$W = \frac{w}{h}, \quad \psi = \frac{z}{h} \quad (3.158)$$

$$n = \frac{a}{b}, \quad m = \frac{h}{a} \quad (3.159)$$

then we get:

$$\begin{align*}
(\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} + n^2 G \frac{\partial^2 U}{\partial \chi^2} + m^{-2} G \frac{\partial^2 U}{\partial \psi^2} + n(\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} + m^{-2}(\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} &= \\
-G(\frac{\partial^2 W}{\partial \phi^2} + \frac{\partial^2 U}{\partial \phi \partial \psi})(\frac{1}{2} \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi}) - G(\frac{\partial W}{\partial \psi} + \frac{\partial U}{\partial \phi})(\frac{1}{2} \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi}) &= \\
G(n\frac{\partial^2 V}{\partial \chi \partial \psi} + n^2 \frac{\partial^2 W}{\partial \chi^2})(\frac{1}{2} \frac{\partial U}{\partial \psi} - \frac{\partial W}{\partial \psi}) - G(n\frac{\partial W}{\partial \psi} + n\frac{\partial^2 U}{\partial \chi \partial \psi})(\frac{1}{2} \frac{\partial U}{\partial \psi} - \frac{\partial W}{\partial \psi}) &= \\
(\lambda + 2G)n^{-2} \frac{\partial^2 W}{\partial \phi^2} + \lambda(\frac{\partial^2 U}{\partial \phi \partial \psi} + n\frac{\partial^2 V}{\partial \chi \partial \psi})(\frac{1}{2} \frac{\partial U}{\partial \psi} - \frac{\partial W}{\partial \psi}) &= \\
(\lambda + 2G)n^{-2} \frac{\partial W}{\partial \phi} + \lambda(\frac{\partial U}{\partial \phi} + n\frac{\partial V}{\partial \chi})(\frac{1}{2} \frac{\partial U}{\partial \psi} - \frac{\partial W}{\partial \psi}) \quad (3.160)
\end{align*}$$

$$\begin{align*}
n(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} + G \frac{\partial^2 V}{\partial \psi^2} + n^2(\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} + m^{-2} G \frac{\partial^2 V}{\partial \psi^2} + n(\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} &= \\
G(\frac{\partial^2 W}{\partial \phi^2} + \frac{\partial^2 U}{\partial \phi \partial \psi})(\frac{1}{2} \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) + G(\frac{\partial W}{\partial \psi} + \frac{\partial U}{\partial \phi})(\frac{1}{2} \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) &= \\
G(n\frac{\partial^2 V}{\partial \chi \partial \psi} + n^2 \frac{\partial^2 W}{\partial \chi^2})(\frac{1}{2} \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) + G(n\frac{\partial W}{\partial \psi} + n\frac{\partial^2 U}{\partial \chi \partial \psi})(\frac{1}{2} \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) &= \\
(\lambda + 2G)n^{-2} \frac{\partial^2 W}{\partial \phi^2} + \lambda(\frac{\partial^2 U}{\partial \phi \partial \psi} + n\frac{\partial^2 V}{\partial \chi \partial \psi})(\frac{1}{2} \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) &= \\
(\lambda + 2G)n^{-2} \frac{\partial W}{\partial \phi} + \lambda(\frac{\partial U}{\partial \phi} + n\frac{\partial V}{\partial \chi})(\frac{1}{2} \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) \quad (3.161)
\end{align*}$$
\[
m^{-1} \left[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} + n(\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} + G \frac{\partial^2 W}{\partial \phi^2} + n^2 m \frac{\partial^2 W}{\partial \chi^2} + m^{-2} (\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} \right] =
\]
\[
m^{-1} \left( (\lambda + 2G) m^2 \frac{\partial^2 U}{\partial \phi^2} + \lambda (nm^2 \frac{\partial^2 V}{\partial \phi \partial \chi} + \frac{\partial^2 W}{\partial \phi \partial \psi}) \right) \frac{1}{2} \left( \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi} \right) +
\]
\[
m^{-1} \left( (\lambda + 2G) m^2 \frac{\partial^2 U}{\partial \phi^2} + \lambda (nm^2 \frac{\partial^2 V}{\partial \phi \partial \chi} + \frac{\partial^2 W}{\partial \phi \partial \psi}) \right) \frac{1}{2} \left( \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi} \right) -
\]
\[
mG(n \frac{\partial^2 U}{\partial \phi \partial \chi} + \frac{\partial^2 V}{\partial \phi^2}) \frac{1}{2} \left( \frac{\partial W}{\partial \phi} - \frac{\partial V}{\partial \phi} \right) - mG(n \frac{\partial^2 U}{\partial \phi \partial \chi} + \frac{\partial^2 V}{\partial \phi^2}) \frac{1}{2} \left( \frac{\partial W}{\partial \phi} - \frac{\partial V}{\partial \phi} \right) +
\]
\[
mG(n \frac{\partial^2 U}{\partial \phi \partial \chi} + \frac{\partial^2 V}{\partial \phi^2}) \frac{1}{2} \left( \frac{\partial W}{\partial \phi} - \frac{\partial V}{\partial \phi} \right) + mG(n \frac{\partial^2 U}{\partial \phi \partial \chi} + \frac{\partial^2 V}{\partial \phi^2}) \frac{1}{2} \left( \frac{\partial W}{\partial \phi} - \frac{\partial V}{\partial \phi} \right) -
\]
\[
m^{-1} \left( (\lambda + 2G)(nm^2) \frac{\partial^2 V}{\partial \chi^2} + \lambda (nm^2 \frac{\partial^2 U}{\partial \phi \partial \chi} + n \frac{\partial^2 W}{\partial \chi \partial \phi}) \right) \frac{1}{2} \left( n \frac{\partial W}{\partial \chi} - n \frac{\partial V}{\partial \psi} \right) -
\]
\[
m^{-1} \left( (\lambda + 2G)(nm^2) \frac{\partial^2 V}{\partial \chi^2} + \lambda (nm^2 \frac{\partial^2 U}{\partial \phi \partial \chi} + n \frac{\partial^2 W}{\partial \chi \partial \phi}) \right) \frac{1}{2} \left( n \frac{\partial W}{\partial \chi} - n \frac{\partial V}{\partial \psi} \right) +
\]
\[
m^{-1} G(\frac{\partial^2 W}{\partial \phi \partial \psi}) \frac{1}{2} \left( \frac{\partial W}{\partial \psi} - \frac{\partial V}{\partial \phi} \right) + m^{-1} G(\frac{\partial^2 W}{\partial \phi \partial \psi}) \frac{1}{2} \left( \frac{\partial W}{\partial \psi} - \frac{\partial V}{\partial \phi} \right) -
\]
\[
m^{-1} G(\frac{\partial^2 W}{\partial \phi \partial \psi}) \frac{1}{2} \left( \frac{\partial W}{\partial \psi} - \frac{\partial V}{\partial \phi} \right) - m^{-1} G(\frac{\partial^2 W}{\partial \phi \partial \psi}) \frac{1}{2} \left( \frac{\partial W}{\partial \psi} - \frac{\partial V}{\partial \phi} \right)
\]
\[(3.162)\]

### 3.5 Boundary conditions

The differential equations of equilibrium which have been derived previously for stresses and displacements within the plate must also be such as to accommodate the conditions of equilibrium with respect to prescribed forces or displacements at the boundary. In thin plate theory the boundary conditions need only be satisfied at the longitudinal and transverse dimensions.
but for thick plates, an additional dimension must be taken into account. In other words, for thick plates there are three boundaries to satisfy. If we consider that the origin of the plate at mid-plate, then the two boundary surfaces should be at \( \psi = \pm 1/2 \) with respect to the non-dimensional \( \psi \) axis upward positive. In this thesis we are considering the case of static deflection due to uniformly distributed loads acting perpendicular to the surface of the plate. For rectangular clamped and simply supported plates, the top and bottom surfaces boundaries conditions are,

\[
\text{at: } \quad \psi = \pm \frac{1}{2}, \quad \sigma_{xz} = 0, \quad \sigma_{yz} = 0, \quad \sigma_{zz} = \pm \frac{q}{2} \quad (3.163)
\]

then the system of equilibrium equations reduces itself to the system of plane stress where,

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \quad (3.164)
\]

\[
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \quad (3.165)
\]

We know from the theory of thin plates that this system can be transformed to the well known differential equation governing the small deflection of thin plates in non-dimensional coordinates: viz.,

\[
\frac{\partial^4 W}{\partial \phi^4} + 2n^2 \frac{\partial^4 W}{\partial \phi^2 \partial \chi^2} + n^4 \frac{\partial^4 W}{\partial \chi^4} = \frac{q a^4}{D h} \quad (3.166)
\]

For the large deflection three equations form a general solution to the problem at the boundary layers. These three equations are the result of a combination between the governing equation with bending and stretching plus the two equilibrium equations (3.160), (3.161). We represent the three
equations in terms of displacements in non-dimensional coordinates namely

\[ \frac{\partial^2 U}{\partial \phi^2} + n^2 \frac{1 - \nu}{2} \frac{\partial^2 U}{\partial \chi^2} + \frac{1 + \nu}{n} \frac{\partial^2 V}{\partial \phi \partial \chi} = -\frac{\partial W}{\partial \phi} \left( \frac{\partial^2 W}{\partial \phi^2} + n^2 \frac{1 - \nu}{2} \frac{\partial^2 W}{\partial \chi^2} \right) - n^2 \frac{1 + \nu}{2} \frac{\partial^2 \phi}{\partial \phi \partial \chi} \]

\[ \frac{n^2 \partial^2 V}{\partial \chi^2} + \frac{1 - \nu}{2} \frac{\partial^2 V}{\partial \phi^2} + \frac{1 + \nu}{2} \frac{\partial^2 U}{\partial \phi \partial \chi} = -\frac{\partial W}{\partial \chi} \left( n^2 \frac{\partial^2 W}{\partial \chi^2} + \frac{1 - \nu}{2} \frac{\partial^2 W}{\partial \phi^2} \right) - \frac{1 + \nu}{2} \frac{\partial^2 \chi}{\partial \phi \partial \chi} \]

\[ \frac{\partial^4 W}{\partial \phi^4} + 2n^2 \frac{\partial^4 W}{\partial \phi^2 \partial \chi^2} + n^4 \frac{\partial^4 W}{\partial \chi^4} = \frac{q a^4}{D h} + 12 \left( \frac{\partial U}{\partial \phi} + \frac{1}{2} \left( \frac{\partial W}{\partial \phi} \right)^2 \right) \left( \frac{\partial^2 W}{\partial \phi^2} + n^2 \frac{\partial^2 W}{\partial \chi^2} \right) + \left( n \frac{\partial V}{\partial \chi} + \frac{n^2 \partial^2 W}{\partial \chi^2} \right)^2 + \frac{\partial^2 W}{\partial \phi^2} + (1 - \nu) \left( n \frac{\partial U}{\partial \chi} + \frac{\partial V}{\partial \phi} + n \frac{\partial W}{\partial \phi} \frac{\partial W}{\partial \chi} \right) \frac{n \partial^2 \chi}{\partial \phi \partial \chi} \]

(3.167)

(3.168)

(3.169)

3.6 Clamped Plate

For the clamped plate, the contour of the plate is fully fixed and all the displacements and rotations are equal to zero,

\[ \text{at: } \phi = \frac{1}{2}, \quad U = V = W = 0 \quad \omega_\phi = 0 \quad (3.170) \]

\[ \text{at: } \chi = \frac{n}{2}, \quad U = V = W = 0 \quad \omega_\chi = 0 \quad (3.171) \]

\[ \text{at: } \psi = \frac{m}{2}, \quad \sigma_{zz} = \pm \frac{q}{2}, \quad \sigma_{zz} = 0, \quad \sigma_{yz} = 0 \quad (3.172) \]
3.7 Simply Supported Plate

In the case of simply supported plates the choice is more complex and the boundary conditions considered here represent the case where the edges are immovable:

\begin{align*}
& \text{at: } \phi = \pm \frac{1}{2}, \quad U = V = W = 0 \quad M_{xx} = 0 \quad (3.173) \\
& \text{at: } \chi = \pm \frac{n}{2}, \quad U = V = W = 0 \quad M_{yy} = 0 \quad (3.174) \\
& \text{at: } \psi = \pm \frac{m}{2}, \quad \sigma_{zz} = \pm \frac{q}{2}, \quad \sigma_{xx} = 0, \quad \sigma_{yz} = 0 \quad (3.175)
\end{align*}
Chapter 4

Formulation of the Solution

4.1 Iterative Solution

In chapter III, the governing equations necessary for the analysis of thick plates were presented, but solving these coupled nonlinear partial differential equations numerically may prove to be extremely difficult. The finite difference technique is used here to transform the differential equations into ordinary algebraic equations in terms of the displacements $U, V,$ and $W$ at specified points. To solve the problem an iterative procedure is used and, in order to perform the numerical analysis efficiently in the programme, the equilibrium equations are arranged in such way so that the solutions are
represented in matrix form:

\[ (\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} + n^2 G \frac{\partial^2 U}{\partial \chi^2} + m^{-2} G \frac{\partial^2 U}{\partial \psi^2} + n (\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} + m^{-2} (\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = \text{RHS} \]

\[ n (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} + G \frac{\partial^2 V}{\partial \phi^2} + n^2 (\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} + m^{-2} G \frac{\partial^2 V}{\partial \psi^2} + n m^{-2} (\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = \text{RHS} \]

\[ \frac{(\lambda + G)}{m} \frac{\partial^2 U}{\partial \phi \partial \psi} + n \frac{(\lambda + G)}{m} \frac{\partial^2 V}{\partial \chi \partial \psi} + \frac{G}{m} \frac{\partial^2 W}{\partial \phi^2} + n^2 \frac{G}{m} \frac{\partial^2 W}{\partial \chi^2} + \frac{(\lambda + 2G)}{m^3} \frac{\partial^2 W}{\partial \psi^2} = \text{RHSZ} \]

Where;

\[ \text{RHSX} = -G \left( \frac{\partial^2 W}{\partial \phi^2} + \frac{\partial^2 U}{\partial \phi \partial \psi} \right) + \frac{1}{2} \left( \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi} \right) - G \left( \frac{\partial W}{\partial \phi} + \frac{\partial U}{\partial \psi} \right) - \frac{1}{2} \left( \frac{\partial^2 U}{\partial \phi^2} - \frac{\partial^2 W}{\partial \phi^2} \right) - 
\]

\[ G \left( \frac{\partial^2 V}{\partial \chi \partial \psi} + n^2 \frac{\partial^2 W}{\partial \chi^2} \right) \frac{1}{2} \left( \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi} \right) - G \left( \frac{\partial V}{\partial \phi} + \frac{\partial W}{\partial \chi} \right) \frac{1}{2} \left( \frac{\partial^2 U}{\partial \chi \partial \phi} - \frac{\partial^2 W}{\partial \chi \partial \phi} \right) - 
\]

\[ \left( \frac{\lambda + 2G}{m^2} \frac{\partial^2 W}{\partial \psi^2} + \lambda \left( \frac{\partial^2 U}{\partial \phi \partial \psi} + n \frac{\partial^2 V}{\partial \chi \partial \psi} \right) \right) \frac{1}{2} \left( \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi} \right) - \]

\[ \left( \frac{\lambda + 2G}{m^2} \frac{\partial^2 W}{\partial \psi^2} + \lambda \left( \frac{\partial^2 U}{\partial \phi \partial \psi} + n \frac{\partial^2 V}{\partial \chi \partial \psi} \right) \right) \frac{1}{2} \left( \frac{\partial^2 U}{\partial \phi \partial \psi} - \frac{\partial^2 W}{\partial \phi \partial \psi} \right) \]

\[ \text{RHSY} = G \left( \frac{\partial^2 W}{\partial \phi^2} + \frac{\partial^2 U}{\partial \phi \partial \psi} \right) + \frac{1}{2} \left( \frac{\partial U}{\partial \chi} - \frac{\partial V}{\partial \chi} \right) + G \left( \frac{\partial W}{\partial \phi} + \frac{\partial U}{\partial \psi} \right) + \frac{1}{2} \left( \frac{\partial^2 W}{\partial \chi \partial \phi} - \frac{\partial^2 V}{\partial \chi \partial \phi} \right) + 
\]

\[ G \left( \frac{\partial^2 V}{\partial \chi \partial \psi} + n^2 \frac{\partial^2 W}{\partial \chi^2} \right) \frac{1}{2} \left( \frac{\partial U}{\partial \chi} - \frac{\partial V}{\partial \chi} \right) + G \left( \frac{\partial V}{\partial \phi} + n \frac{\partial W}{\partial \chi} \right) + \frac{1}{2} \left( \frac{\partial^2 W}{\partial \chi^2} - n \frac{\partial^2 V}{\partial \chi \partial \psi} \right) + 
\]

\[ \left( \frac{\lambda + 2G}{m^2} \frac{\partial^2 W}{\partial \psi^2} + \lambda \left( \frac{\partial^2 U}{\partial \phi \partial \psi} + n \frac{\partial^2 V}{\partial \chi \partial \psi} \right) \right) \frac{1}{2} \left( \frac{\partial U}{\partial \chi} - \frac{\partial V}{\partial \chi} \right) + 
\]

\[ \left( \frac{\lambda + 2G}{m^2} \frac{\partial^2 W}{\partial \psi^2} + \lambda \left( \frac{\partial^2 U}{\partial \phi \partial \psi} + n \frac{\partial^2 V}{\partial \chi \partial \psi} \right) \right) \frac{1}{2} \left( \frac{\partial^2 W}{\partial \chi \partial \psi} - \frac{\partial^2 V}{\partial \chi \partial \psi} \right) \]

\[ \text{RHSZ} = m^{-1} \left( \frac{(\lambda + 2G)}{m^2} \frac{\partial^2 U}{\phi^2} + \lambda (nm^2 \frac{\partial^2 V}{\partial \phi \partial \chi} + \frac{\partial^2 W}{\partial \phi \partial \psi}) \right) \frac{1}{2} \left( \frac{\partial U}{\partial \psi} - \frac{\partial W}{\partial \phi} \right) + 
\]
m^{-1} \left( (\lambda + 2G)m^2 \frac{\partial U}{\partial \phi} + \lambda (nm^2 \frac{\partial V}{\partial \chi} + \frac{\partial W}{\partial \psi}) \right) \frac{1}{2} \left( \frac{\partial^2 U}{\partial \phi \partial \psi} - \frac{\partial^2 W}{\partial^2 \psi} \right) -

mG(n^2 \frac{\partial^2 U}{\partial \phi \partial \chi} + \frac{\partial^2 V}{\partial \phi^2}) \frac{1}{2} (n \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) - mG(n \frac{\partial U}{\partial \chi} + \frac{\partial V}{\partial \phi}) \frac{1}{2} (n \frac{\partial^2 W}{\partial \phi \partial \chi} - \frac{\partial^2 V}{\partial \phi \partial \psi}) +

mG(n^2 \frac{\partial^2 U}{\partial \chi^2} + n \frac{\partial^2 V}{\partial \phi \partial \chi} \frac{1}{2} (\frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi}) + mG(n \frac{\partial U}{\partial \chi} + \frac{\partial V}{\partial \phi}) \frac{1}{2} (n \frac{\partial^2 U}{\partial \phi \partial \chi} - n \frac{\partial^2 V}{\partial \phi \partial \psi}) -

m^{-1} \left( (\lambda + 2G)(nm)^2 \frac{\partial^2 V}{\partial \chi^2} + \lambda (nm^2 \frac{\partial^2 U}{\partial \phi \partial \chi} + n \frac{\partial^2 W}{\partial \phi \partial \psi}) \right) \frac{1}{2} (n \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) -

m^{-1} \left( (\lambda + 2G)nm^2 \frac{\partial V}{\partial \chi} + \lambda (m^2 \frac{\partial U}{\partial \phi} + \frac{\partial W}{\partial \psi}) \right) \frac{1}{2} (n^2 \frac{\partial^2 W}{\partial \phi \partial \chi^2} - n \frac{\partial^2 V}{\partial \phi \partial \psi}) +

m^{-1}G(\frac{\partial^2 W}{\partial \phi \partial \psi} + \frac{\partial U}{\partial \phi^2}) \frac{1}{2} \left( \frac{\partial U}{\partial \phi} - \frac{\partial W}{\partial \phi} \right) + m^{-1}G(\frac{\partial^2 W}{\partial \phi^2} + \frac{\partial V}{\partial \phi^2}) \frac{1}{2} \left( \frac{\partial^2 U}{\partial \phi \partial \psi} - \frac{\partial^2 W}{\partial \phi^2} \right) -

m^{-1}G(n \frac{\partial^2 W}{\partial \chi \partial \psi} + \frac{\partial^2 V}{\partial \psi^2}) \frac{1}{2} (n \frac{\partial W}{\partial \chi} - \frac{\partial V}{\partial \psi}) - m^{-1}G(n \frac{\partial W}{\partial \chi} + \frac{\partial V}{\partial \phi}) \frac{1}{2} (n \frac{\partial^2 W}{\partial \chi \partial \psi} - \frac{\partial^2 V}{\partial \phi \partial \psi})

(4.6)

Satisfying the equilibrium equations and boundary conditions we write:

\[ [A]\{DIS\} = \{FOR\} \]

(4.7)

Where:

\( \{FOR\} = \{FOR\} + \gamma \{RHS\} \)

Before introducing the \{RHS\} vector the problem is reduced to small deflection and the vector \{FOR\} consists in representing the external forces applied on the plate only, where the output \{DIS\} represent the linear displacements and will be cosidered as the first guess into the iterativve procedure. When the \{RHS\} vector is included, the displacement vector \{DIS\} becomes \{DISN\} a nonlinear displacement vector and we resume:

\[ [A]^{-1}\{FOR\} = \begin{cases} \{DIS\} & \text{if } \{RHS\} = 0 \\ \{DISN\} & \text{if } \{RHS\} \neq 0 \end{cases} \]

(4.8)
4.2 Finite difference method

The derivation of the equations of equilibrium and deformation in the analysis of plates leads to a number of partial differential equations in which the unknowns are the internal forces and the deformations. The unknowns occur in differential form by considering the equilibrium and deformation of an infinitesimal plate element. The differential relations have in their turn a well defined physical meaning. They permit the local study of forces or deformations. The complete determination of these phenomena involves the integration of the equations.

In most cases, the solutions of differential or partial differential equations cannot be obtained by means of elementary functions. This arises because, in the most general case, when the radii of curvature of the middle surface and the external loads are not given explicitly, the form of the differential equation or partial differential equation is not known and the general solution cannot be obtained.

In general, solutions to the governing differential equations are difficult to obtain and numerical computations must be resorted to. The approximations in numerical calculation may be made small by suitable choice of the initial scheme for the calculation. The basis of any method of numerical computation lies in not employing infinitely small quantities, but in using very small finite quantities. For this operation however, the form of the problem is modified.

In the first case, the analytical solution leads to a continuous expression for
the unknowns at least within distinct intervals; in other words, having the expression for an unknown as a function of the independent variables, we can determine directly the value of this unknown at any point on the middle surface. On the other hand, the numerical calculation leads to the determination of the values of the unknowns only at the points of a previously established network. In Figure (4.1) we show how the network of modified finite difference is assembled with respect to the origin at the mid-plate. To make everything clear Figure (4.2) shows the three principal planes \( \phi \chi, \chi \psi, \psi \phi \), which are the symmetrical planes of the plate. The same notation is used to simplify the interpretation of the partial derivatives defining the finite difference network within the plate.
Figure (4.1) Three dimensional finite difference network

Figure (4.2) Three principal planes
Figure (4.3) Horizontal partition showing layers of the plate

Figure (4.4) The finite difference mesh-size in the $\phi\chi$ plane
While defining the dimensionless system of notations U, V, W, for the partial derivatives in terms of finite difference equations, in the \( \phi, \chi \) and \( \psi \) directions respectively, the plate is also divided in layers as shown in Figure (4.3). The equations of equilibrium are divided into sub-matrices defining the participation of each function in one principal direction. Here,

- **UU**: represents the participation of the U function in the \( \phi \) direction.
- **UV**: represents the participation of the V function in the \( \phi \) direction.
- **UW**: represents the participation of the W function in the \( \phi \) direction.
- **VU**: represents the participation of the U function in the \( \chi \) direction.
- **VV**: represents the participation of the V function in the \( \chi \) direction.
- **VW**: represents the participation of the W function in the \( \chi \) direction.
- **WU**: represents the participation of the U function in the \( \psi \) direction.
- **WV**: represents the participation of the V function in the \( \psi \) direction.
- **WW**: represents the participation of the W function in the \( \psi \) direction.

For the notation of our grid network, only a quarter of the plate is considered due to the symmetry in the \( xy \) plane, while in the \( z \) direction the advantage of the symmetry is not considered for the general use of the program, in case we consider subsequently a plate with variable thickness.

The increment in the notation Figure (4.8) is considered in a manner to simplify the notation. For example, in the \( \phi \) direction the increment is always unity, in the \( \chi \) direction the increment is equal to the term IJ and in the \( \psi \) direction we have in the plane \( \phi \psi \) the term IK, in the plane \( \chi \psi \) the term JK. These two terms IK, JK are equal in the case of the plate with constant thickness.

As it was described in chapter III the top and bottom layers are determined
by equation governing thin plate theory. The diagram in Figure (4.7) shows the development of this equation and it can be employed at the boundary. As it is known for the boundary conditions of clamped plates, points are added beyond the boundary limit similar to the point inside the plate affected with a plus sign and with a minus sign for the simply supported plate, see Figures (4.5) and (4.6).

\[ w(i-1,j) = -w(i+1,j) \]

\[ w(i-1,j) = \frac{w(i+1,j)}{w(i+1,j)} \]

Figure (4.5) Simply Supported Edge

Figure (4.6) Fixed Edge
Figure (4.7) General Finite Difference Scheme for Thin Plate
Figure (4.8) Clamped Edge Finite Difference Scheme
Figure (4.9) Simply Supported Edge Finite Difference Scheme
Figure (4.10) Three Dimensional Finite Difference Representation
For the general case we represent the partial derivatives determining the equilibrium equations by the following:

\[
(\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = \left[U(I - 1) - 2U(I) + U(I + 1)\right] \frac{(\lambda + 2G)}{\lambda^2} \\
G \frac{\partial^2 U}{\partial \chi^2} = \left[U(I - IJ) - 2U(I) + U(I + IJ)\right] G \frac{2}{\lambda^2} \\
G \frac{\partial^2 U}{\partial \psi^2} = \left[U(I - IK) - 2U(I) + U(I + IK)\right] G \frac{2}{\lambda^2} \\
(\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)\right] \frac{(\lambda + G)}{\lambda \phi \lambda \chi} \\
(\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = \left[W(I) - W(I + 1) - W(I + IK) + W(I + IK + 1)\right] \frac{(\lambda + G)}{\lambda \phi \lambda \psi} \\
(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = \left[U(I) - U(I + 1) - U(I + IJ) + U(I + IJ + 1)\right] \frac{(\lambda + G)}{\lambda \phi \lambda \chi} \\
G \frac{\partial^2 V}{\partial \phi^2} = \left[U(I - 1) - 2U(I) + U(I + 1)\right] G \frac{2}{\lambda^2} \\
(\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = \left[V(I - IJ) - 2V(I) + V(I + IJ)\right] \frac{(\lambda + 2G)}{\lambda^2} \\
G \frac{\partial^2 V}{\partial \psi^2} = \left[V(I - IK) - 2V(I) + V(I + IK)\right] G \frac{2}{\lambda^2} \\
(\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = \left[W(I) - W(I + IJ) - W(I + JK) + W(I + IJ + JK)\right] \frac{(\lambda + G)}{\lambda \chi \lambda \psi} \\
(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = \left[U(I) - U(I + 1) - U(I + IK) + U(I + IK + 1)\right] \frac{(\lambda + G)}{\lambda \phi \lambda \psi} \\
(\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} = \left[V(I) - V(I + IJ) - V(I + JK) + V(I + IJ + JK)\right] \frac{(\lambda + G)}{\lambda \chi \lambda \psi} \\
G \frac{\partial^2 W}{\partial \phi^2} = \left[W(I - 1) - 2W(I) + W(I + 1)\right] G \frac{2}{\lambda^2} \\
G \frac{\partial^2 W}{\partial \chi^2} = \left[W(I - IJ) - 2W(I) + W(I + IJ)\right] G \frac{2}{\lambda^2}
\[(\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = \left[ W(I - IK) - 2W(I) + W(I + IK) \right] \frac{(\lambda + 2G)}{\lambda^2} \]

For the points on the \( \chi \) axis

\[(\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = \left[ 2U(I - 1) - 2U(I) \right] \frac{(\lambda + 2G)}{\lambda^2} \]

\[G \frac{\partial^2 U}{\partial \chi^2} = \left[ U(I - IJ) - 2U(I) + U(I + IJ) \right] \frac{G}{\lambda^2} \]

\[G \frac{\partial^2 U}{\partial \psi^2} = \left[ U(I - IK) - 2U(I) + U(I + IK) \right] \frac{G}{\lambda^2} \]

\[(\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[ V(I) - V(I - 1) - V(I + IJ) + V(I + IJ - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]

\[(\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = \left[ W(I) - W(I - 1) - W(I + IJ) + W(I + IJ - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]

\[(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = \left[ U(I) - U(I - 1) - U(I + IJ) + U(I + IJ - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]

\[G \frac{\partial^2 V}{\partial \phi^2} = \left[ 2V(I - 1) - 2V(I) \right] \frac{G}{\lambda^2} \]

\[(\lambda + 2G) \frac{\partial^2 V}{\partial \psi^2} = \left[ V(I - IJ) - 2V(I) + V(I + IJ) \right] \frac{(\lambda + 2G)}{\lambda^2} \]

\[G \frac{\partial^2 V}{\partial \psi^2} = \left[ V(I - JK) - 2V(I) + V(I + JK) \right] \frac{G}{\lambda^2} \]

\[(\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = \left[ W(I) - W(I + IJ) - W(I + JK) + W(I + IJ + JK) \right] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]

\[(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = \left[ U(I) - U(I - 1) - U(I + IK) + U(I + IK - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]

\[(\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} = \left[ V(I) - V(I + IJ) - V(I + JK) + V(I + IJ + JK) \right] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]

\[G \frac{\partial^2 W}{\partial \phi^2} = \left[ 2W(I - 1) - 2W(I) \right] \frac{G}{\lambda^2} \]

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\[
G \frac{\partial^2 W}{\partial \chi^2} = [W(I - IJ) - 2W(I) + W(I + IJ)] \frac{G}{\lambda_\psi^2}
\]
\[
(\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = [W(I - IK) - 2W(I) + W(I + IK)] \frac{(\lambda + 2G)}{\lambda_\psi^2}
\]

For the points on the \(\phi\) axis
\[
(\lambda + 2G) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I - 1) - 2U(I) + U(I + 1)] \frac{(\lambda + 2G)}{\lambda_\psi^2}
\]
\[
G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - IJ) - 2U(I)] \frac{G}{\lambda_\chi^2}
\]
\[
G \frac{\partial^2 U}{\partial \psi^2} = [U(I - IK) - 2U(I) + U(I + IK)] \frac{G}{\lambda_\psi^2}
\]
\[
(\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I - IJ) + V(I - IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi}
\]
\[
(\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = [W(I) - W(I + 1) - W(I + IK) + W(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi}
\]
\[
(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I - IJ) + U(I - IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi}
\]
\[
G \frac{\partial^2 V}{\partial \phi^2} = [V(I - 1) - 2V(I) + V(I + 1)] \frac{G}{\lambda_\phi^2}
\]
\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = [2V(I - IJ) - 2V(I)] \frac{(\lambda + 2G)}{\lambda_\chi^2}
\]
\[
G \frac{\partial^2 V}{\partial \psi^2} = [V(I - JK) - 2V(I) + V(I + JK)] \frac{G}{\lambda_\psi^2}
\]
\[
(\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = [W(I) - W(I - IJ) - U(I + JK) + U(I - IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi}
\]
\[
(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = [U(I) - U(I + 1) - U(I + IK) + U(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi}
\]
\[
(\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \psi} = [V(I) - V(I - IJ) - V(I + IK) + V(I - IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi}
\]
\[ G \frac{\partial^2 W}{\partial \phi^2} = \left[ W(I - 1) - 2W(I) + W(I + 1) \right] \frac{G}{\lambda_\phi^2} \]
\[ G \frac{\partial^2 W}{\partial \chi^2} = \left[ 2W(I - IJ) - 2W(I) \right] \frac{G}{\lambda_\chi^2} \]
\[ (\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = \left[ W(I - IK) - 2W(I) + W(I + IK) \right] \frac{(\lambda + 2G)}{\lambda_\psi^2} \]

For the central point
\[ (\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = \left[ 2U(I - 1) - 2U(I) \right] \frac{(\lambda + 2G)}{\lambda_\phi^2} \]
\[ G \frac{\partial^2 U}{\partial \chi^2} = \left[ 2U(I - IJ) - 2U(I) \right] \frac{G}{\lambda_\chi^2} \]
\[ G \frac{\partial^2 U}{\partial \psi^2} = \left[ U(I - IK) - 2U(I) + U(I + IK) \right] \frac{G}{\lambda_\psi^2} \]
\[ (\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[ V(I) - V(I - 1) - V(I - IJ) + V(I - IJ - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]
\[ (\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = \left[ W(I) - W(I - 1) - W(I + IK) + W(I + IK - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = \left[ U(I) - U(I - 1) - U(I - IJ) + U(I - IJ - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]
\[ G \frac{\partial^2 V}{\partial \phi^2} = \left[ 2V(I - 1) - 2V(I) \right] \frac{G}{\lambda_\phi^2} \]
\[ (\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = \left[ 2V(I - IJ) - 2V(I) \right] \frac{(\lambda + 2G)}{\lambda_\chi^2} \]
\[ G \frac{\partial^2 V}{\partial \psi^2} = \left[ V(I - IK) - 2V(I) + V(I + IK) \right] \frac{G}{\lambda_\psi^2} \]
\[ (\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = \left[ W(I) - W(I - IJ) - U(I + JK) + U(I - IJ + JK) \right] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = \left[ U(I) - U(I - 1) - U(I + IK) + U(I + IK - 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]
\[
(\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} = \left[ V(I) - V(I - IJ) - V(I + JK) + V(I - IJ + JK) \right] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \\
G \frac{\partial^2 W}{\partial \sigma^2} = \left[ 2W(I - 1) - 2W(I) \right] \frac{G}{\lambda_\phi^2} \\
G \frac{\partial^2 W}{\partial \chi^2} = \left[ 2W(I - IJ) - 2W(I) \right] \frac{G}{\lambda_\phi^2} \\
(\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = \left[ W(I - IK) - 2W(I) + W(I + IK) \right] \frac{(\lambda + 2G)}{\lambda_\psi^2}
\]

For the points on the top and bottom layers, the system of the equilibrium equations expressed in terms of the partial derivatives yields:

\[
\frac{E}{1 - \nu^2} \frac{\partial^2 U}{\partial \phi^2} + n^2 G \frac{\partial^2 U}{\partial \chi^2} + \frac{E}{2(1 - \nu)} n \frac{\partial^2 V}{\partial \phi \partial \chi} = TRHSX \tag{4.9}
\]

\[
n \frac{E}{2(1 - \nu)} \frac{\partial^2 U}{\partial \phi \partial \chi} + G \frac{\partial^2 V}{\partial \phi^2} + n^2 \frac{E}{1 - \nu^2} \frac{\partial^2 V}{\partial \chi^2} = TRHSY \tag{4.10}
\]

Where the right hand sides \( TRHSX \) and \( TRHSY \) are equal:

\[
TRHSX = \frac{E}{1 - \nu^2} \frac{\partial W \partial^2 W}{\partial \phi \partial \phi^2} + n^2 \frac{E}{2(1 - \nu)} \frac{\partial W \partial^2 W}{\partial \phi \partial \chi} - n^2 G \frac{\partial W \partial^2 W}{\partial \phi \partial \chi^2} \\
TRHSY = -n^3 \frac{E}{1 - \nu^2} \frac{\partial W \partial^2 W}{\partial \chi \partial \chi^2} - n \frac{E}{2(1 - \nu)} \frac{\partial W \partial^2 W}{\partial \phi \partial \phi \partial \chi} - n G \frac{\partial W \partial^2 W}{\partial \chi \partial \phi^2} \tag{4.11}
\]

The above system of equations with three unknowns combined with the following principal differential equation governing the lateral displacement create a determined system of three equations (4.9), (4.10), (4.13) with three unknowns.

\[
\frac{\partial^4 W}{\partial \phi^4} + 2n^2 \frac{\partial^4 W}{\partial \phi^2 \partial \chi^2} + n^4 \frac{\partial^4 W}{\partial \chi^4} = \frac{a^4}{D h} + TRHSZ \tag{4.13}
\]
\[
TRHSZ = 12\left(\frac{\partial U}{\partial \phi} + \frac{1}{2} \left(\frac{\partial W}{\partial \phi}\right)^2 + \nu n \frac{\partial V}{\partial \chi} + \frac{n^2}{2} \frac{\partial W}{\partial \chi} \frac{\partial^2 W}{\partial \phi^2} + \frac{\nu}{2} \left(\frac{\partial W}{\partial \phi}\right)^2 + \frac{\nu}{2} \frac{\partial V}{\partial \chi} + \frac{n^2}{2} \left(\frac{\partial W}{\partial \chi}\right)^2 \right) + \frac{n^2}{2} \frac{\partial^2 W}{\partial \chi^2} + (1 - \nu) \left[n^2 \frac{\partial U}{\partial \chi} + \frac{\partial V}{\partial \phi} + n \frac{\partial W}{\partial \phi} \frac{\partial^2 W}{\partial \phi \partial \chi} \right] \right) \quad (4.14)
\]

A general point on the boundary layer will be defined by the expressions below:

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 U}{\partial \phi^2} = \left[U(I - 1) - 2U(I) + U(I + 1)\right] \left(\frac{E}{1 - \nu^2}\right) \frac{1}{\lambda_\phi^2}
\]

\[
G \frac{\partial^2 U}{\partial \chi^2} = \left[U(I - IJ) - 2U(I) + U(I + IJ)\right] \frac{G}{\lambda_\chi^2}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)\right] \left(\frac{E}{2(1 - \nu)}\right) \frac{1}{\lambda_\phi \lambda_\chi}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[U(I) - U(I + 1) + U(I + IJ) + U(I + IJ + 1)\right] \left(\frac{E}{2(1 - \nu)}\right) \frac{1}{\lambda_\phi \lambda_\chi}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[V(I - 1) - 2V(I) + V(I + 1)\right] \frac{G}{\lambda_\phi^2}
\]

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 V}{\partial \chi^2} = \left[V(I - IJ) - 2V(I) + V(I + IJ)\right] \left(\frac{E}{1 - \nu^2}\right) \frac{1}{\lambda_\chi^2}
\]

For the case when the point is on the y axis:

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 U}{\partial \phi^2} = \left[2U(I - 1) - 2U(I)\right] \left(\frac{E}{1 - \nu^2}\right) \frac{1}{\lambda_\phi^2}
\]

\[
G \frac{\partial^2 U}{\partial \chi^2} = \left[U(I - IJ) - 2U(I) + U(I + IJ)\right] \frac{G}{\lambda_\chi^2}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[V(I) - V(I - 1) - V(I + IJ) + V(I + IJ - 1)\right] \left(\frac{E}{2(1 - \nu)}\right) \frac{1}{\lambda_\phi \lambda_\chi}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[U(I) - U(I - 1) + U(I + IJ) + U(I + IJ - 1)\right] \left(\frac{E}{2(1 - \nu)}\right) \frac{1}{\lambda_\phi \lambda_\chi}
\]

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\[
G \frac{\partial^2 V}{\partial \phi^2} = [2V(I-1) - 2V(I)] \frac{G}{\lambda_\phi^2}
\]
\[
(\frac{E}{1-\nu^2}) \frac{\partial^2 V}{\partial \chi^2} = [V(I - IJ) - 2V(I) + V(I + IJ)] (\frac{E}{1-\nu^2}) \frac{G}{\lambda_\chi^2}
\]

and when the point is on the x axis
\[
(\frac{E}{1-\nu^2}) \frac{\partial^2 U}{\partial \phi^2} = [U(I - 1) - 2U(I) + U(I + 1)] (\frac{E}{1-\nu^2}) \frac{G}{\lambda_\phi^2}
\]
\[
G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - IJ) - 2U(I)] \frac{G}{\lambda_\chi^2}
\]
\[
(\frac{E}{2(1-\nu)}) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I - IJ) + V(I - IJ + 1)] (\frac{E}{2(1-\nu)}) \frac{G}{\lambda_\phi \lambda_\chi}
\]
\[
(\frac{E}{2(1-\nu)}) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I - IJ) + U(I - IJ + 1)] (\frac{E}{2(1-\nu)}) \frac{G}{\lambda_\phi \lambda_\chi}
\]
\[
G \frac{\partial^2 V}{\partial \phi^2} = [V(I - 1) - 2V(I) + V(I + 1)] \frac{G}{\lambda_\phi^2}
\]
\[
(\frac{E}{1-\nu^2}) \frac{\partial^2 V}{\partial \chi^2} = [2V(I - IJ) - 2V(I)] (\frac{E}{1-\nu^2}) \frac{G}{\lambda_\chi^2}
\]

and for the central point
\[
(\frac{E}{1-\nu^2}) \frac{\partial^2 U}{\partial \phi^2} = [2U(I - 1) - 2U(I)] (\frac{E}{1-\nu^2}) \frac{G}{\lambda_\phi^2}
\]
\[
G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - IJ) - 2U(I)] \frac{G}{\lambda_\chi^2}
\]
\[
(\frac{E}{2(1-\nu)}) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I - 1) - V(I - IJ) + V(I - IJ - 1)] (\frac{E}{2(1-\nu)}) \frac{G}{\lambda_\phi \lambda_\chi}
\]
\[
(\frac{E}{2(1-\nu)}) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I - 1) - U(I - IJ) + U(I - IJ - 1)] (\frac{E}{2(1-\nu)}) \frac{G}{\lambda_\phi \lambda_\chi}
\]

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\[ G \frac{\partial^2 V}{\partial \phi^2} = [2V(I - 1) - 2V(I)] \frac{G}{\lambda_\phi^2} \]
\[ \left( \frac{E}{1 - \nu^2} \right) \frac{\partial^2 V}{\partial \chi^2} = [2V(I - IJ) - 2V(I)] \frac{E}{\lambda_\chi^2} \]

Now after describing the four cases common for all plates we now attempt to determine the particular points near the boundary. The application is limited to two cases, clamped and simply supported.

4.3 Simply supported plate

If the plate has ideal simply supported edges, it must be free to move along the supported edges in the plane of the plate, that is, the shearing stress along the edges in the plane of the plate is zero.

4.4 Clamped plate

For the clamped plate and simply supported plate with immovable edges all the components \(U, V,\) and \(W\) on the boundary are equal to zero, therefore the points near the boundary their backward finite difference components are equal to zero from the boundary conditions.
Figure (4.11) Clamped Plate

Figure (4.12) Simply Supported Plate
4.4.1 Intermediate surfaces

For the corner point,

\[
(\lambda + 2G) \frac{\partial^2 U}{\partial \phi \partial \chi} = [-2U(I) + U(I + 1)] \frac{(\lambda + 2G)}{\lambda_\phi^2}
\]

\[
G \frac{\partial U}{\partial \chi^2} = [-2U(I) + U(I + IJ)] \frac{G}{\lambda_\chi^2}
\]

\[
G \frac{\partial U}{\partial \psi^2} = [U(I - IK) - 2U(I) + U(I + IK)] \frac{G}{\lambda_\psi^2}
\]

\[
(\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi}
\]

\[
(\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = [W(I) - W(I + 1) - W(I + IK) + W(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi}
\]

\[
(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I + IJ) + U(I + IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi}
\]

\[
G \frac{\partial^2 V}{\partial \phi^2} = [-2U(I) + U(I + 1)] \frac{G}{\lambda_\phi^2}
\]

\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = [-2V(I) + V(I + IJ)] \frac{(\lambda + 2G)}{\lambda_\chi^2}
\]

\[
G \frac{\partial^2 V}{\partial \psi^2} = [V(I - IK) - 2V(I) + V(I + IK)] \frac{G}{\lambda_\psi^2}
\]

\[
(\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = [W(I) - W(I + IJ) - W(I + J K) + W(I + IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi}
\]

\[
(\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = [U(I) - U(I + 1) - U(I + IK) + U(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi}
\]

\[
(\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} = [V(I) - V(I + IJ) - V(I + J K) + V(I + IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi}
\]

\[
G \frac{\partial^2 W}{\partial \phi^2} = [-2W(I) + W(I + 1)] \frac{G}{\lambda_\phi^2}
\]

\[
G \frac{\partial^2 W}{\partial \chi^2} = [-2W(I) + W(I + IJ)] \frac{G}{\lambda_\chi^2}
\]

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\[(\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = \left[ W(I - IK) - 2W(I) + W(I + IK) \right] \frac{(\lambda + 2G)}{\lambda_\psi^2} \]

For points parallel to \( \phi \) axis,

\[ (\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = \left[ U(I - 1) - 2U(I) + U(I + 1) \right] \frac{(\lambda + 2G)}{\lambda_\phi^2} \]

\[ G \frac{\partial^2 U}{\partial \chi^2} = \left[ -2U(I) + U(I + IJ) \right] \frac{G}{\lambda_\chi^2} \]

\[ G \frac{\partial^2 U}{\partial \psi^2} = \left[ U(I - IK) - 2U(I) + U(I + IK) \right] \frac{G}{\lambda_\psi^2} \]

\[ (\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[ V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]

\[ (\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = \left[ W(I - W(I + 1) - W(I + IK) + W(I + IK + 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]

\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = \left[ U(I - 1) - U(I + 1) - U(I + IJ) + U(I + IJ + 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]

\[ G \frac{\partial^2 V}{\partial \phi^2} = \left[ U(I - 1) - 2U(I) + U(I + 1) \right] \frac{G}{\lambda_\phi^2} \]

\[ (\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = \left[ -2V(I) + V(I + IJ) \right] \frac{(\lambda + 2G)}{\lambda_\chi^2} \]

\[ G \frac{\partial^2 V}{\partial \psi^2} = \left[ V(I - IK) - 2V(I) + V(I + IK) \right] \frac{G}{\lambda_\psi^2} \]

\[ (\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = \left[ W(I - W(I + IJ) - W(I + JK) + W(I + IJ + JK) \right] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]

\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = \left[ U(I - U(I + 1) - U(I + IK) + U(I + IK + 1) \right] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]

\[ (\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \phi} = \left[ V(I - V(I + IJ) - V(I + JK) + V(I + IJ + JK) \right] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]

\[ G \frac{\partial^2 W}{\partial \phi^2} = \left[ W(I - 1) - 2W(I) + W(I + 1) \right] \frac{G}{\lambda_\phi^2} \]
\[
G \frac{\partial^2 W}{\partial \chi^2} = \left[-2W(I) + W(I + IJ)\right] \frac{G}{\lambda^2}
\]
\[
(\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = \left[W(I - IK) - 2W(I) + W(I + IK)\right] \frac{(\lambda + 2G)}{\lambda^2}
\]

For points parallel to the \(\phi\) axis and on the \(\chi\) axis,
\[
(\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = \left[2U(I) - 2U(I - 1)\right] \frac{(\lambda + 2G)}{\lambda^2}
\]
\[
G \frac{\partial^2 U}{\partial \chi^2} = \left[-2U(I) + U(I + IJ)\right] \frac{G}{\lambda^3}
\]
\[
G \frac{\partial^2 U}{\partial \psi^2} = \left[U(I - IK) - 2U(I) + U(I + IK)\right] \frac{G}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \phi \partial \chi} = \left[V(I) - V(I - 1) - V(I + IJ) + V(I + IJ - 1)\right] \frac{(\lambda + G)}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \phi \partial \psi} = \left[W(I) - 2W(I - 1) - W(I + IK) + W(I + IK - 1)\right] \frac{(\lambda + G)}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = \left[U(I) - U(I - 1) - U(I + IJ) + U(I + IJ - 1)\right] \frac{(\lambda + G)}{\lambda^3}
\]
\[
G \frac{\partial^2 V}{\partial \phi^2} = \left[2V(I) - 2V(I - 1)\right] \frac{G}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = \left[-2V(I) + V(I + IJ)\right] \frac{(\lambda + 2G)}{\lambda^3}
\]
\[
G \frac{\partial^2 V}{\partial \psi^2} = \left[V(I - JK) - 2V(I) + V(I + JK)\right] \frac{G}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 W}{\partial \chi \partial \psi} = \left[W(I) - W(I + IJ) - W(I + JK) + W(I + IJ + JK)\right] \frac{(\lambda + G)}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 U}{\partial \phi \partial \psi} = \left[U(I) - U(I - 1) - U(I + IK) + U(I + IK - 1)\right] \frac{(\lambda + G)}{\lambda^3}
\]
\[
(\lambda + 2G) \frac{\partial^2 V}{\partial \chi \partial \psi} = \left[V(I) - V(I + IJ) - V(I + JK) + V(I + IJ + JK)\right] \frac{(\lambda + G)}{\lambda^3}
\]

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\[ G \frac{\partial^2 W}{\partial \phi^2} = [2W(I-1) - 2W(I)] \frac{G}{\lambda_\phi^2} \]
\[ G \frac{\partial^2 W}{\partial \chi^2} = [-2W(I) + W(I + IJ)] \frac{G}{\lambda_\chi^2} \]
\[ (\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = [W(I - IK) - 2W(I) + W(I + IK)] \frac{(\lambda + 2G)}{\lambda_\psi^2} \]

For the points parallel to the \( \chi \) axis

\[ (\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = [-2U(I) + U(I + 1)] \frac{(\lambda + 2G)}{\lambda_\phi^2} \]
\[ G \frac{\partial^2 U}{\partial \chi^2} = [U(I - IJ) - 2U(I) + U(I + IJ)] \frac{G}{\lambda_\chi^2} \]
\[ G \frac{\partial^2 U}{\partial \psi^2} = [U(I - IK) - 2U(I) + U(I + IK)] \frac{G}{\lambda_\psi^2} \]
\[ (\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]
\[ (\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = [W(I) - W(I + 1) - W(I + IK) + W(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I + IJ) + U(I + IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]
\[ G \frac{\partial^2 V}{\partial \phi^2} = [-2U(I) + U(I + 1)] \frac{G}{\lambda_\phi^2} \]
\[ (\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = [V(I - IJ) - 2V(I) + V(I + IJ)] \frac{(\lambda + 2G)}{\lambda_\chi^2} \]
\[ G \frac{\partial^2 V}{\partial \psi^2} = [V(I - IK) - 2V(I) + V(I + IK)] \frac{G}{\lambda_\psi^2} \]
\[ (\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = [W(I) - W(I + IJ) - W(I + JK) + W(I + IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = [U(I) - U(I + 1) - U(I + IK) + U(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} = [V(I) - V(I + IJ) - V(I + JK) + V(I + IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]
\[ G \frac{\partial^2 W}{\partial \phi^2} = [-2W(I) + W(I + 1)] \frac{G}{\lambda_\phi^2} \]
\[ G \frac{\partial^2 W}{\partial \chi^2} = [W(I - IJ) - 2W(I) + W(I + IJ)] \frac{G}{\lambda_\chi^2} \]
\[ (\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = [W(I - IK) - 2W(I) + W(I + IK)] \frac{(\lambda + 2G)}{\lambda_\psi^2} \]

For a point parallel to \( \chi \) axis and on the \( \phi \) axis,

\[ (\lambda + 2G) \frac{\partial^2 U}{\partial \phi^2} = [-2U(I) + U(I + 1)] \frac{(\lambda + 2G)}{\lambda_\phi^2} \]
\[ G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - IJ) - 2U(I)] \frac{G}{\lambda_\chi^2} \]
\[ G \frac{\partial^2 U}{\partial \psi^2} = [U(I - IK) - 2U(I) + U(I + IK)] \frac{G}{\lambda_\psi^2} \]

\[ (\lambda + G) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I - IJ) + V(I - IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]
\[ (\lambda + G) \frac{\partial^2 W}{\partial \phi \partial \psi} = [W(I) - W(I + 1) - W(I + IJ) + W(I + JK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I - IJ) + U(I - IJ + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\chi} \]
\[ G \frac{\partial^2 V}{\partial \phi^2} = [-2V(I) + V(I + 1)] \frac{G}{\lambda_\phi^2} \]
\[ (\lambda + 2G) \frac{\partial^2 V}{\partial \chi^2} = [2V(I - IJ) - 2V(I)] \frac{(\lambda + 2G)}{\lambda_\chi^2} \]
\[ G \frac{\partial^2 V}{\partial \psi^2} = [V(I - JK) - 2V(I) + V(I + JK)] \frac{G}{\lambda_\psi^2} \]
\[ (\lambda + G) \frac{\partial^2 W}{\partial \chi \partial \psi} = [W(I) - W(I - IJ) - U(I + JK) + U(I - IJ + JK)] \frac{(\lambda + G)}{\lambda_\chi \lambda_\psi} \]
\[ (\lambda + G) \frac{\partial^2 U}{\partial \phi \partial \psi} = [U(I) - U(I + 1) - U(I + IJ) + U(I + IK + 1)] \frac{(\lambda + G)}{\lambda_\phi \lambda_\psi} \]
\[
(\lambda + G) \frac{\partial^2 V}{\partial \chi \partial \psi} = [V(I) - V(I - IJ) - V(I + IJ) + V(I - IJ + IK)] \frac{(\lambda + G)}{\lambda \chi \lambda \nu} \\
G \frac{\partial^2 W}{\partial \phi^2} = [-2W(I) + W(I + I)] \frac{G}{\lambda \phi^2} \\
G \frac{\partial^2 W}{\partial \chi^2} = [2W(I - IJ) - 2W(I)] \frac{G}{\lambda \phi^2} \\
(\lambda + 2G) \frac{\partial^2 W}{\partial \psi^2} = [W(I - IK) - 2W(I) + W(I + IK)] \frac{(\lambda + 2G)}{\lambda \phi^2}
\]

4.4.2 Boundary surfaces

For the corner point,

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 U}{\partial \phi^2} = [-2U(I) + U(I + I)] \frac{(\frac{E}{1 - \nu^2})}{\lambda \phi^2} \\
G \frac{\partial^2 U}{\partial \chi^2} = [-2U(I) + U(I + IJ)] \frac{G}{\lambda \chi^2} \\
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)] \frac{(\frac{E}{2(1 - \nu)})}{\lambda \phi \lambda \chi} \\
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I + IJ) + U(I + IJ + 1)] \frac{(\frac{E}{2(1 - \nu)})}{\lambda \phi \lambda \chi} \\
G \frac{\partial^2 V}{\partial \phi^2} = [-2V(I) + V(I + I)] \frac{G}{\lambda \phi^2} \\
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 V}{\partial \chi^2} = [-2V(I) + V(I + IJ)] \frac{(\frac{E}{1 - \nu^2})}{\lambda \chi^2}
\]
For points parallel to the $\phi$ axis,

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 U}{\partial \phi^2} = [U(I - 1) - 2U(I) + U(I + 1)] \left(\frac{E}{1 - \nu^2}\right) \frac{\lambda_\phi^2}{\lambda_\phi^2}
\]

\[
G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - 1) - 2U(I)] \frac{G}{\lambda_\chi^2}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I - 1) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)] \left(\frac{E}{2(1 - \nu)}\right) \frac{\lambda_\phi \lambda_\chi}{\lambda_\phi \lambda_\chi}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I - 1) - 2U(I - 1) - U(I + IJ) + U(I + IJ + 1)] \left(\frac{E}{2(1 - \nu)}\right) \frac{\lambda_\phi \lambda_\chi}{\lambda_\phi \lambda_\chi}
\]

\[
G \frac{\partial^2 V}{\partial \phi^2} = [2V(I - 1) - 2V(I)] \frac{G}{\lambda_\phi^2}
\]

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 V}{\partial \chi^2} = [-2V(I) + V(I + IJ)] \left(\frac{E}{1 - \nu^2}\right) \frac{\lambda_\chi^2}{\lambda_\chi^2}
\]

For points parallel to the $\phi$ axis and on the $\chi$ axis,

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 U}{\partial \phi^2} = [2U(I - 1) - 2U(I)] \left(\frac{E}{1 - \nu^2}\right) \frac{\lambda_\phi^2}{\lambda_\phi^2}
\]

\[
G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - 1) - 2U(I)] \frac{G}{\lambda_\chi^2}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I - 1) - V(I - 1) - V(I + IJ) + V(I + IJ - 1)] \left(\frac{E}{2(1 - \nu)}\right) \frac{\lambda_\phi \lambda_\chi}{\lambda_\phi \lambda_\chi}
\]

\[
\left(\frac{E}{2(1 - \nu)}\right) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I - 1) - 2U(I - 1) - U(I + IJ) + U(I + IJ - 1)] \left(\frac{E}{2(1 - \nu)}\right) \frac{\lambda_\phi \lambda_\chi}{\lambda_\phi \lambda_\chi}
\]

\[
G \frac{\partial^2 V}{\partial \phi^2} = [2V(I - 1) - 2V(I)] \frac{G}{\lambda_\phi^2}
\]

\[
\left(\frac{E}{1 - \nu^2}\right) \frac{\partial^2 V}{\partial \chi^2} = [-2V(I) + V(I + IJ)] \left(\frac{E}{1 - \nu^2}\right) \frac{\lambda_\chi^2}{\lambda_\chi^2}
\]
For points parallel to the \( \chi \) axis,

\[
\left( \frac{E}{1 - \nu^2} \right) \frac{\partial^2 U}{\partial \phi^2} = [-2U(I) + U(I + 1)] \left( \frac{E}{1 - \nu^2} \right) \frac{1}{\lambda^2_\phi} \\
G \frac{\partial^2 U}{\partial \chi^2} = [U(I - IJ) - 2U(I) + U(I + IJ)] \frac{G}{\lambda^2_\chi}
\]

\[
\left( \frac{E}{2(1 - \nu)} \right) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I + IJ) + V(I + IJ + 1)] \left( \frac{E}{2(1 - \nu)} \right) \frac{1}{\lambda_\phi \lambda_\chi} \\
\left( \frac{E}{2(1 - \nu)} \right) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I + IJ) + U(I + IJ + 1)] \left( \frac{E}{2(1 - \nu)} \right) \frac{1}{\lambda_\phi \lambda_\chi} \\
G \frac{\partial^2 V}{\partial \phi^2} = [-2V(I) + V(I + 1)] \frac{G}{\lambda^2_\phi} \\
\left( \frac{E}{1 - \nu^2} \right) \frac{\partial^2 V}{\partial \chi^2} = [V(I - IJ) - 2V(I) + V(I + IJ)] \left( \frac{E}{1 - \nu^2} \right) \frac{1}{\lambda^2_\chi}
\]

For the point parallel to the \( \chi \) axis and on the \( \phi \) axis,

\[
\left( \frac{E}{1 - \nu^2} \right) \frac{\partial^2 U}{\partial \phi^2} = [-2U(I) + U(I + 1)] \left( \frac{E}{1 - \nu^2} \right) \frac{1}{\lambda^2_\phi} \\
G \frac{\partial^2 U}{\partial \chi^2} = [2U(I - IJ) - 2U(I)] \frac{G}{\lambda^2_\chi}
\]

\[
\left( \frac{E}{2(1 - \nu)} \right) \frac{\partial^2 V}{\partial \phi \partial \chi} = [V(I) - V(I + 1) - V(I - IJ) + V(I - IJ + 1)] \left( \frac{E}{2(1 - \nu)} \right) \frac{1}{\lambda_\phi \lambda_\chi} \\
\left( \frac{E}{2(1 - \nu)} \right) \frac{\partial^2 U}{\partial \phi \partial \chi} = [U(I) - U(I + 1) - U(I - IJ) + U(I - IJ + 1)] \left( \frac{E}{2(1 - \nu)} \right) \frac{1}{\lambda_\phi \lambda_\chi} \\
G \frac{\partial^2 V}{\partial \phi^2} = [-2V(I) + V(I + 1)] \frac{G}{\lambda^2_\phi} \\
\left( \frac{E}{1 - \nu^2} \right) \frac{\partial^2 V}{\partial \chi^2} = [2V(I - IJ) - 2V(I)] \left( \frac{E}{1 - \nu^2} \right) \frac{1}{\lambda^2_\chi}
\]
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Partial Expression</th>
<th>Finite Difference Representation</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>$W_{,\phi}$</td>
<td>$\frac{W(I+1,1)-W(I-1,1)}{2\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D2</td>
<td>$W_{,\phi\phi}$</td>
<td>$\frac{W(I+1,1)+2W(I,1)-W(I-1,1)}{\lambda_{\phi}^2}$</td>
</tr>
<tr>
<td>D3</td>
<td>$W_{,\chi}$</td>
<td>$\frac{W(I+1,J,1)-W(I-J,1)}{2\lambda_{\chi}}$</td>
</tr>
<tr>
<td>D4</td>
<td>$W_{,\phi\chi}$</td>
<td>$\frac{W(I+1+I,J,1)-W(I+1-I,J,1)-W(I-1+I,J,1)+W(I-1-I,J,1)}{4\lambda_{\phi}\lambda_{\chi}}$</td>
</tr>
<tr>
<td>D5</td>
<td>$W_{,\chi\chi}$</td>
<td>$\frac{W(I+1,J,1)-W(I,J,1)+W(I-J,1)}{\lambda_{\chi}^2}$</td>
</tr>
<tr>
<td>D6</td>
<td>$W_{,\phi\psi}$</td>
<td>$\frac{W(I+I+J,K,1)-W(I+I-J,K,1)-W(I-I+J,K,1)+W(I-I-J,K,1)}{4\lambda_{\phi}\lambda_{\psi}}$</td>
</tr>
<tr>
<td>D7</td>
<td>$W_{,\phi\phi}$</td>
<td>$\frac{W(I+1+I,K,1)-W(I+1-I,K,1)-W(I-1+I,K,1)+W(I-1-I,K,1)}{4\lambda_{\phi}\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D8</td>
<td>$U_{,\phi}$</td>
<td>$\frac{U(I+1,1)-U(I-1,1)}{2\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D9</td>
<td>$V_{,\chi}$</td>
<td>$\frac{V(I+1,I,1)-V(I-I,1)}{2\lambda_{\chi}}$</td>
</tr>
<tr>
<td>D12</td>
<td>$U_{,\psi}$</td>
<td>$\frac{U(I+1,K,1)-U(I-I,K,1)}{2\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D13</td>
<td>$U_{,\phi\psi}$</td>
<td>$\frac{U(I+1+I,K,1)-U(I+1-I,K,1)-U(I-1+I,K,1)+U(I-1-I,K,1)}{4\lambda_{\phi}\lambda_{\psi}}$</td>
</tr>
<tr>
<td>D14</td>
<td>$U_{,\chi\psi}$</td>
<td>$\frac{U(I+1+J,K,1)-U(I+1-J,K,1)-U(I-I+J,K,1)+U(I-I-J,K,1)}{4\lambda_{\chi}\lambda_{\psi}}$</td>
</tr>
<tr>
<td>D15</td>
<td>$U_{,\phi\psi}$</td>
<td>$\frac{U(I+1+I,K,1)-U(I,1)-U(I-I,K,1)}{\lambda_{\phi}^2}$</td>
</tr>
<tr>
<td>D16</td>
<td>$V_{,\phi\psi}$</td>
<td>$\frac{V(I+1+K,J,1)-V(I-J,K,1)}{2\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D17</td>
<td>$V_{,\chi\psi}$</td>
<td>$\frac{V(I+1+J,K,1)-V(I-J-K,1)-V(I-J-K,1)+V(I-J-K,1)}{4\lambda_{\chi}\lambda_{\psi}}$</td>
</tr>
<tr>
<td>D18</td>
<td>$V_{,\phi\phi}$</td>
<td>$\frac{V(I+1+I,K,1)-V(I+1-I,K,1)-V(I-1+I,K,1)+V(I-1-I,K,1)}{4\lambda_{\phi}\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D19</td>
<td>$V_{,\chi\psi}$</td>
<td>$\frac{V(I+1+J,K,1)-V(I-J,K,1)+V(I-J,K,1)}{\lambda_{\chi}^2}$</td>
</tr>
<tr>
<td>D20</td>
<td>$U_{,\chi}$</td>
<td>$\frac{U(I+1,I,1)-U(I-I,1)}{2\lambda_{\chi}}$</td>
</tr>
<tr>
<td>D21</td>
<td>$V_{,\phi}$</td>
<td>$\frac{V(I+1,I,1)-V(I-I,1)}{2\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D22</td>
<td>$W_{,\phi}$</td>
<td>$\frac{W(I+1,K,1)-W(I-I,K,1)}{2\lambda_{\phi}}$</td>
</tr>
<tr>
<td>D23</td>
<td>$U_{,\phi\phi}$</td>
<td>$\frac{U(I+1,1)-2U(I,1)+U(I-1,1)}{\lambda_{\phi}^2}$</td>
</tr>
<tr>
<td>D24</td>
<td>$V_{,\phi\phi}$</td>
<td>$\frac{V(I+1,I,1)-2V(I,1)+V(I-I,1)}{\lambda_{\phi}^2}$</td>
</tr>
<tr>
<td>D25</td>
<td>$W_{,\phi\phi}$</td>
<td>$\frac{W(I+1,K,1)-2W(I,1)+W(I-I,K,1)}{\lambda_{\phi}^2}$</td>
</tr>
<tr>
<td>D26</td>
<td>$U_{,\phi\phi}$</td>
<td>$\frac{U(I+1+I,1)-U(I+1-I,1)-U(I-1+I,1)+U(I-1+I,1)}{4\lambda_{\phi}\lambda_{\chi}}$</td>
</tr>
<tr>
<td>D27</td>
<td>$V_{,\phi\phi}$</td>
<td>$\frac{V(I+1+J,1)-V(I+1-I,1)-V(I-I+J,1)+V(I-I+J,1)}{4\lambda_{\phi}\lambda_{\chi}}$</td>
</tr>
<tr>
<td>D28</td>
<td>$U_{,\chi\chi}$</td>
<td>$\frac{U(I+1,1)-2U(I,1)+U(I-I,1)}{\lambda_{\chi}^2}$</td>
</tr>
<tr>
<td>D29</td>
<td>$V_{,\chi\chi}$</td>
<td>$\frac{V(I+1,I,1)-2V(I,1)+V(I-I,1)}{\lambda_{\chi}^2}$</td>
</tr>
</tbody>
</table>
Chapter 5

Mathematical Solutions

5.1 General

Solving a nonlinear system of equations resulting from nonlinear mechanics analysis is often a difficult task. This chapter will be devoted to the applied mathematics and numerical computations. While there are few standard techniques to solve such problems, four new methods will be described, two of them are iterative solutions and two others direct solutions. These new methods are specially programmed to solve simultaneous equations, using a new approach. The approach of the methods differs from the previous iteratives and direct solutions generally available in the technical literature. For the iterative methods the accuracy depends on the nature of the system of equations, and the initial guess. The iterative methods, in general,
involves less computation than the direct methods. However, the iterative methods are much less efficient when the diagonal elements of the matrix are small in comparison with other nonzero elements of the matrix. When such situations arise, some corrections or transformations to the original system in most cases will correct this deficiency and the iterative method will still yield converging solutions instead of diverging solutions. For the direct methods the original system does not need previous transformations.

5.2 First Iterative Method

Any system of simultaneous equations can be expressed in the matrix form by:

\[ [A] \{X\} = \{B\} \]  \hspace{2cm} (5.1)

where:
A: is a square matrix.
X: is an unknown vector.
B: is a known vector.

Before we start our first iteration a simple transformation is needed. This transformation will make the sum of all row elements equal to unity. To make this transformation possible we have to divide elements in each row by the sum of the elements in that row of the original matrix. If we consider \( a_{ij} \) as the elements of the original matrix and \( \bar{a}_{ij} \) the elements of the
transformed matrix we can write:

$$\bar{a}_{ji} = \frac{a_{ij}}{\sum_{j=1}^{n} a_{ij}}$$  \hspace{1cm} (5.2)

where, \(\sum_{j=1}^{n} a_{ij} \neq 0\)

If the sum of one row is zero we add to this row another row where the sum is not zero. The transformed force vector will be:

$$\bar{b}_{ij} = \frac{b_{ij}}{\sum_{j=1}^{n} a_{ij}}$$  \hspace{1cm} (5.3)

Once this transformation is performed, the iterative procedure proceeds with the first initial guess. Part of this new iterative technique is the choice of the first guess. For the first guess we take \(x_i = \bar{b}_i\). To clarify this choice, we develop below the main idea why we use the transformation before starting the iteration.

The transformation described before is equal to the normalization for the diagonal matrix, and the vector force transformed is the solution for the system. To explain these two rules, let us consider again a 3 by 3 system.

$$\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix}$$

The transformed matrix is:

$$\begin{bmatrix}
\bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\
\bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} \\
\bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33}
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
\bar{b}_1 \\
\bar{b}_2 \\
\bar{b}_3
\end{bmatrix}$$
As we know $\sum_{j=1}^{n} a_{ij} = 1$.

If the original system can be put in the diagonal form, then after normalization it will look like:

$$
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
c_1 \\
c_2 \\
c_3 \\
\end{bmatrix}
$$

Then our first guess is the $c$ vector, the direct solution. For the general case our guess system is:

$$
\begin{bmatrix}
\sum_{j=1}^{3} a_{1j} & 0 & 0 \\
0 & \sum_{j=1}^{3} a_{2j} & 0 \\
0 & 0 & \sum_{j=1}^{3} a_{3j} \\
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix}
= \begin{bmatrix}
\bar{b}_1 \\
\bar{b}_2 \\
\bar{b}_3 \\
\end{bmatrix}
$$

Because the sum is equal to unity, the $x$ vector will be equal to the $\bar{b}$ vector. Then our initial guess vector will be in general equal to the right side vector. Using this as the initial iteration the first calculated vector is:

$$[\tilde{A}][\tilde{b}] = \{b^1\} \quad (5.4)$$

and the next vector:

$$
\begin{bmatrix}
x_1^2 \\
x_2^2 \\
x_3^2 \\
\end{bmatrix}
= \begin{bmatrix}
(\bar{b}_1 - b_1^1)/\tilde{a}_{11} \\
(\bar{b}_2 - b_2^1)/\tilde{a}_{22} \\
(\bar{b}_3 - b_3^1)/\tilde{a}_{33} \\
\end{bmatrix}
+ \begin{bmatrix}
x_1^1 \\
x_2^1 \\
x_3^1 \\
\end{bmatrix}
$$
In general we write:

\[
\begin{bmatrix}
  x_1^{k+1} \\
  x_2^{k+1} \\
  x_3^{k+1}
\end{bmatrix} = \begin{bmatrix}
  (\bar{b}_1 - \bar{b}_1^k)/\bar{a}_{11} \\
  (\bar{b}_2 - \bar{b}_2^k)/\bar{a}_{22} \\
  (\bar{b}_3 - \bar{b}_3^k)/\bar{a}_{33}
\end{bmatrix} + \begin{bmatrix}
  x_1^k \\
  x_2^k \\
  x_3^k
\end{bmatrix}
\]

Where the condensed form gives:

\[
x_i^{k+1} = \frac{(\bar{b}_i - \bar{b}_i^k)}{\bar{a}_{ii}} + x_i^k
\]  \hspace{1cm} (5.5)

If we proceed with another iteration we will get the vector \(b_i^{k+1}\). At this stage we check the mean value of the two last b vectors with the \(\bar{b}\) vector which is:

\[
\frac{b_i^{k+1} + b_i^k}{2} = \bar{b}_i, \implies x_i = \frac{x_i^{k+1} + x_i^k}{2}
\]  \hspace{1cm} (5.6)

Because we do not know the values of the X vector, the B vector will serve as our guideline to reach the required accuracy. Accuracy using iterative solution is specified and if we limit the accuracy to \(\epsilon = 0.0001\) and once this limit is reached the computation stops at that level. In structural problems matrices generally have the following characteristics:

1- In general structural matrices have predominating values at the principal diagonal, a property will favours the application of the iterative method.

2- The accuracy required is not of high order, the maximum is of an order of \(10^{-4}\) which is suitable for any iterative method.
5.3 Second Iterative Method

This method consists of transforming the original system to another system like in the first iterative method. The difference between the two methods will be evident once the following steps are explained.

Let us consider the transformation of the 3 by 3 matrix similar to the first iteration method.

\[
\begin{bmatrix}
\bar{a}_{11} & \bar{a}_{12} & \bar{a}_{13} \\
\bar{a}_{21} & \bar{a}_{22} & \bar{a}_{23} \\
\bar{a}_{31} & \bar{a}_{32} & \bar{a}_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
\bar{b}_1 \\
\bar{b}_2 \\
\bar{b}_3
\end{bmatrix}
\]

As we know \( \sum_{j=1}^{n} \bar{a}_{ij} = 1 \).

From this relation, the first iterative vector to satisfy the first equation will be \( x_1 = x_2 = x_3 = \bar{b}_1 \) and the first row will be

\[
\bar{b}_1 \sum_{j=1}^{3} \bar{a}_{1j} = \bar{b}_1
\]

and for the second equation, for \( i=2 \),

\[
\bar{b}_1 \sum_{j=1}^{3} \bar{a}_{2j} \neq \bar{b}_2
\]

and the difference is;

\[
\Delta \bar{b}_2 = \bar{b}_1 \sum_{j=1}^{3} \bar{a}_{2j} - \bar{b}_2
\]

Thus, we need to correct the previous \( x_i^1 \) vector by the amount of \( \frac{\Delta \bar{b}_2}{\bar{a}_{22}} \) to the corresponding \( x_i \) (i.e \( i = 2 \)) then,
\[ x_1^{12} = \bar{b}_1 \]
\[ x_2^{12} = \bar{b}_1 - \Delta \bar{b}_2 \]
\[ x_3^{12} = \bar{b}_1 \]

Now if we apply this modified \( x_i^{12} \) to the second equation, this latter one will satisfy it exactly.

We pass to the third equation \( i = 3 \)

\[ \bar{a}_{31} + \bar{a}_{32}(\bar{b}_1 - \frac{\Delta \bar{b}_2}{\bar{a}_{22}}) + \bar{a}_{33} \bar{b}_1 \neq \bar{b}_3 \]  \hspace{1cm} (5.10)

\[ (\bar{a}_{31} + \bar{a}_{32} + \bar{a}_{33})\bar{b}_1 - \bar{a}_{32} \frac{\Delta \bar{b}_2}{\bar{a}_{22}} \neq \bar{b}_3 \]  \hspace{1cm} (5.11)

Rewriting equation (5.11) gives:

\[ \bar{b}_1 - \bar{a}_{32} \frac{\Delta \bar{b}_2}{\bar{a}_{22}} \neq \bar{b}_3 \]  \hspace{1cm} (5.12)

the difference \( \Delta \bar{b}_3 \) will be:

\[ \Delta \bar{b}_3 = (\bar{b}_1 - \bar{a}_{32} \frac{\Delta \bar{b}_2}{\bar{a}_{22}}) - \bar{b}_3 \]  \hspace{1cm} (5.13)

To correct the previous vector \( x_i^{12} \) in order to satisfy the third equation, we modify it as shown below by adding to the corresponding \( x_i^{12} \) the appropriate change;

\[ x_1^{13} = \bar{b}_1 \]
\[ x_2^{13} = \bar{b}_1 - \frac{\Delta \bar{b}_2}{\bar{a}_{22}} \]
\[ x_3^{13} = \bar{b}_1 - \frac{\Delta \bar{b}_2}{\bar{a}_{22}} \]

As we finish the first iteration, we go back again to the first equation
repeating the same procedure starting with the last vector.

The notation $x_{i}^{m}$ needs some clarification,

i: defines the position of the unknown.

k: defines the number of iterations.

m: defines the modified vector inside the $k^{th}$ iteration.

The difference between the two iterative methods is clear. In the first method we correct the previous vector, once we have completed all the iterations. In the second method the previous vector is changed internally, when passing from the $i^{th}$ equation to $(i + 1)^{th}$ equation by keeping the $(i + 1)^{th}$ equation satisfied.

5.4 First Direct Method

This direct method transforms the original matrix to a lower triangular matrix and solve the system directly by forward substitution. To transform the original matrix we need to make all the upper triangular elements equal to zero. To explain the new procedure, let us take a system of a 3 by 3 matrix and proceed with direct method of eliminations.

\[
\begin{bmatrix}
  a_{11} & a_{12} & a_{13} \\
  a_{21} & a_{22} & a_{23} \\
  a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
= 
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
\]

To apply this technique we have to start from the last column, and begin
the elimination of the elements. For this we need to create a vector $V$ in such way that it will not affect the previous elimination.

To proceed with the elimination we start with the element $a_{13}$;

$$V = \{v_1, v_2, v_3\}$$  \hspace{1cm} (5.14)

where, $v_1 = -\frac{a_{11} + a_{13}}{a_{13}}$, and $a_{13} \neq 0$, and $v_2 = v_3 = 1$.

Multiplying the system by the vector $V$.

$$\{V\}[A] = \{V\}[B]$$  \hspace{1cm} (5.15)

$$\{V\} = \bar{b}_1$$  \hspace{1cm} (5.16)

and the system becomes,

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & 0 \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ b_2 \\ b_3 \end{bmatrix}$$

To eliminate $a_{23}$

$$V = \{v_1, v_2, v_3\}$$  \hspace{1cm} (5.17)

where, $v_2 = -\frac{a_{21}}{a_{23}}$, and $a_{23} \neq 0$, and $v_1 = v_3 = 1$.

Again, multiplying the new system by the vector $V$ gives.

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} & 0 \\ \bar{a}_{21} & \bar{a}_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \bar{b}_1 \\ \bar{b}_2 \\ b_3 \end{bmatrix}$$

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To eliminate $\bar{a}_{12}$

$$V = \{v_1, v_2, v_3\}$$  \hspace{1cm} (5.18)

where, $v_{12} = -\frac{a_{22}}{\bar{a}_{12}}$, and $\bar{a}_{12} \neq 0$, and $v_2 = 1$, $v_3 = 0$.

Multiplying the new system by the vector $V$ gives:

$$\begin{bmatrix}
    \bar{a}_{11} & 0 & 0 \\
    \bar{a}_{21} & \bar{a}_{22} & 0 \\
    a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    x_2 \\
    x_3
\end{bmatrix} =
\begin{bmatrix}
    \bar{y}_1 \\
    \bar{y}_2 \\
    \bar{y}_3
\end{bmatrix}$$

Once we end with this system, its solution is already known starting with the first row, and forward substitution will determine all the unknowns.

It should be noted that during the elimination, the elements forming the $V$ vector should follow certain rules. To eliminate the elements above any diagonal element $a_{ii}$ in the matrix, the elements forming the $V$ vector from the position $i+1$ to $n$ should be all equal to zero, whereas all remaining terms are equal to one except the element corresponding to the position we are searching to make it zero. The element corresponding to the position we are searching to make it zero is equal to the minus sum of all column elements of the diagonal term excluding the corresponding one, divided by the corresponding element.

For example if we have a 5 by 5 system and we are searching to make the element of the fourth column $a_{34}$ equal to zero, the elements of the $V$ vector will be equal to:

$$\{V\} = \begin{bmatrix} 1 & 1 & -\frac{(a_{14} + a_{24} + a_{44})}{a_{34}} & 1 & 0 \end{bmatrix}$$
5.5 Second Direct Method

This method consists of solving a system of equations using a new technique. The procedure is better described as the technique of change of variables. By changing the variables from $x$ to $y$ results in a diagonal system in terms of $y$. The solution of this system in entails back substitution in the transformed system to solve for $x$. Back substitution would ultimately yield the inverted matrix of the system. To illustrate the validity of the method and the various steps required, the following 3 by 3 matrix is taken as an example:

$$
\begin{bmatrix}
a_{11} & a_{12} & a_{13} \\
a_{21} & a_{22} & a_{23} \\
a_{31} & a_{32} & a_{33}
\end{bmatrix}
\begin{bmatrix}
x_1 \\ x_2 \\ x_3
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\ b_2 \\ b_3
\end{bmatrix}
$$

Starting with the first equation since we want all the elements equal to zero except the diagonal term, we put:

$$x_1 = y_1 - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 \quad (5.19)$$

and by making use of the first equation we will get:

$$a_{11}y_1 = b_1 \quad (5.20)$$

Let us move to the second equation and substitute the value of $x_1$.

$$a_{21}(y_1 - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3) + a_{22}x_2 + a_{23}x_3 \quad (5.21)$$
After collecting terms it becomes:

$$a_{21}y_1 + \left( -\frac{a_{12}}{a_{11}} + a_{22} \right)x_2 + \left( -\frac{a_{13}}{a_{11}} + a_{23} \right)x_3$$  \hspace{1cm} (5.22)

If we want to change the variable and have all the terms zero except the diagonal term, then:

$$x_2 = -\frac{a_{21}}{-\frac{a_{12}}{a_{11}} + a_{22}}y_1 + y_2 - \frac{-\frac{a_{13}}{a_{11}} + a_{23}}{-\frac{a_{12}}{a_{11}} + a_{22}}x_3$$  \hspace{1cm} (5.23)

If we replace the second equation we will get:

$$\left( -\frac{a_{12}}{a_{11}} + a_{22} \right)y_2 = b_2$$  \hspace{1cm} (5.24)

Proceeding to the third equation, we replace the value of $x_1$ and then the value of $x_2$. Once all the calculations are finished we find:

$$\bar{a}_{31}y_1 + \bar{a}_{32}y_2 + \bar{a}_{33}x_3$$  \hspace{1cm} (5.25)

To have all the terms zero except the diagonal term we put:

$$x_3 = \frac{\bar{a}_{31}}{\bar{a}_{33}}y_1 - \frac{\bar{a}_{32}}{\bar{a}_{33}}y_2 + y_3$$  \hspace{1cm} (5.26)

If we replace in the third equation we get:

$$\bar{a}_{33}y_3 = b_3$$  \hspace{1cm} (5.27)

where:

$$\bar{a}_{31} = a_{31} + \frac{(a_{31}a_{12} - a_{32})a_{21}}{-a_{12} + a_{22}a_{11}}$$  \hspace{1cm} (5.28)

$$\bar{a}_{32} = \frac{a_{32} - a_{31}a_{12}}{a_{11}}$$  \hspace{1cm} (5.29)

$$\bar{a}_{33} = \frac{a_{31}a_{12} - a_{32}a_{23}a_{11} - a_{13}}{a_{11}a_{22} - a_{12}a_{11}} + \frac{a_{31}a_{13} + a_{33}a_{11}}{a_{11}}$$  \hspace{1cm} (5.30)
Now we have the first system;

\[
\begin{bmatrix}
  a_{11}y_1 & 0 & 0 \\
  0 & (-\frac{a_{12}}{a_{11}} + a_{22})y_2 & 0 \\
  0 & 0 & \tilde{c}_{33}y_3
\end{bmatrix}
= \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}
\]

And the second system is:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  y_1 & c_{12}x_2 & c_{13}x_3 \\
  c_{21}y_1 & y_2 & c_{23}x_3 \\
  c_{31}y_1 & c_{32}y_2 & y_3
\end{bmatrix}
\]

where the c terms in the second system correspond to the constants of the equations defining the three equation for \(x_1, x_2, x_3\). From the first system we solve for \(y\) and then we back substitute in the second system to determine the \(x\) vector. We can remark now that the second system is in terms of \(x\) and \(y\). Now we pass to the final operation where we have to express the whole system in terms of \(y\) only. The first equation defining \(x_1\) is in term of \(x_2\) and \(x_3\). First we substitute the expression of \(x_2\) in the expression of \(x_1\), then we get:

\[
y_1 + c_{12}(c_{21}y_1 + y_2 + c_{23}x_3) + c_{13}x_3 \quad (5.31)
\]

\[
(c_{12}c_{21} + 1)y_1 + c_{12}y_2 + (c_{12}c_{23} + c_{13})x_3 \quad (5.32)
\]

And the general system would yield:

\[
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = \begin{bmatrix}
  (c_{12}c_{21} + 1)y_1 & c_{12}y_2 & (c_{12}c_{23} + c_{13})x_3 \\
  c_{21}y_1 & y_2 & c_{23}x_3 \\
  c_{31}y_1 & c_{32}y_2 & y_3
\end{bmatrix}
\]
By, substituting $x_3$ in $x_1$ and $x_2$ respectively, row 1 becomes:

$$(c_{12}c_{21} + 1)y_1 + c_{12}y_2 + (c_{12}c_{23} + c_{13})(c_{31}y_1 + c_{32}y_2 + y_3)$$  \hspace{1cm} (5.33)$$

and, after simplification it becomes:

$$[1 + c_{12}c_{21} + c_{31}(c_{12}c_{23} + c_{13})]y_1 + [c_{12} + c_{32}(c_{12}c_{23} + c_{13})]y_2 + (c_{12}c_{23} + c_{13})y_3$$  \hspace{1cm} (5.34)$$

Following the same procedure, row 2 becomes

$$c_{21}y_1 + y_2 + c_{23}(c_{31}y_1 + c_{32}y_2 + y_3)$$  \hspace{1cm} (5.35)$$

and, after collecting the terms we get:

$$(c_{21} + c_{23}c_{31})y_1 + (1 + c_{23}c_{32})y_2 + c_{23}y_3$$  \hspace{1cm} (5.36)$$

The system then becomes:

$$
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix} = 
\begin{bmatrix}
  [((c_{12}c_{21} + 1) + c_{31}(c_{12}c_{23} + c_{13})) [c_{12} + c_{32}(c_{12}c_{23} + c_{13})] (c_{12}c_{23} + c_{13})] & y_1 \\
  (c_{21} + c_{23}c_{31}) & (1 + c_{23}c_{32}) & c_{23} \\
  c_{31} & c_{32} & 1
\end{bmatrix}
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix}
$$

In explicit form this can be written as:

$$\{X\} = [C]\{Y\}$$  \hspace{1cm} (5.37)$$

Once all the terms are in terms of the $y$ variables, we can calculate the $y$ vector from the first system

$$
\begin{bmatrix}
  y_1 \\
  y_2 \\
  y_3
\end{bmatrix} = 
\begin{bmatrix}
  (a_{11})^{-1} & 0 & 0 \\
  0 & (-\frac{a_{12}}{a_{11}} + a_{22})^{-1} & 0 \\
  0 & 0 & (a_{33})^{-1}
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{bmatrix}
$$

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Writing in condensed form:

\[ \{Y\} = [D]\{B\} \]  

(5.38)

and, the x vector is equal to

\[ \{X\} = [C][D]\{B\} \]  

(5.39)

From this relation we conclude that the product of C by D is nothing but the inverse of the A matrix

\[ [A]^{-1} = [C][D] \]  

(5.40)

5.6 Numerical Application

The numerical application carried in appendix A shows the application of the four techniques using a 3 by 3 matrix. The two iterative techniques show that they converge rapidly compared to the well known Gauss-Seidel iterative technique. Using different values of the relaxation factor to improve the Gauss-Seidel technique, the best factor was between 1.60 and 1.80 with 56 iterations, when the Gauss-Seidel technique is applied alone 93 iterations are required. The choice of the relaxation factor to give an optimum solution is very difficult to get if not impossible. In comparison, the two techniques presented herein can give very accurate results with much less computation. An analysis conducted by Bencharif and Ng [61]
shows that the time and storage capacity improved significantly using the direct method of transformation. The comparison of the three techniques is summarized in Table 5.1. To reduce the time, knowing that 144 numerical applications are involved, the two iterative techniques are not convenient. The second direct method is used to carry the solutions of the linear and nonlinear equations of thick rectangular plates for the 144 applications.

<table>
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<tr>
<th>System of Equations</th>
<th>Methods</th>
<th>Computer Time (CPU)</th>
<th>Storage Capacity</th>
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<td>Gauss-Jordan</td>
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<tr>
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<tr>
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</tr>
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<td>12</td>
<td>135×135</td>
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<tr>
<td></td>
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<td>Present Technique</td>
<td>9</td>
<td>135×135</td>
</tr>
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<td>Gauss-Jordan</td>
<td>52</td>
<td>240×240</td>
</tr>
<tr>
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<td>LU decomposition</td>
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<td>2(240×240)</td>
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<td>32</td>
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</tr>
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<td>Present Technique</td>
<td>121</td>
<td>375×375</td>
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Chapter 6

Applications and Results

6.1 General

Using the three dimensional formulation as outlined in chapter III, two numerical examples are presented in this chapter. The two examples consist of the use of plate structures with two different boundary conditions. The first one considered is a fully clamped plate and the second one is a simply supported plate with immovable edges.

A computer program has been developed which is applicable to rectangular plates with two boundary conditions as illustrated in Figures (4.11) and (4.12). The objective of this chapter is to demonstrate the applicability of the method to several types of plates with different aspect ratios ranging from thin plates to thick plates. It should be emphasized here that
the finite difference technique developed in this thesis for the linear and nonlinear modelling of thick plates is, in general, applicable to thick plates of various plan forms and boundary conditions provided appropriate adjustments are made in the input data and the boundary conditions. For the variety of applications considered, the Poisson’s ratio is kept equal to \( \nu = 0.316 \). Three plate aspect ratios have been considered \( \frac{b}{a} = 1.0, 1.5, 2.0 \), as well for the thickness plate ratios \( \frac{h}{a} = 0.025, 0.05, 0.10 \). At the same time two numerical approaches were used to investigate the convergence of results for both the linear and nonlinear analysis of thick plates since no results for such plates are as yet available in the technical literature.

To investigate convergence, four mesh sizes have been used for both clamped and simply supported rectangular plates. In order to have greater confidence in the results obtained, the finite difference was applied to both thin plates and thick plates. Comparison were made with previous investigators for nonlinear deflection results of thin plates and linear deflection results of thick plates. Data for nonlinear thick plates are as yet unavailable and hence no comparison of results can be made at this time. The number of finite difference applications to the thick plate analysis with various parameters are shown in Table 6.1.

### 6.2 Convergence of the Results

To improve the convergence and the accuracy of the results, two numerical finite difference schemes were investigated namely forward and central finite
difference techniques, with four mesh sizes in parallel for each case. For
different ratio of $\frac{h}{2}$, the results of the dimensionless central deflection $W$ for
both linear and nonlinear analysis seem to be very satisfactory for all plates
investigated. Tables 6.2-6.13 show the variation of centre plate deflection
with the dimensionless load $Q = \frac{z_{mm}}{EI}$, the plate aspect ratios, and the mesh
sizes used in this investigation.

To investigate the convergence of the finite difference techniques, four mesh
sizes have been used for both clamped and simply supported rectangular
plates. The convergence as a result of increase in mesh size is summarized
in Table 6.14. We note that the convergence for the simply supported
plates is faster than that of the clamped plates. A maximum mesh size of
125 nodes was adopted yielding 375 simultaneous equations. Convergence
results from Table 6.14 indicate that by changing the mesh size from 80
nodes to 125 nodes the deviation in results does not exceed 4% for all plate
aspect ratios investigated which is very well accepted.
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<th>Equs</th>
<th>a/b</th>
<th>h/a</th>
<th>Schemes</th>
<th>B. Cond.</th>
<th>No. Applications</th>
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Table 6.2, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

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Table 6.3, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

Central Finite Difference $\Delta = 1.0$, $\mu = 0.316$, CC/CC.

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Table 6.4, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

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Table 6.5, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

Central Finite Difference $\frac{b}{a} = 1.5$, $\mu = 0.316$, CC/CC.
Table 6.6, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

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Table 6.7, Linear and Nonlinear $\alpha$ Values for the Dimensionless Central Deflection $W = \alpha h$.

Central Finite Difference $\frac{\partial}{\partial x} = 2.0$, $\mu = 0.316$, CC/CC.
Table 6.8: Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

Forward Finite Difference $\frac{a}{h} = 1.0$, $\mu = 0.316$, SS/SS.

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Table 6.9, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

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Table 6.10, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

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Table 6.12, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$

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Table 6.13, Linear and Nonlinear $\alpha$ Values For The Dimensionless Central Deflection $W = \alpha h$  

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Table 6.14, Convergence of The Linear Dimensionless Center Deflection Using Finite Difference Technique

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<th>Boundary Conditions</th>
<th>Clamped</th>
<th>Simply Supported</th>
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<tr>
<td>20 nodes</td>
<td>14.73% 16.56%</td>
<td>6.86% 0.81%</td>
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<tr>
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<td>18.22%</td>
<td>1.1%</td>
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<tr>
<td>45 nodes</td>
<td>7.16% 7.32%</td>
<td>3.24% 0.38%</td>
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<tr>
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<td>7.92%</td>
<td>0.4%</td>
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<tr>
<td>80 nodes</td>
<td>3.84% 3.71%</td>
<td>1.68% 0.20%</td>
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<tr>
<td></td>
<td>3.99%</td>
<td>0.18%</td>
</tr>
<tr>
<td>125 nodes</td>
<td>3.84% 3.71%</td>
<td>1.68% 0.20%</td>
</tr>
<tr>
<td></td>
<td>3.99%</td>
<td>0.18%</td>
</tr>
</tbody>
</table>

The errors due to finite difference techniques are a combination of discretisation errors and rounding off errors. The rounding off error is the error associated with the accuracy with which the numbers are manipulated in the computations. The discretisation error occurs irrespective of the accuracy of the numerical calculations, and is the result of approximating a continuum which has an infinite number of degrees of freedom with a model having a finite number of degrees of freedom. The discretisation error may tend to zero as the limit when the element size approaches zero. At that stage, the approximate solution is said to converge to the exact solution. To investigate the convergence and accuracy of the method, dimensionless central deflection of various plates were plotted against the number of nodes for the linear analysis in Figures 6.1-6.9 for clamped plates, and Figures
FIGURE 6.1, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.025$
Figure 6.2, Linear Convergence Characteristics of the Clamped Plate. $h/a = 0.050$
FIGURE 6.3, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. h/a=0.100
FIGURE 6.4, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.025$
FIGURE 6.5, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.050$
FIGURE 6.6, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. h/a=0.100
FIGURE 6.7, LINEAR CONVERGENCE CHARACTERISTICS OF
THE CLAMPED PLATE. $h/a=0.025$
FIGURE 6.8, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.050$
FIGURE 6.9, LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.100$
FIGURE 6.10, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.025$
FIGURE 6.11, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE: $h/a=0.050$
FIGURE 6.12, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.100$
FIGURE 6.13, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.025$
FIGURE 6.14, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.050$
FIGURE 6.15, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a = 0.100$
\( \frac{\alpha}{\alpha_{\text{central}}} \) VALUES FOR CENTRAL DEFLECTION

\[ \text{NUMBER OF NODES} \]

\[ \begin{align*}
\square &= \text{Forward Finite Difference} \ a/b = 2.0 \\
\circ &= \text{Central Finite Difference} \ a/b = 2.0
\end{align*} \]

**Figure 6.16, Linear Convergence Characteristics of the Simply Supported Plate, \( h/a = 0.025 \)**
FIGURE 6.17, LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.050$
Figure 6.18, Linear Convergence Characteristics of the Simply Supported Plate. $h/a = 0.100$
FIGURE 6.19, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CALMPED PLATE. $h/a=0.025$
FIGURE 6.20, NON–LINEAR CONVERGENCE CHARACTERISTICS OF THE CALMPED PLATE. $h/a=0.050$
FIGURE 6.21, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CALMPED PLATE. h/a=0.100
FIGURE 6.22, NON-LINEAR CONVERGENCE CHARACTERISTICS
OF THE CALMPED PLATE. $h/a=0.025$
FIGURE 6.23, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.050$
FIGURE 6.24, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CLAMPED PLATE. $h/a=0.100$
FIGURE 6.25, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CALMPED PLATE. $h/a=0.025$
FIGURE 6.26, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CALMPED PLATE. \( h/a = 0.050 \)
FIGURE 6.27, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE CALMPED PLATE. $h/a=0.100$
FIGURE 6.28, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.025$
**FIGURE 6.29, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. h/a=0.050**
Figure 6.30, Non-linear Convergence Characteristics of the Simply Supported Plate. \( h/a = 0.100 \)
FIGURE 6.31, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. h/a=0.025
FIGURE 6.32, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. h/a=0.050
FIGURE 6.33, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.100$
FIGURE 6.34, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.025$
FIGURE 6.35, NON LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. \( h/a = 0.050 \)
 FIGURE 6.36, NON-LINEAR CONVERGENCE CHARACTERISTICS OF THE SIMPLY SUPPORTED PLATE. $h/a=0.100$
6.10-6.18 for simply supported plates.

Figures 6.19-6.36 show the convergence of the dimensionless nonlinear central deflection versus the loading. For all the figures cited the solid line and the dash line represent the forward and central finite difference techniques respectively. The two techniques show a difference ranging from 0 to 4%.

6.3 Comparison of Results

In order to show that the finite difference technique adopted is a good numerical scheme for the deflection analysis of thick nonlinear plates where no data is available, nonlinear thick plates results were reduced to thin plates to the limit $\frac{h}{a} = 0.025$ and results are compared with nonlinear analysis results of thin plates where the Von-Karman equations are used. In addition linear thick plates results were also compared with other investigators whenever possible.

6.3.1 Linear Analysis

For the linear analysis of thick plates, comparison has been made for the case of the square plate. The results presented by two investigators Srinivas [31] and Rajabhai [38] show good agreement with the present analysis for the case of clamped thick plates as shown in Figure 6.37. In the case of simply supported square plates the present analysis is compared with four
investigators Srinivas [30], [31], Rajabhou [38], Reissner [20], MIF [43] and
good agreements are also found as shown in Figure 6.38.

6.3.2 Nonlinear Analysis

In Figure 6.39, the finite difference results presented here for clamped thin
square plates are compared with those by Way [1] using polynomials and
by Levy [3] using series solution. As can be observed from the diagram,
the agreement is good. The difference between the present analysis and
those of Way [1] and Levy [3] is more evident once the dimensionless cen-
tral deflection approaches the thickness of the plate. At this point, it is
well reported [54] that for thin plate theory using the Von-Karman equa-
tions, membrane forces resist virtually all the loading when the deflection is
greater than the thickness, and since the participation of the inplane forces
in the three dimensional analysis is higher than in the two dimensional
analysis, the three dimensional thick plate analysis would give the plate
more resistance to the external loading. Rectangular thin plate analysis
compared with Way [1], and Levy [4] for the plate aspect ratios 1.5 and
2.0 Figures 6.40-6.41 show that the two dimensional analysis is more rigid.
For square simply supported thin plates, the results are compared with
those of Chia [46]. Again, good agreements are obtained as shown in Fig-
ure 6.42. Large deflection results for thick rectangular plates for both the
simply supported and clamped boundary conditions based on the present
finite difference technique are presented in Figures 6.43-6.48. Even though
no nonlinear thick plate results are as yet available, based on convergence analysis and good comparison of results of the same techniques applied to large deflection of thin plate as well as the small deflection of thick plates, the author is confident that the large deflection results for thick rectangular plates as presented in Figures 6.43-6.48 are of acceptable engineering accuracy.
FIGURE 6.37, COMPARISON OF THE LINEAR DEFLECTIONS OF SQUARE CLAMPED THICK PLATES.
FIGURE 6.38, COMPARISON OF THE LINEAR DEFLECTIONS OF SQUARE SIMPLY SUPPORTED THICK PLATES.

Plate Ratio $h/a=0.05$
Plate Ratio $h/a=0.10$
- Srinivas [31]
- Rajabhau [38]
- Srinivas [30]
- Reissner [20]
- MIF [43]
- Present Solutions
FIGURE 6.39, COMPARISON OF THE NONLINEAR DEFLECTIONS OF SQUARE CLAMPED THIN PLATES. \( a/b = 1.0 \)
\[ \text{DIMENSIONLESS LOAD } q = \frac{q^* a^4}{E^* h^4} \]

\[ 0.0 \quad 50.0 \quad 100.0 \quad 150.0 \quad 200.0 \quad 250.0 \quad 300.0 \]

\[ 0.0 \quad 1.0 \quad 1.5 \quad 2.0 \]

\( \text{\square} = \text{Way [1] Polynomials } a/b=1.5 \)

\( \text{\bigcirc} = \text{Levy [4] Series Solution } a/b=1.5 \)

\( \text{\bigtriangleup} = \text{Present Solution } h/a=0.025, a/b=1.5 \)

FIGURE 6.40, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR CLAMPED THIN PLATES, \( a/b=1.5 \)
FIGURE 6.41, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR CLAMPED THIN PLATES. a/b=2.0

- Way [1] Polynomials a/b=2.0
- Present Solution h/a=0.025, a/b=2.0
FIGURE 6.42, COMPARISON OF THE NONLINEAR DEFLECTIONS OF SQUARE SIMPLU SUP. THIN PLATES. a/b=1.0
FIGURE 6.43, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR CLAMPED PLATES. $a/b=1.0$
FIGURE 6.44, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR CLAMPED PLATES. \( a/b = 1.5 \)
FIGURE 6.45, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR CLAMPED PLATES. $a/b=2.0$
FIGURE 6.46, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR SIMPLE SIMPLY SUPPORTED PLATES. a/b=1.0
FIGURE 6.47, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR SIMPLY SUPPORTED PLATES. a/b=1.5
FIGURE 6.48, COMPARISON OF THE NONLINEAR DEFLECTIONS OF RECTANGULAR SIMPLY SUPPORTED PLATES. $a/b = 2.0$
Chapter 7

Conclusions and Recommendations

7.1 Conclusions

Three dimensional nonlinear analysis of thick plates was formulated using finite difference techniques. To use the finite difference modelling, the plate was divided in 5 layers, with the top and bottom layers satisfying the thick plate equilibrium equations at the boundaries.

The thick plate analysis considered here is quite different from the Reissner's and Mindlin's theories. In this analysis the general three dimensional equations of equilibrium are involved, whereas in Reissner and Mindlin theories the vertical shear is considered, but the analysis conducted is a two
dimensional analysis. Two finite difference techniques were used, namely, forward finite difference and central finite difference to support the analysis. Both those techniques give excellent convergence pattern and acceptable results. It can be concluded that the finite difference technique as adopted here represents quite well the thick plate equilibrium equations. No results for similar nonlinear analysis of thick plates are available in the technical literature to confirm the validity of the solutions. However, for comparison purposes, the large deflections differential equations for thick plates were simplified to solve small deflection problem of thick plates and to analyse the large deflection problem of thin plates. Limited results are available by other researchers in the area of small deflection of thick plates and large analysis of thin plates. All comparisons of results indicate that the present finite difference modelling yields good agreement with other investigators. As a result of the present research, the following main conclusions can be drawn:

(1)- A three dimensional analysis of thick plates was formulated as three very complex coupled partial differential equations. These equations are solved by finite difference modelling for both small and large deflections.

(2)- By dividing the thickness of the plate into layers and by adopting a new finite difference modelling technique, the problem of small and large deflection of thick plates was successfully completed.

(3)- A computer program using forward and central finite difference techniques included in the appendix creates the general matrix, the linear and nonlinear force vectors from relatively simple input. Each node is identified with a function label. The submatrices are formed automatically in
the general matrix to save time and space.

(4)- Analytical nonlinear results compared with large deflection of thin plate theory show excellent agreement with earlier investigators.

(5)- Linear analysis of thick plates show excellent convergence and for square thick plates, results are in excellent agreement with previous results available in the technical literature.

(6)- For nonlinear analysis results of thick plates, no data is yet available in the literature and hence no comparison can be made. However, convergence and comparison of limited data seem to indicate the results are of acceptable accuracy for engineering purposes.

7.2 Recommendations

Further works can be recommended for research based on this study.

(1)- Application of improved finite difference methods involving more accurate approximations to the derivatives.

(2)- The function labels can be modified to make them applicable to orthotropic and nonhomogeneous materials.

(3)- Due to the absence of experimental sources to guarantee the accuracy and practicability of the results, an experimental analysis of such plates from the convergence and stability point of view should be useful.

(4)- The analysis could be extended for the analysis of pre- and postbuckling of thick plates, vibration, and thick plates with variable stiffness.

(5)- The computer programme developed here was mainly aimed to solve
particular problems. It is more general than needed for the problems solved here. Programme although used only for rectangular thick plate with simply supported and clamped boundary conditions, input can be adjusted to solve thick plates of other plan forms and other boundary conditions.

(6)- An analysis using finite element method in three dimensional analysis applied to a simple cantilever [64] can be extended to thick plates for the analysis of linear and nonlinear deflection. The boundary integral method might also be considered. Considering the same case and comparing numerical results on convergence, accuracy, would be of interest.
Bibliography


Appendix A

Numerical Examples

A.1 Examples

A.1.1 First iterative method

Let us consider the 3 by 3 system of equations:

\[ [A]{X} = {B} \] \hspace{1cm} (A.1)

\[
\begin{bmatrix}
20 & -16 & 2 \\
-16 & 24 & -8 \\
8 & -32 & 20
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} =
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

For the application of the iterative methods, this system need to be transformed.

177
First row: \(20 - 16 + 2 = 6\) \hspace{1cm} (A.2)

Second row: \(-16 + 24 - 8 = 0\) \hspace{1cm} (A.3)

The sum is zero, and by adding the first row to the second row will get:

\[4 + 8 - 6 = 6\] \hspace{1cm} (A.4)

Third row: \(-8 - 32 + 20 = -4\) \hspace{1cm} (A.5)

The transformed system will be:

\[
\begin{bmatrix}
\frac{20}{6} & \frac{-16}{6} & \frac{2}{6} \\
\frac{4}{6} & \frac{8}{6} & \frac{-6}{6} \\
\frac{-8}{-4} & \frac{-32}{-4} & \frac{20}{-4}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
\frac{1}{6} \\
\frac{2}{6} \\
\frac{1}{4}
\end{bmatrix}
\]

\[
\begin{bmatrix}
3.3333 & -2.66667 & 0.33333 \\
0.66667 & 1.33333 & -1.00000 \\
-2.00000 & 8.00000 & -5.00000
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
0.16667 \\
0.33333 \\
-0.25000
\end{bmatrix}
\]

\[\tilde{A}\{\tilde{X}\} = \{\tilde{B}\}\] \hspace{1cm} (A.6)

The initial guess is: \(X^0 = \tilde{B}\)

\[
\begin{bmatrix}
3.3333 & -2.66667 & 0.33333 \\
0.66667 & 1.33333 & -1.00000 \\
-2.00000 & 8.00000 & -5.00000
\end{bmatrix}
\begin{bmatrix}
0.16667 \\
0.33333 \\
-0.25000
\end{bmatrix}
=
\begin{bmatrix}
-0.41667 \\
0.80556 \\
3.58333
\end{bmatrix}
\]

The first iterative vector is:

178
\[
\begin{bmatrix}
x_1^1 \\
x_2^1 \\
x_3^1 \\
\end{bmatrix} = \begin{bmatrix}
(0.16667 - (-0.41667))/3.3333 \\
(0.33333 - (0.80556))/1.3333 \\
(-0.25000 - (3.58333))/ -5.0000 \\
\end{bmatrix} + \begin{bmatrix}
0.16667 \\
0.33333 \\
-0.25000 \\
\end{bmatrix} = \begin{bmatrix}
0.34167 \\
-0.02083 \\
0.51667 \\
\end{bmatrix}
\]

The second iterative vector is:
\[
\begin{bmatrix}
3.3333 & -2.66667 & 0.33333 \\
0.66667 & 1.33333 & -1.00000 \\
-2.00000 & 8.00000 & -5.00000 \\
\end{bmatrix} \begin{bmatrix}
0.34167 \\
-0.02083 \\
0.51667 \\
\end{bmatrix} = \begin{bmatrix}
1.36667 \\
-0.31667 \\
-3.43333 \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
x_1^2 \\
x_2^2 \\
x_3^2 \\
\end{bmatrix} = \begin{bmatrix}
(0.16667 - (1.36667))/3.3333 \\
(0.33333 - (-0.31667))/1.33333 \\
(-0.25000 - (-3.43333))/ -5.00000 \\
\end{bmatrix} + \begin{bmatrix}
0.34167 \\
-0.02083 \\
0.51667 \\
\end{bmatrix} = \begin{bmatrix}
-0.01833 \\
0.46667 \\
-0.12000 \\
\end{bmatrix}
\]

From this stage we see that \( \frac{B_1 + B_2}{2} \neq \bar{B} \) is not satisfied then we continue our iteration till this relation is verified. For this particular problem, the above relation is satisfied at the 58th iteration. And the X and B vectors are:

\[
\{X^{57}\} = \begin{bmatrix}
0.74170 \\
0.55517 \\
1.42091 \\
\end{bmatrix}, \quad \{B^{57}\} = \begin{bmatrix}
-1.13222 \\
0.85287 \\
3.64666 \\
\end{bmatrix}
\]

\[
\{X^{58}\} = \begin{bmatrix}
0.35204 \\
0.94483 \\
0.64159 \\
\end{bmatrix}, \quad \{B^{58}\} = \begin{bmatrix}
1.46555 \\
-0.18621 \\
-4.14666 \\
\end{bmatrix}
\]

Then, if
\[
\left\{ \frac{B^{57} + B^{58}}{2}\right\} = \begin{bmatrix} 0.16667 \\ 0.33333 \\ -0.25000 \end{bmatrix}
\]

So the solution vector is:

\[
\left\{ \frac{X^{57} + X^{58}}{2}\right\} = \begin{bmatrix} 0.54687 \\ 0.75000 \\ 1.03125 \end{bmatrix}
\]

### A.1.2 Second iterative method

As mentioned before, this method transforms the original system as shown in the first method. Then the transformed system is:

\[
[\bar{A}]{X} = \{\bar{B}\} \tag{A.7}
\]

The first initial guess is:

\[
\{X^{11}\} = \begin{bmatrix} 0.16667 \\ 0.16667 \\ 0.16667 \end{bmatrix}
\]

And this vector will satisfy the first equation.

\[
0.16667(3.33333 - 2.66667 + 0.33333) = 0.16667 \tag{A.8}
\]
and for the second equation,

\[ 0.16667(0.66667 + 1.33333 - 1.00000) \neq 0.33333 \]  \hspace{1cm} (A.9)

The vector does not satisfy it, then we correct this first vector,

\[
\{X^{12}\} = \begin{bmatrix}
0.16667 \\
0.16667 - \frac{0.16667 - 0.33333}{1.33333} \\
0.16667
\end{bmatrix} = \begin{bmatrix}
0.16667 \\
0.29167 \\
0.16667
\end{bmatrix}
\]

If we apply this vector, the second equation is satisfied.

Proceeding to the third equation,

\[ 0.16667(-2.00000) + 0.29167(8.00000) - 5.00000(0.16667) \neq -0.25000 \]  \hspace{1cm} (A.10)

as a result, we need to correct this vector to satisfy the third equation.

\[
\{X^{13}\} = \begin{bmatrix}
0.16667 \\
0.29167 \\
0.16667 - \frac{0.16667 + 0.33333}{-5.00000}
\end{bmatrix} = \begin{bmatrix}
0.16667 \\
0.29167 \\
0.45000
\end{bmatrix}
\]

At this stage the first iteration is finished and we have to go back to the first equation and repeat the same procedures using this last vector.

By using this new approach we need only 19 iteration to solve this particular example.

\[
\{X^{19}\} = \begin{bmatrix}
0.54687 \\
0.75000 \\
1.03125
\end{bmatrix}
\]
A.1.3 First direct method

Let us consider the same system,

\[
\begin{bmatrix}
20 & -16 & 2 \\
-16 & 24 & -8 \\
8 & -32 & 20
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

we want to transform this system to a lower triangular system and solve it by forward substitution.

First we eliminate the element above the diagonal element $a_{33}$. We start with $a_{13} = 2$. The corresponding vector to eliminate this element is:

\[
\{V\} = \begin{bmatrix}
-6.0 & 1.0 & 1.0
\end{bmatrix}
\]

Premultiplying the system by the vector $V$ will give a new equation.

\[-128x_1 + 88x_2 + 0x_3 = -4 \quad (A.11)\]

And the new system will look like:

\[
\begin{bmatrix}
-128 & 88 & 0 \\
-16 & 24 & -8 \\
8 & -32 & 20
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= 
\begin{bmatrix}
-4 \\
1 \\
1
\end{bmatrix}
\]

Proceeding to the element $a_{33} = -8$, the corresponding vector is:

\[
\{V\} = \begin{bmatrix}
1.0 & 2.5 & 1.0
\end{bmatrix}
\]

Premultiplying the system by the vector $V$ will give the new equation.

\[-160x_1 + 116x_2 + 0x_3 = -0.5 \quad (A.12)\]
resulting in the new system:

\[
\begin{bmatrix}
-128 & 88 & 0 \\
-160 & 116 & 0 \\
8 & -32 & 20
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
-4 \\
-0.5 \\
1
\end{bmatrix}
\]

Proceeding to the last term \( a_{12} = 88 \), and the corresponding vector is:

\[
\{V\} = \begin{bmatrix}
-\frac{116}{88} & 1.0 & 0.0
\end{bmatrix}
\]

Premultiplying the system by the vector \( V \) will give the new equation.

\[
8.72727x_1 + 0x_2 + 0x_3 = 4.77273
\]  (A.13)

And the final system is:

\[
\begin{bmatrix}
8.72727 & 0 & 0 \\
-160 & 116 & 0 \\
8 & -32 & 20
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
=
\begin{bmatrix}
4.77273 \\
-0.5 \\
1
\end{bmatrix}
\]

By forward substitution we determine the \( X \) vector.

\[
x_1 = \frac{4.77273}{8.72727} = 0.54688
\]  (A.14)

\[
x_2 = \frac{-0.5 - (-160 \times 0.54688)}{116} = 0.75001
\]  (A.15)

\[
x_3 = \frac{1.0 - (8 \times 0.54688) - (-32.0 \times 0.75001)}{20} = 1.03126
\]  (A.16)

which is the solution of the system.
A.1.4 Second direct method

Let us consider the same 3 by 3 system of equation:

\[
\begin{bmatrix}
20 & -16 & 2 \\
-16 & 24 & -8 \\
8 & -32 & 20
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}
= \begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

We start with the first unknown, we put \( x_1 \) equal to:

\[ x_1 = y_1 - \frac{-16}{20} x_2 - \frac{2}{20} x_3 \quad (A.17) \]

substituting it into the three equations, the first equation becomes:

\[ 20(y_1 - \frac{-16}{20} x_2 - \frac{2}{20} x_3) - 16x_2 + 2x_3 = 1 \quad (A.18) \]

\[ 20y_1 = 1 \quad (A.19) \]

the second equation becomes:

\[ -16(y_1 - \frac{-16}{20} x_2 - \frac{2}{20} x_3) + 24x_2 - 8x_3 = 1 \quad (A.20) \]

\[ -16y_1 + 11.2x_2 - 6.4x_3 = 1 \quad (A.21) \]

From this expression we put the value of \( x_2 \) equal to:

\[ x_2 = \frac{-16}{11.2} y_1 + y_2 - \frac{-6.4}{11.2} x_3 \quad (A.22) \]

substituting it in the previous equation will give:

\[ -16y_1 + 11.2(\frac{-16}{11.2} y_1 + y_2 - \frac{-6.4}{11.2} x_3 - 6.4x_3) = 1 \quad (A.23) \]
\[ 11.2y_2 = 1 \]  
(A.24)

and after substituting the value of \( x_1 \) and then \( x_2 \), the third equation becomes:

\[
8(y_1 - \frac{-16}{20} x_2 - \frac{2}{20} x_3) - 32x_2 + 20x_3 = 1
\]
(A.25)

\[
8y_1 - 25.6x_2 + 19.2x_3 = 1
\]
(A.26)

then,

\[
8y_1 - 25.6\left(-\frac{16}{11.2} y_1 + y_2 - \frac{6.4}{11.2} x_3\right) + 19.2x_3 = 1
\]
(A.27)

\[
-28.57143y_1 - 25.6y_2 + 4.57143x_3 = 1
\]
(A.28)

From this we put the value of \( x_3 \) equal to:

\[
x_3 = -\frac{-28.57143}{4.57143} y_1 - \frac{-25.6}{4.57143} y_2 + y_3
\]
(A.29)

After we substitute it in the previous equation we get:

\[
4.57143y_3 = 1
\]
(A.30)

we have now formed two systems of equations the first system from (1.59), (1.64), and (1.70),

\[
\begin{bmatrix}
20 & 0 & 0 \\
0 & 11.2 & 0 \\
0 & 0 & 5.37143
\end{bmatrix}
\begin{bmatrix}
y_1 \\
y_2 \\
y_3
\end{bmatrix}
= 
\begin{bmatrix}
1 \\
1 \\
1
\end{bmatrix}
\]

and the second system from (1.57), (1.62), and (1.69)

\[
x_1 = y_1 - \frac{-16}{20} x_2 - \frac{2}{20} x_3
\]
(A.31)

\[
x_2 = -\frac{-16}{11.2} y_1 + y_2 - \frac{-6.4}{11.2} x_3
\]
(A.32)

\[
x_3 = -\frac{-28.57143}{4.57143} y_1 - \frac{-25.6}{4.57143} y_2 + y_3
\]
(A.33)
By replacing the $x_2$ expression in the $x_1$ expression,

$$x_1 = y_1 - \frac{-16}{20}(-\frac{-16}{11.2} y_1 + y_2 - \frac{-6.4}{11.2} x_3) - \frac{2}{20} x_3 \quad (A.34)$$

$$x_1 = 2.14286 y_1 + 0.8 y_2 + 0.35714 x_3 \quad (A.35)$$

And, by replacing the $x_3$ expression in the $x_2$ expression and, with the new $x_1$ expression.

$$x_2 = 1.42857 y_1 + y_2 + 0.57143(6.25 y_1 + 5.6 y_2 + y_3) \quad (A.36)$$

$$x_2 = 5.00001 y_1 + 4.20001 y_2 + 0.57143 y_3 \quad (A.37)$$

Then

$$x_1 = 2.14286 y_1 + 0.8 y_2 + 0.35714(6.25 y_1 + 5.6 y_2 + y_3) \quad (A.38)$$

$$= 4.37499 y_1 + 2.79998 y_2 + 0.35714 y_3 \quad (A.39)$$

The system is now reduced to:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4.37499 & 2.79998 & 0.35714 \\ 5.00001 & 4.20001 & 0.57143 \\ 6.25 & 5.6 & 1.0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

By replacing the first system in the second system we get:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4.37499 & 2.79998 & 0.35714 \\ 5.00001 & 4.20001 & 0.57143 \\ 6.25 & 5.6 & 1.0 \end{bmatrix}^{-1} \begin{bmatrix} 20 & 0 & 0 \\ 0 & 11.2 & 0 \\ 0 & 0 & 5.37143 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

or we can write;

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4.37499 & 2.79998 & 0.35714 \\ 5.00001 & 4.20001 & 0.57143 \\ 6.25 & 5.6 & 1.0 \end{bmatrix} \begin{bmatrix} \frac{1}{20} & 0 & 0 \\ 0 & \frac{1}{11.2} & 0 \\ 0 & 0 & \frac{1}{5.37143} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

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Thus the final system is:

\[
\begin{bmatrix}
  x_1 \\ x_2 \\ x_3
\end{bmatrix} =
\begin{bmatrix}
  0.21875 & 0.25000 & 0.07812 \\
  0.25000 & 0.37500 & 0.12500 \\
  0.31250 & 0.50000 & 0.21875
\end{bmatrix}
\begin{bmatrix}
  1 \\
  1 \\
  1
\end{bmatrix}
\]

This system gives the exact solution of the problem the 3 by 3 matrix. The solution from this system is therefor:

\[
\begin{align*}
x_1 &= 0.54687 \\
x_2 &= 0.75000 \\
x_3 &= 1.03125
\end{align*}
\]
Appendix B

Program Elements
Two programs are written for the three dimensional finite difference techniques described in this thesis. The two programs are used to compute the linear and nonlinear deflection of clamped and simply supported thick plates.

The program DIFF9 FORTRAN considers forward finite difference. The program DIFF91 FORTRAN considers central finite difference.

For the two programs, the submatrices described in chapter IV are formulated automatically in the general matrix AA to save the storage.

Nine subroutines are involved in each program, namely:

SUBROUTINE FORDIF OR SUBROUTINE CENDIF
SUBROUTINE GENR
SUBROUTINE SCHEM
SUBROUTINE BOUNDA
SUBROUTINE SOLINV
SUBROUTINE MATMUL
SUBROUTINE FORCE
SUBROUTINE BOUNDF
SUBROUTINE NOLINE

In these programs, double precision arithmetic is used for all the calculations.
B.1 Description of Input Data

1-Free format card to define, the number of points to generate the scheme (N), the increment number (INC), the relaxation factor (GAMA), the number of iteration maximum (IMAX), the type of boundary condition used (KD).

2-Master Control Card (I5,F10.4,F5.3,F10.5,2F5.3,2I5,F10.5,I5)
One Card to define the number of nodes, the modulus of elasticity, Poisson's ratio, the principal length, the ratio a/b, the ratio h/a, number of mesh sizes in x,y directions, number of mesh sizes in z direction, the convergence ratio, number of loads used.

Columns 1-5: number of nodes in the plate [NOD].
6-15: Modulus of elasticity (E).
16-20: Poisson's ratio (P).
21-30: The principal length (XA).
31-35: The ratio a/b (AB).
36-40: The ratio h/a (AH).
41-45: Number of mesh sizes in the x and y direction (NTX).
46-50: Number of mesh sizes in z direction (NTZ).
51-60: Convergence ratio (DELTA).
61-65: Number of loads (NIL).

3-Element Data (F5.2,4I5,3I5).
One Card for each node in order.
Columns 1-5: Load factor [FF(NOD,1)].
FF(NOD,1)=1.0 loaded.
FF(NOD,1)=0.0 not loaded.
3-10 : Incrementation in the Y direction (IJKL(I,1)).
11-15 : Incrementation in the plan XZ (IJKL(I,2)).
16-20 : Incrementation in the plan YZ (IJKL(I,3)).
21-25 : Label of the functions related to each node (IJKL(I,4)).
Nodes relationship for the boundary layers [NP(13)].
26-30 : Row position of the node to generate the scheme (IM).
31-35 : Column position to generate the scheme (JM).
36-40 : To generate the appropriate boundary conditions,
ITEST=0 Clamped plate
ITEST=1 Simply Supported plate
For clamped plate the points beyond the edges are affected with a sign (+).
For simply supported plate the points beyond the edges are affected with a sign (-).
Appendix C

Input-Output

C.1 Input-Output: Clamped Plate 20 nodes, Using DIFF9

C.2 Input-Output: Simply supp. Plate 45 nodes, Using DIFF91
0.01126E+02 0.19216E+02 0.34417E+02 0.02962E+02

0.00000E+00 0.10000E+00

Delt = 0.00000E+00

Mesh Size on Z Axis = 0.00000E+00

Mesh Size on X Axis = 0.00000E+00

Rat10/H4 = 1.0000

Rat10/1/B = 1.0000

L0NTOMNAL LENGTH = 1.0000

POISSON'S RATIO = 0.316

Young's Modulus = 0.00000

Number of Nodes = 20

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

0 0 0 0 0 0 0 0 0 0

CALCULATION RESULTS FOR AN UNREMARK

TOP LAYER SCHEME GENERATED BY AN UNMARK

NUMERICAL INTEGRAL BEAM PLATE

FILE DIFFERENTIAL MONITOR SYSTEM

PAGE 0001
Appendix D

Computer Programmes

D.1 DIFF9 FORTRAN

D.2 DIFF91 FORTRAN
SUBROUTINE DETERMINING THE FINE

*****************************************************************************

END

STOP

SUBROUTINE MULTIPLICATION

*****************************************************************************

SUBROUTINE PROGRAM FOR CALCULATING MATRIX

*****************************************************************************

FILE: OFF. IFFA
DIMENSION VN(11,11)
IMPLICIT REAL(A-H,O-Z)
SUBROUTINE GENR(VN,IC,ICD)

*****************************************************************************
* DIFFERENCE SCHEME  FOR BIHARMONIC OPERATOR  *
*****************************************************************************
**DIFFERENTIAL EQUATION**

**SUBROUTINE DIFFERENTIAL EQUATION OPERATIONS**

**SUBROUTINE PROGRAM GENERATING THE FINITE ELEMENT**

---

END

RETURN

(\n\nI2 = N \n\nI2 = I2 - 1)

CONTINUE
CONVERSION IN THE GLOBAL MATRIX, M.M.

***************

***************

END
RETURN

B
CONTINUE

RETURN

RETURN

FILE: DIPGA FORMAIN

THE GROUNDS MONITOR SYSTEM

W/M/P CONVEYOR MONITOR SYSTEM
AA(12,1-1)=AA(12,1-1)+G2/(TX*TZ)  DIF16510
AA(12,1+1K)=AA(12,1+1K)-G2/(TX*TZ)  DIF16520
AA(12,1+1K-1)=AA(12,1+1K-1)-G2/(TX*TZ)  DIF16530
AA(12,11)=AA(12,11)+G2/(TY*TZ)  DIF16540
AA(12,11-1J)=AA(12,11-1J)+G2/(TY*TZ)  DIF16550
AA(12,11+1K-1J)=AA(12,11+1K-1J)+G2/(TY*TZ)  DIF16560

CCCCccc
AA(12,12-1)=AA(12,12-1)+2.0*G/(TX*TZ*HA)  DIF16570
AA(12,12)=AA(12,12)-2.0*G/(TX*TZ*HA)  DIF16580
AA(12,12-1J)=AA(12,12-1J)-2.0*G/(TY*TZ*HA)  DIF16590
AA(12,12)=AA(12,12)-2.0*G/(TY*TZ*HA)  DIF16600
AA(12,12-1K)=AA(12,12-1K)+G1/(TZ*TZ*HA))  DIF16610
AA(12,12)=AA(12,12)-2.0*G1/(TZ*TZ*HA))  DIF16620
AA(12,12-1K)=AA(12,12-1K)+G1/(TZ*TZ*HA))  DIF16630

GO TO 2
DIF16640
DIF16650
DIF16660
DIF16670
DIF16680
DIF16690
DIF16700
DIF16710
DIF16720
DIF16730
DIF16740
DIF16750
DIF16760
DIF16770
DIF16780
DIF16790
DIF16800
DIF16810
DIF16820
DIF16830
DIF16840
DIF16850
DIF16860
DIF16870
DIF16880
DIF16890
DIF16900
DIF16910
DIF16920
DIF16930
DIF16940
DIF16950
DIF16960
DIF16970
DIF16980
DIF16990
DIF17000
DIF17010
DIF17020
DIF17030
DIF17040
DIF17050
AA(I1,I1-I1)=AA(I1,I1-I1)-G22/(TX*TY)
AA(I1,I1-I1-1)=AA(I1,I1-I1-1)-G22/(TX*TY)
AA(I1,I1-I1)=AA(I1,I1-I1)+2.0*G11/TZ**2
AA(I1,I1)=AA(I1,I1)-2.0*G11/TZ**2
CC
AA(I1,I1-I1)=AA(I1,I1-I1)+2.0*G11/TZ**2
AA(I1,I1)=AA(I1,I1)-2.0*G11/TZ**2
C
IR2=ISIGN(1,JK)
C
AA(I1,I1-(1-IR2)*ABS(JK))=AA(I1,I1-(1-IR2)*ABS(JK))-(G3)/TZ**2
AA(I1,I1+IR2*ABS(JK))=AA(I1,I1+IR2*ABS(JK))-2.0*(G3)/TZ**2
C
CALL SCHE(NA,IM,JM,NOD,NODE,TEST,CP)
DO 67 II=1,13
NPA(II)=ABS(NP(II))
IF(NP(II).EQ.0) GO TO 67
IF(NP(II).NE.0) GO TO 68
68 AA(I2,NPA(II)+2*NOD)=AA(I2,NP(II)+2*NOD)+(NP(II)/NPA(II))*R(II)
67 CONTINUE
GO TO 2
C
CCC
CORNER POINT (B.L)
109 AA(I1,I1)=AA(I1,I1)-2.0*G11/TX**2
AA(I1+1,I1)=AA(I1+1,I1)+G11/TX**2
AA(I1,I1+1)=AA(I1,I1+1)-2.0*G11/TX**2
AA(I1+1,I1+1)=AA(I1+1,I1+1)+G11/TX**2
C
IR1=ISIGN(1,IK)
C
AA(I1,I1-(1-IR1)*ABS(IK))=AA(I1,I1-(1-IR1)*ABS(IK))-(G3)/TZ**2
AA(I1,I1+IR1*ABS(IK))=AA(I1,I1+IR1*ABS(IK))-2.0*(G3)/TZ**2
AA(I1,I1+(1-IR1)*ABS(IK))=AA(I1,I1+(1-IR1)*ABS(IK))+(G3)/TZ**2
AA(I1,I1)=AA(I1,I1)+G22/(TX*TY)
AA(I1+1,I1)=AA(I1+1,I1)+G22/(TX*TY)
AA(I1,I1+1)=AA(I1,I1+1)+G22/(TX*TY)
AA(I1+1,I1+1)=AA(I1+1,I1+1)+G22/(TX*TY)
C
IR2=ISIGN(1,JK)
C
AA(I1+1-I1-I1-1)*ABS(JK))=AA(I1+1-(1-IR2)*ABS(JK))-(G3)/TZ**2
AA(I1+1+IR2*ABS(JK))=AA(I1+1+IR2*ABS(JK))-2.0*(G3)/TZ**2
C
CALL SCHE(NA,IM,JM,NOD,TEST,CP)
DO 69 II=1,13
NPA(II)=ABS(NP(II))
DIF1820
DIF18210
DIF18220
DIF18230
DIF18240
DIF18250
DIF18260
DIF18270
DIF18280
DIF18290
DIF18300
DIF18310
DIF18320
DIF18330
DIF18340
DIF18350
DIF18360
DIF18370
DIF18380
DIF18390
DIF18400
DIF18410
DIF18420
DIF18430
DIF18440
DIF18450
DIF18460
DIF18470
DIF18480
DIF18490
DIF18500
DIF18510
DIF18520
DIF18530
DIF18540
DIF18550
DIF18560
DIF18570
DIF18580
DIF18590
DIF18600
DIF18610
DIF18620
DIF18630
DIF18640
DIF18650
DIF18660
DIF18670
DIF18680
DIF18690
DIF18700
FILE D89 FORMATTED (8.80000)  (M.A.D.)  (J.J.1)  (J.J.1)  (J.J.1)  (J.J.1)  (J.J.1)


CALL G209(NINC,KA)

IF(CARAME) CONTINUE

N(J)=J

DO 90 J=1,J

IF(G209(J)) PRINT 803

IF(G209(J)) PRINT 801

IF(G209(J)) PRINT 802

ENDDO

DO 220 J=1,J

IF(G209(J)) PRINT 802

ENDDO

ENDCC

ENDCC

ENDCC

ENDCC

ENDCC

ENDCC
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<th>Line 3</th>
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<td>(Zi+Xl+Z)/10=10</td>
</tr>
<tr>
<td>(Zi+Xl+Z)/10=10</td>
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**CC GENERAL POINTS NEAR A AXIS**

- 60 TO 70

- 90
C C C C C C C C C C C C C C C C C C C C C C C C

NOTES ON THE X AXIS NEAR THE CENTRE

GO TO 20

FILE: OFIPG1 FORTRAN

VM/SP CONVERSATIONAL MONITOR SYSTEM
GENERAL POINTS NEAR A AXIS (G)  

C CEC GO TO 30

C (XI,Z) = (((XI+1),(SI),1))

C CEC POINTS PARALLEL TO X AXIS NEAR A AXIS (G)

C CEC GO TO 30

C (XI,Z) = (((XI+1),(SI),1))

C CEC POINTS PARALLEL TO Y AXIS ON THE X AXIS

C CEC GO TO 30

C (XI,Z) = (((XI+1),(SI),1))
CC POINTS ON THE X AXES NEAR THE CENTRE (G.L)

00 00 30
00 00 20
00 00 10
00 00 00

CC POINTS ON THE Y AXES NEAR THE CENTRE (G.L)

00 00 30
00 00 20
00 00 10
00 00 00

GENERAL POINTS NEAR X & Y AXES (G.L)

00 00 30
00 00 20
00 00 10
00 00 00

FILE: DPI2G1
FORMATTED
Points on the Axes (b, t)  

GO TO 2
FILE: DIPFG
PERSON: SYSTEM