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ORDER IDEALS IN A $\mathcal{C}^*$-ALGEBRA

A thesis submitted

by

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to

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in the subject of

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ABSTRACT

Let $A$ be a $C^*$-algebra. Since the bidual of $A$ can be considered as a $W^*$-algebra, this enables us to prove the following duality theorems:

(i) There exists a bijection between the norm-closed 2-sided ideals of $A$ and the norm-closed invariant order ideals of $A$.

(ii) There exists a bijection between the norm-closed left ideals of $A$ and the norm-closed order ideals of $A$.

(iii) There exists an order inverting bijection between the norm-closed 2-sided ideals of $A$ and the weak*-closed invariant faces of $S(A)$, where $S(A)$ is the state space of $A$.

The object of the thesis is to verify the above observations and to give Størmer's solution to J. Dixmier's problem: if $N$ and $M$ are norm-closed 2-sided ideals of $A$, then $(N + M)^+ = N^+ + M^+$, where $N^+$ and $M^+$ denote the positive parts of $N$ and $M$ respectively.
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INTRODUCTION

Let $A$ be a C*-algebra. Since the bidual of $A$ can be considered as a $W^*$-algebra (§ 2.2), this enables us to employ $W^*$-algebra methods to study some properties of ideals of $A$ and, in particular, to answer in the affirmative the following question asked by J. Dixmier in [3, p.20]: If $M$ and $N$ are norm-closed 2-sided ideals of $A$, is it true that $(M + N)^+ = M^+ + N^+$, where $M^+$ and $N^+$ denote the positive parts of $M$ and $N$ respectively? The solution to this problem given here is due to E. Størmer [14].

Chapter I deals with general properties of positive linear functionals on a C*-algebra $A$ which we shall frequently apply throughout the thesis. It is well known that for each positive linear functional on $A$ there corresponds a cyclic representation of $A$ which is unique up to equivalence, and conversely. This is described in detail in § 1.2. In Chapter II, we define the four basic topologies on $B(H)$, the algebra of all continuous linear operators on the complex Hilbert space, and establish some useful properties of $W^*$-algebras.

In Chapter III, we deal with the structure of norm-closed ideals in a C*-algebra obtained by E. G. Effros in [5]. We show that there exists a bijection between the family of all norm-closed 2-sided ideals of $A$ and the family of all norm-closed invariant order ideals of $A$ [Theorem (3.2.5)].
similarly, there exists a bijection between the family of all
norm-closed left ideals of $A$ and the family of all norm-
closed order ideals of $A$ [Theorem (3.2.8)]. In §3.3 we
prove the existence and uniqueness of a polar decomposition
for a normal linear functional on a $W^*$-algebra. More
precisely, if $f$ is a normal linear functional on a
$W^*$-algebra $A$, then there exists a partial isometry $U$ in $A$
such that $f = U|f|$, where $|f| = U^*f$ is a positive linear
functional and $\|U^*f\| = \|f\|$. Using this polar decomposition,
we show in §3.4 that if $N$ is a weak*-closed order ideal in
the dual $A'$ of a $C^*$-algebra $A$, then $N = N^{\perp\perp}$, where
$N^{\perp} = \{T \in A': T \geq 0 \text{ and } f(T) = 0 \text{ if } T \in N\}$, and
$N^{\perp\perp} = (N^{\perp})^{\perp} = \{f \in A': f \geq 0 \text{ and } f(T) = 0 \text{ if } T \in N^{\perp}\}.
We apply the equality $N = N^{\perp\perp}$ in Chapter IV to prove the
following duality theorem: There exists an order-inverting
bijection between norm-closed 2-sided ideals of $A$ and weak*-closed
invariant faces of $S(A)$, where $S(A)$ denotes the
state space of $A$. We use this duality theorem to solve
J. Dixmier's problem which is given in Theorem (4.2.2).
CHAPTER I

REPRESENTATION OF A C*-ALGEBRA

§1.1. Positive linear functionals.

Let $A$ be an algebra over the field $C$ of complex numbers. A mapping $x \rightarrow x^*$ from $A$ into itself is called an involu- tion if (i) $(x^*)^* = x$, (ii) $(\lambda x)^* = \overline{\lambda} x^*$, (iii) $(x + y)^* = x^* + y^*$, (iv) $(xy)^* = y^* x^*$ for all $x, y$ in $A$ and for all $\lambda$ in $C$. An algebra with an involution defined on it is called a $*$-algebra. A Banach $*$-algebra is a Banach algebra with an involution, and a $C^*$-algebra is a Banach $*$-algebra $A$ such that $\|x\|^2 = \|x^* x\|$ for all $x \in A$.

Let $A$ be a $*$-algebra. An element $x$ of $A$ is called hermitian if $x^* = x$, and it is called unitary if $x^* x = x x^* = 1$. An element $x$ is called positive if $x = y^* y$ for some $y \in A$. The set of all hermitian elements of $A$ will be denoted by $A_h$, the set of all unitary elements by $A_u$, and the set of all positive elements by $A^+$.

Let $A$ be a $C^*$-algebra. For each $x \in A$, let $x_1 = \frac{1}{2} (x + x^*)$ and $x_2 = -\frac{i}{2} (x - x^*)$. Then $x_1, x_2$ are hermitian and $x = \frac{1}{2} (x_1 + i x_2)$. If $x$ is hermitian, then $x^2 = x x^*$ is positive and hence has a positive square root $(x^2)^{\frac{1}{2}}$ [3, p.12]. Let $x^+ = \frac{1}{2} ((x^2)^{\frac{1}{2}} + x)$, $x^- = \frac{1}{2} ((x^2)^{\frac{1}{2}} - x)$. 
Then $x^+, x^-$ are positive and $x = x^+ - x^-$. Moreover, $x^+x^- = 0$. For each hermitian $x$ with $\|x\| \leq 1$, let $u = x + i(1 - x^2)^{\frac{1}{2}}$. It is easy to verify that $uu^* = uu = 1$. Therefore $u$ is unitary and $x = \frac{1}{2}(u + u^*)$. From these considerations we see that $A$ is spanned by either $A_h$, $A_u$ or $A^+$.

Let $A'$ be the dual (conjugate) space of $A$. An element $f \in A'$ is called positive if $f(x^*x) \geq 0$ for all $x \in A$. For each $f \in A'$, define $f^*$ by $f^*(x) = \overline{f(x^*)}$ for all $x \in A$; $f^*$ belongs to $A'$ and is called the adjoint linear functional of $f$. If $f^* = f$, $f$ is called hermitian.

For an element $f \in A'$, let $f_1 = \frac{1}{2}(f + f^*)$ and $f_2 = -\frac{i}{2}(f - f^*)$. Then $f_1, f_2$ are hermitian and $f = f_1 + f_2$. Let $f, g \in A'$. If $f - g$ is positive, we say that $f$ majorizes $g$ and write $f \geq g$. A positive element $f$ of $A'$ is called a state of $A$ if $\|f\| = 1$.

**Theorem (1.1.1).** If $f$ is a positive linear functional on a $C^*$-algebra $A$, then for all $x, y \in A$, we have:

1. $|f(y^*x)|^2 \leq f(x^*x)f(y^*y)$;
2. $f(y^*x)^* = \overline{f(x^*y)}$;
3. $f(x^*) = \overline{f(x)}$;
4. $f$ is bounded and $|f(x)|^2 \leq \|f\|f(x^*x)$;
5. if $A$ has an identity element, then $\|f(x)\|^2 \leq f(1)f(x^*x)$.

**Proof.** [11, p.213] For all $x, y \in A$, and for all $\alpha, \beta \in \mathbb{C}$,

$$0 \leq f((\alpha x + \beta y)^*(\alpha x + \beta y))$$
\begin{align*}
(1) & \quad = |\alpha|^2 f(x^*x) + \alpha \beta f(y^*x) + \overline{\alpha} \beta f(x^*y) + |\beta|^2 f(y^*y).
\end{align*}

Since $f(x^*x) \geq 0$ and $f(y^*y) \geq 0$ are real, $\alpha \beta f(y^*x) + \overline{\alpha} \beta f(x^*y)$ is real for all $\alpha, \beta$. Take $\alpha = \beta = 1$, then $f(y^*x) + f(x^*y)$ is real and so $\text{Im}(f(y^*x)) = - \text{Im}(f(x^*y))$.

Take $\alpha = 1$, $\beta = i$, then $-i f(y^*x) + if(x^*y)$ is real and so $\text{Re}(f(y^*x)) = \text{Re}(f(x^*y))$. Therefore $f(y^*x) = \overline{f(x^*y)}$.

This proves (ii). To prove (i), let $\alpha$ be real and $\beta = f(y^*x)$, then by (1) we have

\begin{align*}
\alpha^2 f(x^*x) + |f(y^*x)|^2 + \alpha f(y^*x) f(x^*y) + |f(y^*x)|^2 f(y^*y) \\
= \alpha^2 f(x^*x) + 2 \alpha |f(y^*x)|^2 + |f(y^*x)|^2 f(y^*y) \geq 0.
\end{align*}

Since this holds for all real $\alpha$, the discriminant of the quadratic is negative or zero, whence $|f(y^*x)|^2 \leq f(x^*x)f(y^*y)$.

This proves (i).

(iv): [11, p.246] We prove that $f$ is bounded. For this it is sufficient to prove boundedness on $A_h$. Note that, if $h \in A_h$, then there exists two hermitian elements $h_+, h_-$ with non-negative spectra such that $h = h_+ - h_-$, $h_+ h_- = h_- h_+ = 0$ [11, p.243]. Let $h_a = h_+ + h_-$. It follows that $\|h\| = \|h_a\|$, $h \leq h_a$ and, since $f$ is positive, $|f(h)| \leq f(h_a)$. Therefore, if $f$ is not bounded on $A_h$, then there exists a sequence $h_n$ of elements of $A_h$ such that $h_n \geq 0$, $\|h_n\| = 1$, and $f(h_n) \geq 2^n$ for each $n$.

Define $h = \sum 2^{-n} h_n$. Then $h \in A_h$ and $h - \sum_{k=1}^n 2^{-k} h_k = \sum_{k=n+1}^\infty 2^{-k} h_k = u_n$, say. Then, by [11, Lemma (4.7.10) and Corollary (4.7.13)], $u_n \geq 0$. Therefore, $f(h) \geq \sum_{k=1}^n 2^{-k} f(h_k) \geq n$ for all $n$. This is impossible since $f(h)$ is a finite
number; hence \( f \) must be bounded. Now, let \( \{e_\lambda\} \) be an approximate identity in \( A \) consisting of hermitian elements [11, Theorem (4.8.14)]. Then, by (i), 
\[
|f(e_\lambda x)|^2 \leq f(e_\lambda^2) f(x^*x)
\]
\[
\leq \|f\| f(x^*x).
\]
Since \( f \) is bounded, \( \lim_{\lambda} f(e_\lambda x) = f(x) \).
Hence 
\[
|f(x)|^2 \leq \|f\| f(x^*x).
\]
This proves (iv). (iii) follows from (ii) using (iv) and the approximate identity in \( A \). If \( A \) has an identity element, put \( y = 1 \) in (i) to obtain (v).

The inequality (i) in Theorem (1.1.1) is called the Cauchy-Schwartz inequality.

**THEOREM (1.1.2).** Let \( A \) be a \( C^* \)-algebra with identity. A linear functional \( f \) on \( A \) is positive if and only if \( \|f\| = f(1) \).

**PROOF.** [10, pp.189-190; 11, p.247]. Let \( f \) be a positive linear functional on \( A \), and let \( x = x^* \) in \( A \) with 
\[
\|x\| < 1.
\]
Then by [3, Lemma (2.1.3)], \( 1 - x \) is of the form \( y^*y \) for some \( y \in A \). Hence 
\[
f(1 - x) = f(y^*y) \geq 0.
\]
Consequently, \( |f(1)| \geq f(x) \). Substituting here \(-x\) in place of \( x \), we have \( |f(1)| \geq -f(x) \). Hence \( |f(x)| \leq f(1) \). Next, let \( x \) be an arbitrary hermitian element and put 
\[
x_1 = (\|x\| + \varepsilon)^{-1} x, \quad \text{where} \quad \varepsilon > 0.
\]
It is clear that \( x_1 = x_1^* \) and \( \|x_1\| < 1 \), and so
\[
|f(x_1)| \leq f(1) \quad \text{or} \quad |f(x)| \leq f(1)(\|x\| + \varepsilon).
\]
Hence \( |f(x)| \leq f(1)\|x\| \) since \( \varepsilon \) is arbitrary.

Now suppose that \( x \) is an arbitrary element of \( A \). Then \( x^*x \) is a hermitian element; consequently,
(2) \[ f(x^*x) \leq f(1) \|x^*x\| \leq f(1) \|x\|^2. \]

But, as \( f \) is positive, by Theorem (1.1.1) (v) and (2), we have

\[ |f(x)|^2 \leq f(1) f(x^*x) \leq f(1) \|x\|^2, \]

and so \( |f(x)| \leq f(1) \|x\| \). Hence \( \|f\| \leq f(1) \). On the other hand, by the definition of \( \|f\| \), \( f(1) \leq \|f\| \). Hence \( \|f\| = f(1) \).

Conversely, suppose \( \|f\| = f(1) \). Without loss of generality, we may assume that \( f(1) = 1 \). Write \( f = f_1 + if_2 \) where \( f_1, f_2 \) are hermitian. As \( f(1), f_1(1) \) and \( f_2(1) \) are real, \( f_2(1) = 0 \). We now show that \( f_2 = 0 \). To do this, let \( x \) be an arbitrary hermitian element of \( A \), and set \( u = \lambda - ix \) where \( \lambda \) is an arbitrary real number. Then

(3) \[ \|u\|^2 = \|u^*u\| \leq \lambda^2 + \|x\|^2. \]

Also,

\[ |f(u)|^2 = |(f_1 + if_2)(\lambda + ix)|^2 = \lambda^2 + 2\lambda f_2(x) + f_2^2(x) + f_1^2(x). \]

Therefore, from (3) we obtain

\[ \|u\|^2 \leq |f(u)|^2 - 2\lambda f_2(x) - f_2^2(x) - f_1^2(x) + \|x\|^2 \leq |f(u)|^2 - 2\lambda f_2(x) + \|x\|^2 \leq \|u\|^2 - 2\lambda f_2(x) + \|x\|^2. \]

Hence \( 2\lambda f_2(x) \leq \|x\|^2 \). Since \( \lambda \) and \( x \) are arbitrary, \( f_2 = 0 \), and so \( f = f_1 \). Thus \( f \) is a hermitian linear functional. Next, we shall show that \( f \) is positive.
Suppose on the contrary that there exists an element $x$ of $A$ such that $f(x^*x) < 0$. We can assume that $0 \leq x^*x \leq 1$ so that $0 \leq 1 - x^*x \leq 1$, and hence $\|1 - x^*x\| \leq 1$. Then
\[
1 = f(1) = f(1 - x^*x + x^*x) = f(1 - x^*x) + f(x^*x)
\]
\[
< f(1 - x^*x) \leq \|1 - x^*x\| \leq 1 \text{ since } f(x^*x) < 0.
\]
This is a contradiction and hence $f$ is positive.

**Theorem (1.1.3).** For each positive linear functional on a C*-algebra $A$, we have:

(i) $|f(y^*xy)| \leq \|x\|f(y^*y)$ for all $x, y \in A$;

(ii) $\|f\| = \sup_{\|y\| \leq 1} f(y^*y)$.

**Proof.** [3, p.23] For each $y \in A$ define a mapping $F$ on $A$ by $F(x) = f(y^*xy)$ for all $x \in A$. Then $F$ is a continuous linear functional on $A$ and it is positive since for all $x \in A,$

\[
F(x^*x) = f(y^*(x^*x)y) = f((xy)^*(xy)) \geq 0.
\]

Hence by [3, Proposition (2.1.5)], $\|F\| = f(y^*y),$ and
\[
|f(y^*xy)| = |F(x)| \leq \|F\|\|x\| = f(y^*y)\|x\| \text{ for all } x, y \in A.
\]
This proves (i).

Next, we shall prove (ii). It follows from Theorem (1.1.1) (iv) that $|f(y)|^2 \leq \|f\|f(y^*y)$ for all $y \in A$. By taking the supremum over all $y \in A$ with $\|y\| \leq 1$ on both sides of this inequality, we obtain
\[
\|f\|^2 \leq \|f\|\sup_{\|y\| \leq 1} f(y^*y), \text{ hence } \|f\| \leq \sup_{\|y\| \leq 1} f(y^*y). \text{ On the other }
\]

\[
\|y\| \leq 1
\]
hand, for each \( y \in A \) with \( \|y\| \leq 1 \), we have
\[
|f(y^* y)| \leq \|f\| \|y^* y\| = \|f\| \|y\|^2 \leq \|f\|.
\]
Consequently, we have
\[
\|f\| = \sup \{ f(y^* y) : \|y\| \leq 1 \}.
\]

Let \( A \) be a Banach \(*\)-algebra and \( A_h \) be the set of all hermitian elements of \( A \). Then \( A_h \) is a real Banach space. Let \( f \) be a hermitian continuous linear functional on \( A \) and \( f|_{A_h} \) denote the restriction of \( f \) to \( A_h \).
Then the mapping \( f \mapsto g = f|_{A_h} \) is a continuous isomorphism between the set of all hermitian continuous linear functionals on \( A \) and the set of all real-valued continuous linear functionals on \( A_h \). In fact, we have \( \|f\| = \|g\| \)
([3], p.5).

**THEOREM (1.1.4).** Let \( A \) be a C*-algebra and \( g \) a hermitian continuous linear functional on \( A \). Then there exist two positive linear functionals \( f \) and \( f' \) on \( A \) such that \( g = f - f' \) and \( \|g\| = \|f\| + \|f'\| \); the positive linear functionals \( f \) and \( f' \) are uniquely determined by these properties.

**PROOF.** [6, pp.98-99] Let \( B \) be the set of all positive linear functionals on \( A \) with norm \( \leq 1 \). Then, by ([3], Proposition (2.5.5), p.37), \( B \) is \( \sigma(A', A) \)-compact (i.e., \( B \) is weak*-compact). Since \( \sigma(A_h', A_h) \) is the topology on \( A_h' \) induced by the topology \( \sigma(A', A) \) on \( A' \), by the remark above, \( B \) can be considered as a \( \sigma(A_h', A_h) \)-compact convex subset of the dual space \( A_h' \).
of \( A_h \). The polar \( B^0 \) of \( B \) in \( A_h \) is equal to the closed unit ball \( S \) of \( A_h \). In fact, let \( C(B) \) be the set of all continuous real-valued functions on \( B \). For each \( x \) in \( A_h \), define a continuous real-valued function \( F_x \) on \( B \) by \( F_x(f) = f(x) \) for all \( f \) in \( B \). Then, by ([3], Proposition (2.6.3), p.39), the mapping \( \varphi: x \rightarrow F_x \) is an isometric isomorphism of \( A_h \) into \( C(B) \). Hence \( S \) is isometrically isomorphic to the set

\[
\left\{ F_x \in C(B); \|F_x\| \leq 1 \right\} = \left\{ F_x \in C(B); \sup_{f \in B} |F_x(f)| \leq 1 \right\}
\]

\[
= \left\{ F_x \in C(B); \sup_{f \in B} |f(x)| \leq 1 \right\}.
\]

Now, by definition, \( B^0 = \left\{ x \in A_h; \sup_{f \in B} |f(x)| \leq 1 \right\} \).

Therefore \( B^0 = S \). The closed unit ball \( S' \) of \( A_h' \) is equal to the set \( \left\{ f' \in A_h'; \sup_{x \in S} |f'(x)| \leq 1 \right\} \), which is the polar of \( S \) in \( A_h' \), i.e., \( S' = S^0 \). Thus \( S' \) is the bipolar of \( B \) in \( A_h' \). By ([13], Corollary 1, p.36), \( S' \) is the \( \sigma(A_h', A_h) \)-closed absolutely convex envelope of \( B \). Since \( B \) is \( \sigma(A', A) \)-closed and convex then, by ([15], Theorem (3.4-F), p.132), \( S' \) is the absolutely convex envelope of \( B \). Therefore, by ([15], Theorem (3.4-E), p.132), each element of \( S' \) is of the form \( \alpha u - \beta v \) with \( u, v \) in \( B \) and \( \alpha, \beta \) positive scalars.
such that $\alpha + \beta \leq 1$ \cite[p.132]{15}. Let us assume that $\|g\| = 1$. Then the restriction $g_h$ of $g$ to $A_h$ is in $S'$. Thus $g_h = f - f'$ with $f = \alpha u$, $f' = \beta v$. Hence $g = f - f'$ where $f \geq 0$, $f' \geq 0$. We also have $1 = \|g\| \leq \|f\| + \|f'\| \leq \alpha + \beta \leq 1$. Hence $\|g\| = \|f\| + \|f'\|$. The uniqueness of such $f$, $f'$ follows from \cite[Theorem 1, p.97]{6}.

\S 1.2. Representations associated with positive linear functionals.

Let $H$ be a complex Hilbert space and $B(H)$ the Banach algebra of all bounded linear operators on $H$ into itself with the operator bounded norm. $B(H)$ is a $C^*$-algebra with the involution given by the adjoint.

A homomorphism $h$ from a $C^*$-algebra $A$ into a $C^*$-algebra is called a $*$-homomorphism if $h(x^*) = h(x)^*$ for all $x \in A$. A $*$-homomorphism of $A$ into $B(H)$ is called a representation of $A$ on $H$. Let $L: x \mapsto L(x)$ be a representation of a $C^*$-algebra $A$ on the Hilbert space $H$. Then $L$ is called a cyclic representation if there exists a vector $\varphi$ in $H$ such that the set $\{L(x)\varphi : x \in A\}$ is dense in $H$. The vector $\varphi$ is called a cyclic vector for $L$. A representation $L$ of $A$ on $H$ is said to be faithful if it is one-to-one.

Let $\varphi$ be a non-zero vector of $H$ and $L$ a representation of $A$ on $H$. For each $x \in A$, let $p(x) = \langle L(x)\varphi, \varphi \rangle$. Then $p$ is a positive linear...
functional on $A$ since

$$p(x^*x) = (L(x^*x)\varphi, \varphi)$$

$$= (L(x)^*L(x)\varphi, \varphi)$$

$$= (L(x)\varphi, L(x)\varphi) \geq 0$$

for all $x \in A$.

**Theorem (1.2.1).** Let $L$ be a representation of a $C^*$-algebra $A$ on a Hilbert space $H$. Then

$$\|L(x)\| \leq \|x\|$$

for all $x \in A$. If $L$ is a faithful representation, then $\|L(x)\| = \|x\|$ for all $x \in A$.

**Proof.** By [3, Proposition (1.3.7)], we have

$$\|L(x)\| \leq \|x\|$$

for all $x \in A$. Now if $L$ is faithful by [3, Proposition (1.8.1)], we have $\|L(x)\| = \|x\|$ for all $x \in A$.

**Theorem (1.2.2).** Let $L$ and $L'$ be two cyclic representations of $A$ on the Hilbert space $H$ and $H'$, and let $\varphi$, $\varphi'$ be cyclic vectors for $L$ and $L'$, respectively. For each $x \in A$, let

$$p(x) = (L(x)\varphi, \varphi), \quad p'(x) = (L'(x)\varphi', \varphi').$$

If $p(x) = p'(x)$ for all $x \in A$, then $L$ and $L'$ are equivalent, i.e., there exists an isometric isomorphism $U$ on $H$ onto $H'$ such that $UL(x) = L'(x)U$ for all $x \in A$. 
PROOF. [10, p. 242] Let \( u \) be the mapping of the set \( \{L(x) \varphi : x \in A\} \) into the set \( \{L'(x) \varphi' : x \in A\} \) defined as follows: for each \( x \in A \), let

\[
u(L(x) \varphi) = L'(x) \varphi'.\]

Thus, if we let \( L(x) \varphi = \xi \) and \( L'(x) \varphi' = \xi' \), we have \( u(\xi) = \xi' \). It is easy to see that \( \|u(\xi)\| = \|\xi\| \); in fact,

\[
\|\xi\|^2 = (\xi, \xi) = (L(x) \varphi, L(x) \varphi) \\
= (L(x \ast x) \varphi, \varphi) \\
= (L'(x) \varphi', L'(x) \varphi') \\
= (\xi', \xi') \\
= \|\xi'\|^2 = \|u(\xi)\|^2.
\]

Since the sets \( \{L(x) \varphi : x \in A\} \) and \( \{L'(x) \varphi' : x \in A\} \) are dense in \( H \) and \( H' \) respectively, \( u \) can be extended to an isometric isomorphism \( U \) of \( H \) onto \( H' \). Moreover, for each \( y \in A \), we have

\[
UL((y) \xi) = UL((y)(L(x) \varphi)) \\
= UL(yx) \varphi \\
= L'(yx) \varphi' \\
= L'(y)L'(x) \varphi' \\
= L'(y)U(\xi)
\]

which shows that \( UL(y) = L'(y)U \) for all \( y \in A \). Hence \( L \) and \( L' \) are equivalent and this completes the proof.
The theorem just proved shows that a cyclic representation $L$ of a C*-algebra $A$ is determined up to an equivalence by the positive linear functional $p(x) = (L(x)y, y)$. The question arises whether or not, for each positive linear functional $p$ on $A$, there exists a representation $L^p$ of $A$ such that $p(x) = (L^p(x)y, y)$ for all $x \in A$ and for some non-zero vector $y$ of $H$. The answer to this question is in the affirmative. We shall construct $L^p$ as follows [10, pp. 242-245]:

Let $\tilde{A}$ be the C*-algebra obtained from $A$ by the adjunction of an identity and $\tilde{p}$ the canonical extension of $p$ to $\tilde{A}$ [3, Propositions (1.3.8) and (2.1.4)]. For all $x, y \in \tilde{A}$, let

$$(x, y) = \tilde{p}(y^*x).$$

It is clear that $(x, y)$ is linear in $x$ and conjugate linear in $y$. Since $\tilde{p}$ is positive, $\tilde{p}(y^*x) = \overline{\tilde{p}(x^*y)}$ by Theorem (1.1.1); i.e., $(x, y) = (y, x)$. The set $N = \{x \in A: (x, x) = 0\}$ is a left ideal of $\tilde{A}$. Let $H' = \tilde{A}/N$ and, for each $x \in \tilde{A}$, let $x'$ be the coset in $H'$ determined by $x$. $H'$ with the inner product defined by $(x', y') = (x, y)$ is a pre-Hilbert space. Let $H$ be its completion; $H$ is a Hilbert space. Since $N$ is a left ideal of $\tilde{A}$, for each $x \in \tilde{A}$, we may define a linear operator $L^\tilde{p}(x)$ on $H'$ by setting $L^\tilde{p}(x)(y') = (xy)'$ for all $y' \in H'$. We shall show that $L^\tilde{p}(x)$ is bounded. By definition of the inner product $(, )$ on $H'$, we have
\[ \|\widetilde{L^p}(x)(y')\|^{2} = (\widetilde{L^p}(x)(y'), \widetilde{L^p}(x)(y')) = ((xy)', (xy)'), \]
\[ = \tilde{p}((xy)^{*}(xy)) \leq \tilde{p}(1) \| (xy)^{*}(xy) \| \]
\[ = \tilde{p}(1) \| xy \|^{2} \leq \tilde{p}(1) \| x \|^{2} \| y \|^{2} \]
\[ = \tilde{p}(1) \| x \|^{2} \| y' \|^{2} \leq \| x \|^{2} \| y' \|^{2}. \]

Thus \( \widetilde{L^p}(x) \) is bounded and

\[ \| \widetilde{L^p}(x) \| \leq \| x \|. \]

Consequently, this operator can be uniquely extended to a bounded operator on \( H \) having the same operator bound. We denote this extension by the same symbol \( \widetilde{L^p}(x) \).

Therefore, the inequality (3) also remains valid for the norm of the operator \( \widetilde{L^p}(x) \) on \( H \). The mapping \( x \to \widetilde{L^p}(x) \) is a representation of \( \tilde{A} \) on \( H \). In fact,

\[ \widetilde{L^p}(\alpha x + \beta y)(z') = ((\alpha x + \beta y)z')' = \alpha(xy)' + \beta(yz)', \]
\[ = \alpha \widetilde{L^p}(x)(z') + \beta \widetilde{L^p}(y)(z'). \]

It remains to prove that \( (\widetilde{L^p}(x))^{*} = \widetilde{L^p}(x^{*}) \), i.e.,

that \( (\widetilde{L^p}(x)y', z') = (y', \widetilde{L^p}(x^{*})z') \) for all \( y', z' \in H \).

But

\[ (\widetilde{L^p}(x)y', z') = (y', \widetilde{L^p}(x^{*})z') = \tilde{p}(x^{*}xy) = \tilde{p}((x^{*}z)^{*}y) \]
\[ = (y, x^{*}z) = (y', (x^{*}z)'), \]
\[ = (y', \widetilde{L^p}(x^{*})z'). \]

Thus \( \widetilde{L^p} \) is a representation. Moreover, \( \widetilde{L^p} \) is cyclic.

To see this, let \( \psi = 1 + N \in H' \) and let \( x \) be any element of \( \tilde{A} \). Then \( \widetilde{L^p}(x)(\psi) = (x\psi)' = x' \). Consequently, the set of all vectors of the form \( \widetilde{L^p}(x)(\psi) \) for all \( x \in \tilde{A} \) coincides with the set \( H' \) of all equivalence classes.
Since \( H' \) is dense in \( H \), it follows that \( \psi \) is a cyclic vector and \( (L^P(x)\psi, \psi) = ((x)\psi, \psi) = (x, 1) = p(x) \). The restriction of \( L^P \) to \( A \) is clearly a representation of \( A \) on \( H' \); we denote this representation by \( L^P \). Since the set \( L^P(A) \) is the canonical image of \( A \) in \( H' \), by [3, Proposition (2.1.5)(vii)]\(^1\), \( L^P(A) \) is dense in \( H' \) and hence dense in \( H \). Thus \( \psi \) is a cyclic vector for \( L^P \) restricted to \( A \). Also, \( \|L^P(x)\| \leq \|x\| \) for all \( x \in A \). We have thus proved the following theorem:

**Theorem (1.2.3).** Let \( A \) be a C*-algebra. To every cyclic representation \( L \) of \( A \), with cyclic vector \( \psi \), there corresponds a positive functional \( p(x) = (L(x)\psi, \psi) \). The representation \( L \) is defined uniquely up to equivalence by the functional \( p \). Conversely, to every positive linear functional \( p \) on \( A \) there corresponds a cyclic representation \( L^P \) and a cyclic vector \( \psi \) such that \( p(x) = (L^P(x)\psi, \psi) \) for all \( x \in A \). Moreover, \( \|L^P(x)\| \leq \|x\| \) for all \( x \in A \).

Let \( \{H_\lambda \}_{\lambda \in \Lambda} \) be a family of Hilbert spaces and let \( H \) be the family of all functions \( (\xi_\lambda) \) defined on \( \Lambda \) such that:

(i) for each \( \lambda \in \Lambda \), \( \xi_\lambda \in H_\lambda \);

(ii) \( (\xi_\lambda) \) contains at most a countable number of elements which are different from zero;

(iii) \( \sum_\lambda \|\xi_\lambda\|^2 < \infty \).

We define addition, multiplication by scalars and inner product in \( H \) by the following formulae:
\[(\xi_\lambda) + (\eta_\lambda) = (\xi_\lambda + \eta_\lambda),\]
\[\alpha(\xi_\lambda) = (\alpha \xi_\lambda),\]
\[((\xi_\lambda), (\eta_\lambda)) = \sum_\lambda (\xi_\lambda, \eta_\lambda).\]

Then with these operations, \(H\) is a Hilbert space, called the Hilbert sum (or direct sum) of the Hilbert spaces \(H\) and we denote it by \(H = \bigoplus_{\lambda \in \Lambda} H_\lambda\).

Let \(A\) be a C*-algebra, \(\{H_\lambda\}_{\lambda \in \Lambda}\) a family of Hilbert spaces and \(\{L_\lambda\}_{\lambda \in \Lambda}\) a family of representations of \(A\) such that \(L_\lambda\) is a representation of \(A\) on \(H_\lambda\) for each \(\lambda \in \Lambda\). Let \(H\) be the Hilbert sum of \(H_\lambda\). Let \((\xi_\lambda) \in H\) and \(x \in A\). Then, since for each \(x \in A\) and each \(\lambda \in \Lambda\),

\[\|L_\lambda(x)\| \leq \|x\|,\]

it follows that \((L_\lambda(x)\xi_\lambda) \in H\); in fact

\[\|L_\lambda(x)\xi_\lambda\|^2 = ((L_\lambda(x)\xi_\lambda), (L_\lambda(x)\xi_\lambda)) = \sum_\lambda (L_\lambda(x)\xi_\lambda, L_\lambda(x)\xi_\lambda) = \sum_\lambda \|L_\lambda(x)\xi_\lambda\|^2 \leq \sum_\lambda \|L_\lambda(x)\|^2 \|\xi_\lambda\|^2 \leq \|x\|^2 \|\xi_\lambda\|^2.\]

Now, let \(L(x)\) be the operator on \(H\) defined by \(L(x)(\xi_\lambda) = (L_\lambda(x)\xi_\lambda)\). Then it is easy to see that \(L(x)\) is a bounded linear operator on \(H\) for every \(x \in A\), with \(\|L(x)\| \leq \|x\|\). Since each \(L_\lambda\) is a representation of \(A\) on \(H\), it is easily seen that \(L\) is a representation of \(A\) on \(H\) which we call the Hilbert sum (or direct sum) of the representations \(L_\lambda\) and denote it by \(L = \bigoplus_{\lambda \in \Lambda} L_\lambda\).

**Theorem (1.2.4).** Let \(Q\) be the set of all positive linear functionals defined on a C*-algebra \(A\). For each \(p \in Q\), let \(L^p\) be the corresponding representation of \(A\), \(H^p\) its underlying Hilbert space and \(\varphi^p\) the cyclic vector.
\( L = \bigoplus_{p \in Q} L^p \), the Hilbert sum of \( L^p \), and \( H = \bigoplus_{p \in Q} H^p \), the Hilbert sum of \( H^p \). Then the norm closure of \( \{L(x)H: x \in A\} \) is \( H \).

**Proof.** [3, p.43] Considering \( y^p \) as a vector of \( H \), we have

\[
\text{cl}(L(A)H) \supseteq \text{cl}(L(A)y^p) = H^p \quad (p \in Q)
\]

whence

\[
H \supseteq \text{cl}(L(A)H) \supseteq \bigoplus_{p \in Q} H^p = H,
\]

where \( \text{cl}(L(A)H) \) and \( \text{cl}(L(A)y^p) \) denote the norm closures of \( L(A)H \) in \( H \) and \( L(A)y^p \) in \( H^p \) respectively.

**Definition.** The mapping \( L \) defined in Theorem (1.2.4) is called the universal representation of \( A \), and \( H \) is called the underlying Hilbert space of \( L \).

**Remark:** By [3, Proposition (2.7.3)], \( \|L(x)\| = \|x\| \) for all \( x \in A \).
CHAPTER II

TOPOLOGIES ON $\mathcal{B}(H)$

§ 2.1. The dual space of $\mathcal{B}(H)$.

Let $H$ be a complex Hilbert space and $\mathcal{B}(H)$ the
$C^*$-algebra of all bounded linear operators on $H$ into
itself with the operator bounded norm. The topologies on
$\mathcal{B}(H)$ defined by the following semi-norms are called
respectively, the strong, weak, ultrastrong and ultraweak
topologies:

(i) $T \rightarrow \| T \xi \| \quad (T \in \mathcal{B}(H), \xi \in H)$  \hspace{1cm} \text{(strong)}

(ii) $T \rightarrow |(T \xi, \eta)| \quad (T \in \mathcal{B}(H), \xi, \eta \in H)$  \hspace{1cm} \text{(weak)}

(iii) $T \rightarrow \left( \sum_{i=1}^{\infty} \| T \xi_i \|^2 \right)^{\frac{1}{2}} \quad (T \in \mathcal{B}(H), \xi_i \in H,$

$\sum_{i=1}^{\infty} \| \xi_i \|^2 < \infty) \quad \text{(ultrastrong)}$

(iv) $T \rightarrow \left| \sum_{i=1}^{\infty} (T \xi_i, \eta_i) \right| \quad (T \in \mathcal{B}(H), \xi_i, \eta_i \in H,$

$\sum_{i=1}^{\infty} \| \xi_i \|^2 < \infty, \sum_{i=1}^{\infty} \| \eta_i \|^2 < \infty) \quad \text{(ultraweak)}$

These topologies are related as follows:

Norm topology $\triangleright$ ultrastrong topology $\triangleright$ strong topology
$\triangleright$$\triangleright$
ultraweak topology $\triangleright$ weak topology

where "$\triangleright$" means stronger than. For the proof of most of
these relations, see [10, pp.441-444].

Since, for all $\xi, \eta$ in $H$ and $T \in \mathcal{B}(H)$, we have
\[
4(T \xi, \eta) = (T(\xi + \eta), \xi + \eta) - (T(\xi - \eta), \xi - \eta) + i(T(\xi + i\eta), \xi + i\eta) - i(T(\xi - i\eta), \xi - i\eta),
\]
it follows that the weak and ultraweak topologies are also defined by the seminorms of the form $T \rightarrow |(T \xi, \xi)|$
$(\xi \in H)$ and $T \rightarrow \left| \sum_{i=1}^{\infty} (T \xi_i, \xi_i) \right|$ $(\xi_i \in H, \sum_{i=1}^{\infty} ||\xi_i||^2 < \infty)$ respectively.

The above topologies on $\mathcal{B}(H)$ are all compatible with the vector space structure of $\mathcal{B}(H)$, but not in general with the algebraic structure of $\mathcal{B}(H)$. In particular, $(S, T) \rightarrow ST$ is not always strongly continuous, however, if we denoted by $B_1(H)$ the unit ball of $\mathcal{B}(H)$, the equality
\[
ST - S_0T_0 = S(T - T_0) + (S - S_0)T_0
\]
shows that the mapping $(S, T) \rightarrow ST$ on $B_1(H) \times \mathcal{B}(H)$ into $\mathcal{B}(H)$ is strongly continuous. This mapping is also ultrastrongly continuous. The mappings $S \rightarrow ST, T \rightarrow ST$ on $\mathcal{B}(H)$ into itself are weakly, ultraweakly and ultrastrongly continuous. The mapping $T \rightarrow T^*$ is only weakly and ultraweakly continuous [7, Chapter 12].

**Theorem (2.1.1).** Strong (resp. weak) and ultrastrong (resp. ultraweak) topologies coincide on each bounded
subset $M$ of $B(H)$.

PROOF. [2, p.36] Without loss of generality, we may assume that $M = \{T \in B(H): \|T\| < 1\}$. Let $N$ be an ultrastrong neighbourhood of 0 in $M$. We may assume that

$$N = \{T \in M: \sum_{i=1}^{\infty} \langle T \xi_i, \xi_i \rangle < 1, \xi_i \in H, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty\}.$$

Since $\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty$, there exists a positive integer $n$ such that $\sum_{i=n}^{\infty} \|\xi_i\|^2 < \frac{1}{2}$. Let $N' = \{T \in M: \sum_{i=1}^{n-1} \|T \xi_i\|^2 < \frac{1}{2}\}$.

It is clear that $N'$ is a strong neighbourhood of 0 in $M$. For each $T \in N'$, we have

$$\sum_{i=1}^{\infty} \|T \xi_i\|^2 = \sum_{i=1}^{n-1} \|T \xi_i\|^2 + \sum_{i=n}^{\infty} \|T \xi_i\|^2 < \frac{1}{2} + \|T\| \sum_{i=n}^{\infty} \|\xi_i\|^2$$

$$< \frac{1}{2} + \frac{1}{2} = 1.$$

Thus $T \in N$ and $N' \subseteq N$. Hence $N$ is a strong neighbourhood of 0 in $M$ and hence the strong topology coincides with the ultrastrong topology on the bounded set $M$.

Next, we shall show that the ultraweak and weak topologies coincide on $M$. Let

$$N = \{T \in M: \sum_{i=1}^{\infty} \langle T \xi_i, \xi_i \rangle < 1, \sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty, \xi_i \in H\}$$

be an ultraweak neighbourhood of 0 in $M$. Since

$$\sum_{i=1}^{\infty} \|\xi_i\|^2 < \infty,$$

there exists an integer $m$ such that

$$\sum_{i=m}^{\infty} \|\xi_i\|^2 < \frac{1}{2}.$$

Let $N' = \{T \in M: \sum_{i=1}^{n} \langle T \xi_i, \xi_i \rangle < \frac{1}{2}\}$. 

Then \( N' \) is a weak neighbourhood of \( 0 \) in \( M \). For each \( T \in N' \), we have
\[
\left| \sum_{i=1}^{n} (T \xi_i, \xi_i) \right| \leq \left| \sum_{i=1}^{n} (T \xi_i, \xi_i) \right| + \left| \sum_{i=n+1}^{\infty} (T \xi_i, \xi_i) \right| < \frac{1}{2} + \|T\| \sum_{i=n+1}^{\infty} \|\xi_i\|^2 < \frac{1}{2} + \frac{1}{2} = 1.
\]

Hence \( T \in N \) and \( N' \subset N \). This shows that ultraweak topology coincides with the weak topology on \( M \).

**NOTATION:**

(i) For each pair of vectors \( \xi, \eta \) of \( H \) and each \( T \in B(H) \), let \( \omega_{\xi, \eta}(T) = (T \xi, \eta) \) and \( \omega_{\eta, \xi} = \omega_{\xi, \eta} \). It is clear that \( \omega_{\xi, \eta} \) is weakly continuous positive linear functional on \( B(H) \).

(ii) Let \( X' \) denote the dual (conjugate) space of a locally convex topological vector space \( X \).

(iii) For each \( \ast \)-subalgebra \( A \) of \( B(H) \), let \( A_\omega \) denote the set of all weakly continuous linear functionals on \( A \) and \( A_\ast \) denote the (norm) closure of \( A_\omega \) in \( A' \).

(iv) For each couple of vectors \( \xi, \eta \) of \( H \), let \( [\xi, \eta] \) denote linear mapping of rank one on \( H \) into itself given by \( [\xi, \eta]\gamma = (\gamma, \xi)\eta \) for all \( \gamma \in H \). Since \( \| [\xi, \eta]\gamma \| \leq \|\xi\|\|\eta\| \|\gamma\| \), \( [\xi, \eta] \) is continuous and hence it is completely continuous and \( [\xi, \eta]^\ast = [\eta, \xi] \).

(v) We shall write \( B \) for \( B(H) \) throughout this chapter.

**THEOREM (2.1.2).** Let \( f \) be a linear functional on \( B(H) \).

Then the following statements are equivalent:
(i) \( f = \sum_{i=1}^{\infty} \omega_i \varphi_i, \psi_i \) with \( \sum_{i=1}^{\infty} \| \varphi_i \|^2 < \infty \), \( \sum_{i=1}^{\infty} \| \psi_i \|^2 < \infty \).

(ii) \( f \) is ultraweakly continuous.

(iii) \( f \) is ultrastrongly continuous.

Moreover, if \( f \) is ultrastrongly continuous, then \( f \in B_x \).

**Proof.** [3, p.38] (i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii) is clear. We have only to prove that (iii) \( \Rightarrow \) (i). Suppose that \( f \) is ultrastrongly continuous. Since \( B(H) \) with the ultrastrong topology is a locally convex space, then there exists a ultrastrongly continuous seminorm \( p \) on \( B(H) \) such that \( |f(T)| \leq p(T) \) for all \( T \in B(H) \) [15, p.143]. That is, there exists a sequence \( \{ \varphi_i \} \) of vectors of \( H \) with \( \sum_{i=1}^{\infty} \| \varphi_i \|^2 < \infty \) such that \( |f(T)| \leq \left( \sum_{i=1}^{\infty} \| T \varphi_i \|^2 \right)^{\frac{1}{2}} \) for all \( T \in B(H) \). Let \( H_i = \bigcap_{i=1}^{\infty} H_i \) where \( H_i = H \) for all \( i \). For each \( T \in B(H) \), we define an operator \( \overline{T} \) on \( \overline{H} \) by \( \overline{T}(\overline{\varphi}) = (T \varphi_i) \) for all \( \overline{\varphi} = (\varphi_i) \) in \( \overline{H} \). It is clear that the set \( M = \{ \overline{T}(\overline{\varphi}) : T \in B(H) \} \) is a subspace of \( \overline{H} \). Define a linear functional \( \Theta \) on \( M \) by \( \Theta(\overline{T}(\overline{\varphi})) = f(T) \). Then as \( |\Theta(\overline{T}(\overline{\varphi}))| = |f(T)| \leq \left( \sum_{i=1}^{\infty} \| T \varphi_i \|^2 \right)^{\frac{1}{2}} \) \( = \| (T \varphi_i) \| = \| \overline{T}(\overline{\varphi}) \| \) for all \( \overline{\varphi} = (\varphi_i) \in \overline{H} \). \( \Theta \) is continuous on \( M \). Then by Hahn-Banach Theorem, \( \Theta \) can be extended to a continuous linear functional \( \Theta' \) on \( \overline{H} \). By Riesz representation theorem, there exists \( \overline{\eta} = (\eta_i) \in \overline{H} \):

\[
\int f(T) = \Theta(\overline{T}(\overline{\varphi})) = \langle \overline{T}(\overline{\varphi}), \overline{\eta} \rangle = \sum_{i=1}^{\infty} \langle T \varphi_i, \eta_i \rangle.
\]
Hence, \( f = \sum_{i=1}^{\infty} \omega_{\varphi_i}, \eta_i \). This proves (i).

Finally, suppose that \( f \) is ultrastrongly continuous, i.e., \( f = \sum_{i=1}^{\infty} \omega_{\varphi_i}, \eta_i \) and \( f_n = \sum_{i=1}^{n} \omega_{\varphi_i}, \eta_i \) where \( (\varphi_i) \) and \( (\eta_i) \) are sequences of vectors of \( H \) with \( \sum_{i=1}^{\infty} \|\varphi_i\|^{2} \) and \( \sum_{i=1}^{\infty} \|\eta_i\|^{2} < \infty \). Then \( f_n \) is weakly continuous since each \( \omega_{\varphi_i}, \eta_i \) is weakly continuous. Now,

\[
\| f - f_n \| = \sup_{\|T\| \leq 1} |f(T) - f_n(T)| = \sup_{\|T\| \leq 1} \left| \sum_{i=n+1}^{\infty} (T\varphi_i, \eta_i) \right|
\leq \sum_{i=n+1}^{\infty} \|\varphi_i\| \|\eta_i\| \leq \sum_{i=1}^{\infty} \|\varphi_i\| \|\eta_i\| \to 0 \text{ as } n \to \infty
\]

since \( \sum_{i=1}^{\infty} \|\varphi_i\|^{2} < \infty \) and \( \sum_{i=1}^{\infty} \|\eta_i\|^{2} < \infty \).

We observe that if \( X \) is a linear space and \( \mathcal{T}_1, \mathcal{T}_2 \) are two locally convex topologies on \( X \) such that \( (X, \mathcal{T}_1) \) and \( (X, \mathcal{T}_2) \) have the same continuous linear functionals, then a convex set is closed in \( (X, \mathcal{T}_1) \) iff it is closed in \( (X, \mathcal{T}_2) \) \([4, \text{ p.418}]\). It follows form Theorem (2.1.2) that

**COROLLARY (2.1.3).** The ultraweak and ultrastrong closure of a convex set in \( B(H) \) coincide.

**THEOREM (2.1.4).** Let \( \xi_i, \eta_i, \xi_j, \eta_j, i = 1,2, \ldots, m, j = 1,2, \ldots, n \) be vectors of \( H \) and \( g = \sum_{i=1}^{m} [\xi_i, \eta_i], g' = \sum_{j=1}^{n} [\xi'_j, \eta'_j] \). If \( g = g' \), then \( \sum_{i=1}^{m} \omega_{\xi_i}, \eta_i = \sum_{j=1}^{n} \omega_{\xi_j'}, \eta_j' \).
PROOF. [2, p. 39] For each couple of vector \( r, r' \) of \( H \), we have

\[
\sum_{i=1}^{m} ([r, r'] \xi_i, \eta_i) = \sum_{i=1}^{m} ((\xi_i, r') y_i, \eta_i) = \sum_{i=1}^{m} (y_i, (r, \xi_i) \eta_i)
\]

\[
= (y', \sum_{i=1}^{m} [\xi_i, \eta_i] y) = (y', \sum_{j=1}^{n} [\xi_j, \eta_j] y')
\]

\[
= \sum_{j=1}^{n} (y', (r, \xi_j') \eta_j) = \sum_{j=1}^{n} (r, \xi_j') \xi_j, \eta_j
\]

Let \( F \) be the set of all operators of finite rank on \( H \). Since the set of all operators which commutes with all elements of \( F \) is CI, where \( I \) is the identity operator in \( B(H) \) and \( C \) is the field of complex numbers, \( F \) is weakly dense in \( B(H) \) [2, p. 44]. By the weak continuity and linearity, we have

\[
\sum_{j=1}^{m} (T \xi_j, \eta_j) = \sum_{k=1}^{n} (T \xi_k', \eta_k')
\]

for all \( T \in B(H) \), i.e.,

\[
\sum_{j=1}^{m} \omega \xi_j, \eta_j = \sum_{k=1}^{n} \omega \xi_k', \eta_k'.
\]

THEOREM (2.1.5). Let \( f \) be a weakly continuous linear functional on \( B \). Then there exist two orthonormal systems

\( (\xi_1, \xi_2, \ldots, \xi_n) \) and \( (\eta_1', \eta_2', \ldots, \eta_n') \) and positive scalars \( \lambda_i \) \( (i = 1, 2, \ldots, n) \) such that

\[
f = \sum_{i=1}^{n} \lambda_i \xi_i, \xi_i' \quad \text{and} \quad \|f\| = \sum_{i=1}^{n} \lambda_i.
\]

PROOF. [3, p. 39] Since \( f \) is weakly continuous,

\[
f = \sum_{j=1}^{m} \omega \xi_j, \eta_j \quad [4, p. 42].\]

The mapping \( g = \sum_{j=1}^{m} [\xi_j, \eta_j] \)
is continuous, and is of finite rank, hence it is completely continuous. Then there exist two orthonormal systems \((e_i), (e'_i), i = 1, 2, \ldots, n\) and positive scalars \(\lambda_i\) such that

\[ g = \sum_{i=1}^{n} \lambda_i [e_i, e'_i] = \sum_{i=1}^{n} [\lambda_i e_i, e'_i] \]

[13, p.18]. By Theorem (2.1.4), we have

\[ f = \sum_{i=1}^{n} \omega_i [e_i, e'_i] = \sum_{i=1}^{n} \lambda_i \omega_i e_i, e'_i. \]

Thus,

\[ |f(T)| \leq \sum_{i=1}^{n} \lambda_i \|T\| \|e_i\| \|e'_i\| = \|T\| \sum_{i=1}^{n} \lambda_i. \]

Hence \(\|f\| \leq \sum_{i=1}^{n} \lambda_i\). Let \(T' = \sum_{i=1}^{n} [e_i, e'_i]\). Then \(\|T'\| \leq 1\).

In fact, let \((e_\alpha)_{\alpha \in \Delta}\) be an orthonormal basis of \(H\), which contains \((e_i), i = 1, 2, \ldots, n\). For each \(\xi\) in \(H\), write \(\xi = \sum_{k=1}^{\infty} (\xi, e_k)e_k\), then \(T' (\xi) = \sum_{i=1}^{n} (\xi, e_i)e'_i\).

\[ \|T' (\xi)\|^2 \leq \|\xi\|^2. \]

Therefore we have

\[ f(T') = \sum_{i=1}^{n} \lambda_i, \text{ i.e., } \|f\| \geq \sum_{i=1}^{n} \lambda_i. \]

**THEOREM (2.1.6).** Each element of \(B_\kappa\) is ultraweakly continuous.

**PROOF.** [2, p.40] Let \(f\) be an element of \(B_\kappa\) and let \(\{f_k\}\) be a sequence of weakly continuous linear functionals such that \(f = \sum_{k=1}^{\infty} f_k\) and \(\|f_k\| \leq 2^{-k}\). By Theorem (2.1.5),
\[
f_k = \sum_{k=1}^{n_k} \lambda_i^k \omega_i^k e_i^k \quad \text{with } \|e_i^k\| = \|e_i^1\| = 1, \quad \lambda_i^k > 0 \text{ and }
\]
\[
\sum_{i=1}^{n_k} \lambda_i^k = \|f_k\| \leq 2^{-k}. \quad \text{Hence } f = \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n_k} \lambda_i^k \omega_i^k e_i^k, (\lambda_i^k)^{1/2} e_i^1 \right),
\]
\[
\sum_{k=1}^{\infty} \left( \sum_{i=1}^{n_k} \lambda_i^k \right) \leq \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n_k} \lambda_i^k \right) \leq \sum_{k=1}^{\infty} 2^{-k} < \infty. \quad \text{Similarly } \sum_{k=1}^{\infty} \left( \sum_{i=1}^{n_k} \|\lambda_i^k\| \|e_i^k\|^2 \right) < \infty.
\]

Hence \( f \) is ultraweakly continuous.

COROLLARY (2.1.7). \( B_\Sigma \) coincides with the set of all ultraweakly continuous linear functionals on \( B(H) \).

PROOF. By Theorem (2.1.2) and Theorem (2.1.6).

THEOREM (2.1.8). A linear functional on \( B(H) \) is weakly continuous if and only if it is strongly continuous.

PROOF. \([4, \text{p.477}]\) Since the strong topology on \( B(H) \) is stronger than the weak topology on \( B(H) \), a weakly continuous linear functional on \( B(H) \) is strongly continuous.

Conversely, let \( f \) be a strongly continuous linear functional on \( B(H) \). Then there exists a finite subset \( \{\varphi_1, \varphi_2, \ldots, \varphi_n\} \) of \( H \) and an \( \varepsilon > 0 \) such that \( \|T \varphi_i\| < \varepsilon, \ T \in B(H), \ i = 1, 2, \ldots, n \) implies \( |f(T)| < 1 \).

Consider the Hilbert space \( H_n = H \otimes H \otimes \ldots \otimes H \) of \( n \)-tuples \( \psi = (\psi_1, \psi_2, \ldots, \psi_n) \) with \( \psi_i \in H, \ i = 1, 2, \ldots, n \); the norm in \( H_n \) is \( \|\psi\| = \max_{1 \leq i \leq n} \|\psi_i\| \). Define \( S: B(H) \to H_n \)

by \( T \mapsto (T \varphi_1, T \varphi_2, \ldots, T \varphi_n) \) for all \( T \in B(H) \). For each \( \varphi = (\varphi_1, \varphi_2, \ldots, \varphi_n) \), put \( F(\varphi) = f(T) \). Then \( F \) is
a functional on \( S(B(H)) \). Since \( \|\hat{\mathcal{F}}\| = \sup_i \|T\gamma_i\| < \delta \) implies \( |f(T)| < \delta \). Hence \( F \) is bounded and so continuous on \( S(B(H)) \).

By Hahn-Banach Theorem, \( F \) has a continuous extension \( F_1 \) defined on all of \( H_n \). Now each \( \psi \) in \( H_n \) can be uniquely in the form \( \psi = \psi_1 + \psi_2 + \cdots + \psi_n \) with \( \psi_i \in H \), \( i = 1, 2, \ldots, n \) and so

\[
F_1(\psi) = F_1(\psi_1, 0, \ldots, 0) + \cdots + F_1(0, \ldots, 0, \psi_n) = \gamma_1^*(\psi_1) + \cdots + \gamma_n^*(\psi_n)
\]

where \( \gamma_i^* = F_1|_{H_i}, \ i = 1, 2, \ldots, n \). By Riesz's representation theorem, there exist \( \xi_1, \xi_2, \ldots, \xi_n \) of \( H \) such that

\[
\psi_i^*(\psi_1) = (\psi_1, \xi_i), \ i = 1, 2, \ldots, n;
\]

consequently,

\[ f(T) = F_1(S(T)) \text{ has the form } f(T) = \sum_{i=1}^{n} \gamma_i^*(T\psi_i) = \sum_{i=1}^{n} \gamma_i^*(T\psi_i, \xi_i) = \sum_{i=1}^{n} \omega_{\psi_i, \xi_i}(T) \text{ and it is clear that } f \]

is weakly continuous.

**COROLLARY (2.1.9).** Let \( M \) be an ultraweakly closed subspace of \( B(H) \), and \( f \) a linear functional on \( M \). Then the following statements are equivalent:

(i) \( f \) is weakly continuous.

(ii) \( f \) is strongly continuous.

(iii) \( f = \sum_{i=1}^{n} \omega_{\psi_i, \xi_i} \gamma_i \) where \( \psi_i, \xi_i, \ i = 1, 2, \ldots, n \) are vectors of \( H \).

**COROLLARY (2.1.10).** The weak and strong closures of a convex set in \( B(H) \) coincide.
PROOF. This follows from Theorem (2.1.8) and the remark preceding Corollary (2.1.3).

**THEOREM (2.1.11).** $B(H)$ is the dual of $B_{\ast}$ (considered as a Banach space).

**PROOF.** [2, p.37] Let $E$ be the dual space of $B_{\ast}$. For each $T \in B(H)$, define $\hat{T}$ in $E$ by $\hat{T}(f) = f(T)$ for all $f \in B_{\ast}$.

Let $\Theta$ be a linear mapping on $B(H)$ into $E$ such that $\Theta(T) = \hat{T}$. Then $\Theta$ is a continuous linear function on $(B(H), \sigma(B(H), B_{\ast}))$ into $(E, \sigma(E, B_{\ast}))$. In fact, for each $T_0 \in B(H)$, the set

$$N = \{G \in E : |\hat{T}_0(f) - G(f)| < \varepsilon, f \in B_{\ast}, \varepsilon > 0\}$$

is a neighbourhood of $\Theta(T_0) = \hat{T}_0$ in $\sigma(E, B_{\ast})$. Let

$$N' = \{T \in B(H) : |f(T_0) - f(T)| < \varepsilon, f \in B_{\ast}\}.$$ 

Then $N'$ is a neighbourhood of $T_0$ in $\sigma(B(H), B_{\ast})$. Since $\hat{T}(f) = f(T)$, $\Theta(N') \subset N$ and so $\Theta$ is continuous. By Corollary (2.1.7), $\sigma(B(H), B_{\ast})$ is just the ultraweak topology on $B(H)$. If $\Theta(T) = \hat{T} = 0$ for some $T \in B(H)$, then $f(T) = 0$ for all $f \in B_{\ast}$. As $(B(H), \sigma(B(H), B_{\ast}))$ is a locally convex Hausdorff space, $T = 0$. Hence $\Theta$ is one-to-one.

Let $B_1$ and $E_1$ be the closed unit balls of $B(H)$ and $E$ respectively. Then $B_1$ is $\sigma(B(H), B_{\ast})$-compact [1, pp.65-66].

Since $\Theta$ is continuous and

$$\|\Theta(T)\| = \|\hat{T}\| = \sup_{\|f\| \leq 1} |\hat{T}(f)| = \sup_{\|f\| \leq 1} |f(T)| \leq \|T\|,$$

$\Theta(B_1)$ is $\sigma(E, B_{\ast})$-compact and contained in $E_1$. Moreover $\Theta(B_1)$ is dense in $E_1$. In fact, if it were not, there
would exist $e \in B_1$, $\varepsilon > 0$ and $f \in B_\infty$, $\|f\| = 1$, such that $|e(f) - \hat{T}(f)| > \varepsilon$ for all $T \in B_1$. Since $B_1$ is ultraweakly compact, there is $T_0 \in B_1$ with $|f(T_0)| = 1$. We may assume that $e(f) - f(T_0) > \varepsilon$ (otherwise consider $-f$). Hence $e(f) > \varepsilon + f(T_0) > 1$. But $|e(f)| \leq \|e\| \|f\| \leq 1$, a contradiction. Thus $\Phi(B_1) = E_\perp$ because $\Phi(B_1)$ is closed. Moreover, $\Phi$ is isometric; for if it were not, there would exist a $T \in B_1$ such that $\|\Phi(T)\| < \|T\|$. Hence $\Phi(\|\Phi(T)\|^{-1} T) \in E_\perp$ and $(\Phi(T))^{-1} T \notin B_1$. Therefore $\Phi$ is not one-to-one which is a contradiction. Thus $\Phi(B(H)) = E$, that is, $B(H)$ is isometrically isomorphic to $E$.

**THEOREM (2.1.12).** Let $A$ be a weakly closed $\ast$-subalgebra of $B(H)$. Then $A_\infty$ is the norm closure of $A_\omega$ and $A$ is the dual space of $A_\infty$.

**PROOF.** [2, p. 41] By Theorem (2.1.2), each element of $A_\infty$ is of the form $\sum_{i=1}^{\infty} \omega_{\alpha_i, \beta_i}$ with $\sum_{i=1}^{\infty} \|\alpha_i\|^2 < \infty$, $\sum_{i=1}^{\infty} \|\beta_i\|^2 < \infty$. Also, each element of $A_\omega$ is of the form $\sum_{i=1}^{\infty} \omega_{\gamma_i, \beta_i}$.

Hence $A_\infty$ is the norm closure of $A_\omega$. By Corollary (2.1.7), $\sigma(B(H), B_\infty)$-closed. By Theorem (2.1.11), $B(H)$ is the dual space of $B_\infty$. Hence $A$ is just the vector subspace of $B(H)$ orthogonal to the set $A^\perp = \{ f \in B_\infty : f(T) = 0 \text{ for all } T \in A\}$, that is, $A = (A^\perp)^\perp$ [15, p. 232]. The mapping $T \rightarrow \mathcal{T}$ where $T \in (B_\infty)^\perp = B(H)$ and $\mathcal{T}$ is defined by $\mathcal{T}(f + A) = T(f)$ ($f \in B_\infty$), is an isometric isomorphism of $(A^\perp)^\perp = A$ onto
$(B^*/A^*)'$. We write $A \cong (B^*/A^*)'$. Thus $A' \cong (B^*/A^*)\Pi$; this is to say $B^*/A^*$ is a subspace of $A'$. Since every nonzero $f \in B^*/A^* \subseteq A'$ is ultraweakly continuous on $A$ and $f(A) \neq 0$, we see that $f \in A_{\infty}$. On the other hand, for each $g \in A_{\infty}$, $g$ can be extended to an ultraweakly continuous linear functional on $B(H)$. Hence $B^*/A^* \cong A_{\infty}$.

**Theorem (2.1.3).** Let $M$ be an ultraweakly closed subspace of $B(H)$, and $K$ a convex subset of $M$, and for each real number $r > 0$, let $M_r = \{T \in M: \|T\| \leq r\}$. Then the following statements are equivalent:

(i) $K$ is ultraweakly closed.

(ii) $K$ is ultrastrongly closed.

(iii) $K \cap M_r$ is ultraweakly closed for each $r$.

(iv) $K \cap M_r$ is weakly closed for each $r$.

(v) $K \cap M_r$ is ultrastrongly closed for each $r$.

(vi) $K \cap M_r$ is strongly closed for each $r$.

**Proof.** [2, p.41] (i) $\iff$ (ii) and (iii) $\iff$ (v) follow from Corollary (2.1.3). (iii) $\iff$ (iv) and (v) $\iff$ (vi) follow from Theorem (2.1.1) and the remark preceding Corollary (2.1.3).

(i) $\iff$ (iii) Since $M$ is ultraweakly closed in $B(H)$, it is norm-closed in $B(H)$. Hence $M$ is a Banach space. By Theorem (2.1.12), $M$ is the conjugate space of the Banach space $M_{\infty}$. It is easy to see that $\sigma(M, M_{\infty})$ is equal to the ultraweak topology on $M$. But by Krein-Šmulian Theorem [4, p.429], $K$ is weak*-closed if and only if $K \cap M_r$ is
weak*-closed for every $r > 0$. This completes the proof.

**Theorem (2.1.14).** Let $M$ be an ultraweakly closed subspace of $B(H)$, $M_1$ the closed unit ball in $M$ and $f$ a linear functional on $M$. Then the following statements are equivalent:

(i) $f$ is ultraweakly continuous.

(ii) $f$ is ultrastrongly continuous.

(iii) $f = \sum_{i=1}^{\infty} \omega_{i,1}^\prime \eta_i$ with $\sum_{i=1}^{\infty} \|f_i\| < \infty$, $\sum_{i=1}^{\infty} \|\eta_i\| < \infty$,

$\eta_i, \gamma_i \in H$, $i = 1,2, \ldots$.

(iv) The restriction of $f$ to $M_1$ is ultraweakly continuous.

(v) The restriction of $f$ to $M_1$ is weakly continuous.

(vi) The restriction of $f$ to $M_1$ is ultrastrongly continuous.

(vii) The restriction of $f$ to $M_1$ is strongly continuous.

**Proof.** [2, pp.41-42] (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) and (iv) $\Leftrightarrow$ (vi) follow from Theorem (2.1.2). (iv) $\Leftrightarrow$ (v) and (vi) $\Leftrightarrow$ (viii) follow from Theorem (2.1.1). To prove (vi) $\Leftrightarrow$ (i), let $f$ be a linear functional on $M$ such that its restriction to $M_1$ is ultrastrongly continuous. Let $K$ be the hyperplane generated by $f(T) = 0$. Then $K$ is an ultrastrongly closed convex subset of $M$. Hence by Theorem (2.1.13)(iii), $K \cap M_1$ is ultraweakly closed and so $f$ is ultraweakly continuous. Since (ii) $\rightarrow$ (iv) is trivial, this completes the proof.
§ 2.2. Definition of a $W^*$-algebra.

Let $A$ be a $*$-subalgebra of $B(H)$. Then the weak, ultraweak, strong and ultrastrong closures of $A$ coincide \cite[p.43]{2}. A $W^*$-algebra (von Neumann algebra, or ring of operators) is a $*$-subalgebra of $B(H)$ with identity which is closed with respect to the weak topology (and hence closed with respect to the ultraweak, strong, and ultrastrong topologies) on $B(H)$. Note that every $W^*$-algebra is a $C^*$-algebra.

If $A$ is a subset of $B(H)$, the set $A^\circ$ consisting of all elements of $B(H)$ which commute with all elements of $A$ is called the commutant of $A$. $(A^\circ)^\circ = A^{cc}$ is called the bicommutant of $A$. It is clear that $A^{cc}$ is weakly closed and $A \subseteq A^{cc}$. $A$ is a $W^*$-algebra if and only if $A^{cc} = A$ \cite[p.448]{10}.

If $A$ is a $W^*$-algebra, the Banach space $A_*$ of ultraweakly continuous linear functionals on $A$ is called the predual of $A$. By virtue of Theorem (2.1.14), $A_*$ is also the Banach space of ultrastrongly continuous linear functionals on $A$. The elements of $A_*$ are called normal linear functionals.

§ 2.3. The bidual of a $C^*$-algebra.

**THEOREM (2.3.1).** Let $A$ be a $C^*$-algebra contained in $B(H)$, and let $\tilde{A}$ be the smallest $W^*$-algebra containing $A$.

Suppose that each continuous linear functional $f$ on $A$ is
ultrastrongly continuous and hence has a unique ultrastrong continuous extension \( \hat{f} \) to \( \hat{A} \). Then the following statements are true:

(i) The mapping \( f \to \hat{f} \) is an isometric isomorphism from the Banach space \( A' \) onto the predual \( \hat{A}_\pi \) of \( \hat{A} \).

(ii) For each \( T \in \hat{A} \), let \( \hat{T} \) be the linear functional \( f \to \hat{f}(T) \) on \( A' \). Then \( T \to \hat{T} \) is an isometric isomorphism on \( \hat{A} \) onto the bidual \( A'' \) of \( A \). The restriction of the mapping \( T \to \hat{T} \) to \( A \) is a canonical injection on \( A \) into \( A'' \).

**Proof.** [3, pp.235-236] (i): If \( f \in A' \), then by hypothesis, \( \hat{f} \in \hat{A}_\pi \), the predual of \( \hat{A} \). Suppose that \( A_1 \) and \( \hat{A}_1 \) are the closed unit balls of \( A \) and \( \hat{A} \), respectively. Then \( A_1 \) is ultrastrongly dense in \( \hat{A}_1 \) by Kaplansky Density Theorem [2, p.46] and, since the weak and ultrastrong closures of \( A \) coincide [2, p.43], it follows that \( \sup \{ |f(T)| : T \in A_1 \} = \sup \{ |\hat{f}(S)| : S \in \hat{A}_1 \} \), i.e., \( \|f\| = \|\hat{f}\| \). Thus \( f \to \hat{f} \) is an isometric isomorphism on \( A' \) into \( \hat{A}_\pi \). That it is onto \( \hat{A}_\pi \) follows from the fact that every element of \( \hat{A}_\pi \) is the continuous ultrastrong extension of its restriction to \( A \).

(ii): By (i) and Theorem (2.1.12), \( T \to \hat{T} \) is an isometric isomorphism on \( \hat{A} \) onto \( A'' \). If \( T \in A \), then clearly \( \hat{T} \) is the continuous linear functional \( f \to f(T) \) on \( A' \). Thus \( T \to \hat{T} \) maps \( A \) canonically into \( A'' \).

**Theorem (2.3.2).** If \( A \) is a \( W^* \)-algebra, then each element in the predual \( A_\pi \) of \( A \) is a linear combination of normal positive functionals on \( A \).
PROOF. [2, p. 54] Let $f$ be an element of $\mathbb{A}_\omega$. Then by Theorem (2.1.14),

$$f = \sum_{j=1}^{\infty} \omega_{\xi_j} \eta_j$$

for some sequences $(\xi_j)$, $(\eta_j)$ of vectors of $H$ with $\sum_{j=1}^{\infty} \| \xi_j \|^2 < \infty$, $\sum_{j=1}^{\infty} \| \eta_j \|^2 < \infty$.

Now, for all $T \in B(H)$, we have

$$4(T \xi_j, \eta_j) = (T(\xi_j + \eta_j), \xi_j + \eta_j) - (T(\xi_j - \eta_j), \xi_j - \eta_j)$$

$$+ i(T(\xi_j + i\eta_j), \xi_j + i\eta_j) - i(T(\xi_j - i\eta_j), \xi_j - i\eta_j).$$

Thus,

$$f = \frac{1}{4} \left( \sum_{j=1}^{\infty} \omega_{\xi_j + \eta_j} - \sum_{j=1}^{\infty} \omega_{\xi_j - \eta_j} + i \sum_{j=1}^{\infty} \omega_{\xi_j + i\eta_j} - i \sum_{j=1}^{\infty} \omega_{\xi_j - i\eta_j} \right).$$

As each functional of the form $\omega_{\xi}$ is positive, the theorem is proved.

THEOREM (2.3.3). Let $A$ be a $C^*$-algebra and $L$ the universal representation of $A$ and $\widetilde{A}$ the weak closure of $L(A)$, which is a $W^*$-algebra on the underlying Hilbert space $H$ of $L$. Then the following statements are true:

(i) Every positive normal functional on $\widetilde{A}$ is of the form $\omega_g (g \in H)$. Every ultraweakly continuous linear functional on $\widetilde{A}$ is weakly continuous.

(ii) If $f \in A'$, then there exists a unique weakly continuous linear functional $\widetilde{f}$ on $\widetilde{A}$ such that $\widetilde{f}(L(T)) = f(T)$ for all $T \in A$.

(iii) The mapping $f \mapsto \widetilde{f}$ is an isometric isomorphism of $A'$ onto the predual of $\widetilde{A}$ which transforms the set of all positive linear functionals on $A$ into the set of all normal
positive linear functionals on $\tilde{A}$. We have $\tilde{f}^* = \hat{f}^*$ for all $f \in A'$.

(iv) For each $S \in \tilde{A}$, let $\hat{S}$ be the linear functional $f \rightarrow \tilde{f}(S)$ on $A'$. Then the mapping $\gamma$ defined by $\gamma(S) = \hat{S}$ is an isometric isomorphism of $\tilde{A}$ onto $A''$ whose composition with $L$ (i.e., $\gamma \circ L$) is the canonical injection of $A$ into $A''$.

(v) This isomorphism $\gamma$ is bicontinuous for the weak (operator) topology on $A$ and the $\sigma(A'', A')$-topology on $A''$.

**Proof.** [3, pp. 236-237] For each $f \in A'$, define a functional $f'$ on $L(A)$ by

$$f'(L(T)) = f(T)$$

for all $T \in A$. Then $f'$ is linear and continuous since $f$ is linear and $|f'(L(T))| = |f(T)| \leqslant \|f\| \|T\| = \|f\| \|L(T)\|$ for all $T \in A$. Then $\|f'\| \leqslant \|f\|$. Conversely, since $|f(T)| = |f'(L(T))| \leqslant \|f'\| \|L(T)\| = \|f'\| \|T\|$, we have $\|f\| \leqslant \|f'\|$ whence $\|f\| = \|f'\|$.

Also, for each $f'$ in $L(A)'$, the relation (1) defines a continuous linear functional $f$ on $A$ such that $\|f\| = \|f'\|$. Therefore the mapping $\gamma$ defined by $\gamma(f) = f'$ is an isometric isomorphism of $A'$ onto $L(A)'$. Since $L(T)$ is positive if and only if $T$ is positive, it follows that $f'$ is positive if and only if $f$ is positive.

(i) Let $\tilde{f}$ be a normal positive linear functional on $\tilde{A}$. Then the restriction $f' = f|L(A)$ of $\tilde{f}$ to $L(A)$ is positive on $L(A)$. Hence $f = \gamma^{-1}(f')$ is positive on $A$. Let $L^f$ and $\varphi_f$ be the representation and vector defined by $f$. Then we have $f(T) = (L^f(T)\varphi_f, \varphi_f)$ for all $T$ in $A$.

Since $L = \bigoplus_{f \in Q} L^f$, where $Q$ is the set of all positive linear functionals on $A$, we have $f(T) = (L(T)\varphi, \varphi)$, where $\varphi = \varphi_f$ is considered as a vector in $H$. Thus
\( f'(L(T)) = f(T) = (L(T)\varphi, \varphi) \) for all \( T \) in \( A \). Since, by
Theorem (1.2.4) and [2, Corollary 1, p.44], \( L(A) \) is ultra-
weakly dense in \( \tilde{A} \), we have by the ultraweak continuity of \( f' \)
that \( f(S) = (S\varphi, \varphi) \) for all \( S \in \tilde{A} \), i.e., \( \tilde{f} = \omega_\varphi \). Hence \( \tilde{f} \)
is weakly continuous. Therefore it follows from Theorem
(2.3.2) that each ultraweakly continuous linear functional on
\( \tilde{A} \) is weakly continuous. This proves (i).

(ii). If \( h' \) is a positive functional on \( L(A) \), then
\( h = \varphi^{-1}(h') \) is positive on \( A \) and so \( h \in Q \), where \( Q \) is
the set of all positive linear functionals on \( A \). Therefore
\( h'(L(T)) = h(T) = (L(T)\varphi_h, \varphi_h) \). Hence \( h' \) is weakly
continuous on \( L(A) \). Since each \( g \) in \( L(A)' \) is of the
form \( g = g_1 + ig_2 \), where \( g_1, g_2 \) are hermitian, by Theorem
(1.1.7), \( g \) is a linear combination of positive functionals
on \( L(A) \) and therefore weakly continuous. It follows that
if \( f \in A' \), then \( f' = \varphi(f) \) is weakly continuous on \( L(A) \).
Hence \( f' \) can be extended to a unique weakly continuous
linear functional \( \tilde{f} \) on \( \tilde{A} \). Moreover, we have \( f(L(T)) =
f'(L(T)) = f(T) \) for all \( T \in A \). This proves (ii).

(iii). If \( f' \in L(A)' \), then \( f' \) is weakly continuous
and hence is ultraweakly continuous. Let \( \tilde{f} \) be the unique
extension of \( f' \) to \( \tilde{A} \) by weak continuity. By Theorem
(2.3.1), the mapping \( f' \rightarrow \tilde{f} \) is an isometric isomorphism
of \( L(A)' \) onto the predual of \( \tilde{A} \). Since the mapping \( f \rightarrow f' \)
is an isomorphism of \( A' \) onto \( L(A)' \), \( f \rightarrow \tilde{f} \) is an
isometric isomorphism of \( A' \) onto the predual of \( \tilde{A} \). If \( f \)
is positive, then, by the same argument as in (ii),
\( f' = \varphi(f) = \omega_\varphi \) on \( L(A) \). Hence by the weak continuity of \( f' \)
\( \hat{f} = \omega_{\hat{f}} \) on \( \hat{A} \). Therefore \( \hat{f} \) is positive and normal. If \( f \in A' \), then \( f^* \in A'^* \). Since \( (f^*)' = (f')^* \), \( f'^* \in L(A)' \).

We have \( (f'^*)(L(T)) = \overline{f^*(T) = \overline{f(T^*)} = \overline{f'(L(T^*))} = \hat{f}^*(L(T)) \)
for all \( T \in A \). Hence, by the weak continuity, \( \hat{f}^*(S) = f^*(S) \) for all \( S \in \hat{A} \). This proves (iii).

(iv) By (iii), \( \Psi: S \to \hat{S} \) is an isometric isomorphism of \( \hat{A} \) into \( A'' \). To see that \( \Psi \) is onto \( A'' \), let \( \varphi \in A'' \) and define a linear functional \( l_{\varphi} \) on \( \hat{A}^* \) by \( l_{\varphi}(\hat{f}) = \varphi(f) \). It is clear that \( l_{\varphi} \in (\hat{A}^*)' \) with \( \|l_{\varphi}\| = \|\varphi\| \). Now by [3, A23], \( l_{\varphi} \) can be uniquely and isometrically identified with an element \( \hat{S} \) of \( \hat{A} \) such that \( l_{\varphi}(\hat{f}) = \hat{f}(S) \) for all \( \hat{f} \in \hat{A}^* \). Hence \( \hat{S}(f) = \varphi(f) \) for all \( f \in A' \) and so \( \hat{S} = \varphi \), which shows that \( \Psi \) is onto \( A'' \). Since \( y \to L(y) \) is an isometric mapping of \( A \) onto \( L(A) \), it follows that \( \Psi \circ L \) is the canonical injection of \( A \) into \( A'' \).

(v) From (i) we know that the predual of \( \hat{A} \) is also the set of all weakly continuous linear functionals on \( \hat{A} \). By (iii), every weakly continuous linear functional on \( \hat{A} \) has the form \( \hat{f} \) with \( f \in A' \). Now a sub-base of 0-neighbourhoods \( N \) for the weak topology in \( \hat{A} \) is the family of sets of the form

\[
N = \{ S \in \hat{A}: |\hat{f}(S)| < \varepsilon, \varepsilon > 0, f \in A' \},
\]

and a sub-base of 0-neighbourhoods \( N' \) for the topology \( \sigma(A'', A') \) in \( A'' \) is the family of sets of the form

\[
N' = \{ S \in A'': |\hat{S}(f)| < \varepsilon, \varepsilon > 0, f \in A' \}.
\]

Since the isomorphism \( \Psi: S \to \hat{S} \) transforms the elements of \( N \) into the elements of \( N' \) and its inverse transforms the
elements of $N'$ into the elements of $N$, it follows that 
$\psi : S \rightarrow \hat{S}$ is bicontinuous.

REMARK:

By (iv), we can identify $A^*$ with $\hat{A}$. Since $\hat{A}$ is a
W$^*$-algebra, $A^*$ can be considered as a W$^*$-algebra. Hence
by (iii), the predual $(A^*)^*$ of the W$^*$-algebra $A^*$ can be
identified with the dual $A'$ of $A$. It follows from this
identification that the weak$^*$-topology $\sigma(A^*, A')$ on $A^*$
coincides with the ultraweak topology $\sigma(A^*, A'')$ on $A^*$. 
Consequently, by (v); these topologies coincide with the
weak (operator) topology on the W$^*$-algebra $A^*$. When $A$ is
regarded as imbeded in $A^*$, the restriction of $\sigma(A^*, A')$
to $A$ coincides with the weak topology $\sigma(A, A')$ on $A$.
In particular, if $K$ is a norm-closed convex subset of $A$,
then it is relatively closed in $A$ under the weak topology
$[4$, p.422], and hence relatively closed in $A$ under the
ultraweak topology. $A$ is weakly$^*$ dense in $A^*$ $[4$, p.425]
thus weakly dense in $A^*$. As the weak and strong closures
of a convex set in a W$^*$-algebra coincide, $A$ is strongly
dense in $A^*$. 
CHAPTER III

CLOSED ORDER IDEALS IN A C*-ALGEBRA AND ITS DUAL

§ 3.1. Definitions and basic concepts.

In what follows, for each closed subspace \( X \) of a complex Hilbert space \( H \), let \( P_X \) be the (orthogonal) projection of \( H \) onto \( X \), i.e., \( P_X(H) = X \). Let \( T \) be an element of \( B(H) \). It is clear that the kernel of \( T \), \( \ker T = \{ \xi \in H : T(\xi) = 0 \} \), is a closed subspace of \( H \). If \( X = (\ker T)^\perp \), the orthogonal complement of \( \ker T \), then the projection \( E = P_X \) is called the support of \( T \). It is easy to see that \( E \) is the smallest projection in \( B(H) \) such that \( TE = ET = T \) [10, p.445]. Let \( M \) be the closure of \( T(H) \) and let \( F = P_M \), then \( F \) is the support of \( T^* \) since \( (\ker T^*)^\perp = M \). Hence, if \( T \) is hermitian, then its support \( E_T \) is the projection on the closure of the range \( T(H) \) of \( T \). If \( T \) is an element of a C*-algebra \( A \), then \( E_T \) is the smallest projection in \( A^* \) (considered as a W*-algebra) such that \( TE_T = E_T T = T \).

Let \( p \) be a positive normal functional on a W*-algebra \( A \) and let \( N \) be the set of all projections \( G \in A \) such that \( p(G) = 0 \). Then, by [2, Proposition 3, p.61], \( N \) contains a projection \( F \) such that \( G \leq F \) for all \( G \in N \) and \( p(TF) = p(FT) = 0 \) for all \( T \in A \). Clearly \( F \) is the only such projection in \( N \). The projection \( E = I - F \) is called
the support of \( p \). It is easy to see that \( p(T) = p(ETE) \) for all \( T \in A \).

Let \( U \in B(H) \) and let \( E \) be the support of \( U \). Then \( U \) is called partially isometric if \( U \) is isometric on \( X = E(H) \); \( U(H) = U(X) \) is a closed vector subspace of \( H \). Let \( M = U(X) \) and \( F = P_M \). We called \( E \) (resp. \( X \)) the initial projection (resp. initial subspace) of \( U \), and \( F \) (resp. \( M \)) the final projection (resp. final subspace) of \( U \). \( U^* \) is also partially isometric, having initial projection \( F \) and final projection \( E \). We have \( U^*U = E \) and \( UU^* = F \).

It follows that an element \( V \in B(H) \) is a partial isometry iff \( V = VV^*V \) \([7, \text{pp. 62-63}]\).

Let \( T \in B(H) \), \( E \) the support of \( T \), \( F \) the support of \( T^* \), \( X = E(H) \), \( M = F(H) \), and \( |T| = (T^*T)^{\frac{1}{2}} \). Then \( |T| \) has the support \( E \) and the closure of \( |T| (H) \) is equal to \( X \). The linear mapping \( |T| \ell \rightarrow T \ell \) \((\ell \in H)\) is isometric from \( |T| (H) \) onto \( T(H) \) and therefore can be extended to an isometric linear operator \( V \) on \( X \) onto \( M \). Let \( U \) be a partially isometric operator on \( H \) with the support \( E \) such that \( U \) coincides with \( V \) on \( X \). Then \( T = U|T| \). This representation of \( T \) as the product of the unique partially isometry \( U \) and the positive operator \( |T| = (T^*T)^{\frac{1}{2}} \) is called the polar decomposition of \( T \). The equality \( T^* = U^*(U|T|U^*) \) is the polar decomposition of \( T^* \). We also have \( |T^*| = U|T|U^* \) and \( |T| = U^*|T^*|U \) \([7, \text{pp. 68-69}]\).

For each subset \( N \) of a \( C^* \)-algebra \( A \), let
$N^{1} = \{ f \in A^{*} : f \geq 0, f(T) = 0, T \in \mathbb{N} \}$. Then
$N^{1\perp} = (N^{1})^{\perp} = \{ T \subseteq A^{*} : T \geq 0, f(T) = 0, f \in N^{1} \}$. The set
$N^{0} = \{ f \in A^{*} : \text{Re} f(T) \leq 1, T \in \mathbb{N} \}$ is called the polar of $N$.

It is easy to see that if $N$ is balanced, and in particular a subspace of $A$, then $N^{0} = \{ f \in A^{*} : |f(T)| \leq 1, T \in \mathbb{N} \}$.

§ 3.2. Order ideals in a C*-algebra.

Let $A$ be a C*-algebra, $A_{h}$ the set of all hermitian elements of $A$ and $A^{+}$ the set of all positive elements of $A$. It is easy to see that $A_{h}$ is a partially ordered vector space with the partial order defined by the cone $A^{+}$. A subset $N$ of $A^{+}$ is called an order ideal of $A$ if $N + N \subseteq N$, $\lambda N \subseteq N$ for all $\lambda \geq 0$ and such that, if $S \subseteq N$ and $T \subseteq A$ with $0 \leq T \leq S$, then $T \subseteq N$.

Let $A$ be a C*-algebra. For each positive operator $T$ of $A$, let $(T)$ be the norm-closure of the smallest order ideal containing $T$, i.e., $(T)$ is the norm-closure of all positive operators $S$ of $A$ such that $S \leq \lambda T$ for some positive scalar $\lambda$. For each $S \subseteq A$ and $f \subseteq A^{*}$, let $Sf$ and $fS$ be defined by $(Sf)(T) = f(ST)$, $(fS)(T) = f(TS)$ for all $T \subseteq A$. If $A$ is a W*-algebra and if $E_{f}$ denotes the support of the positive normal functional $f$ on $A$, then $E_{f}f = fE_{f} = f$. A subspace $M$ of the dual $A'$ of $A$ is said to be left invariant if for all $f \in M$ and $S \subseteq A$, $Sf \subseteq M$.

THEOREM (3.2.1). Let $A$ be a C*-algebra and $T$ a positive element of $A$. Then $(T) = \{ T^{1\perp} \}$.

PROOF. [5, p. 393] It is clear that $\{ T^{1\perp} \}$ is a norm-closed order ideal containing $T$. Hence $(T) \subseteq \{ T^{1\perp} \}$. 
Conversely, let \( S \in \{T\}^{+1} \) and let \( E_T, E_S \) be supports of \( T \) and \( S \) respectively. We wish to show that \( E_S \leq E_T \). Suppose on the contrary that \( E_S > E_T \). Then there exists a vector \( \xi \in H \) such that \( E_T(\xi) = 0 \) and \( E_S(\xi) \neq 0 \). Since \( TE_T = T \) and \( T(\xi) = TE_T(\xi) = 0 \), \( \omega_T(T) = (T \xi, \xi) = 0 \).

Since \( \omega_T(\xi) \) is weakly continuous, it is continuous. Therefore \( \omega_T(\xi) \in \{T\}^{+1} \) but \( E_S(\xi) \neq 0 \) and \( \ker S = \ker E_S \); hence \( \omega_T(S) = (S \xi, \xi) \neq 0 \), which contradicts the fact that \( S \in \{T\}^{+1} \). Therefore \( E_S \leq E_T \). For each positive integer \( n \geq 1 \), define the function \( f_n \) on the reals by \( f_n(t) = 0 \) for \( t \leq 0 \), \( f_n(t) = nt \) for \( 0 \leq t \leq n^{-1} \) and \( f_n(t) = 1 \) for \( t \geq n^{-1} \). Then \( \{f_n(T)\} \) is an increasing sequence of positive operators with supremum \( E_T \). Hence \( \{f_n(T)\} \) converges to \( E_T \) strongly [2, p.331]. As the map \( (U, V) \rightarrow UV \) is strongly continuous on bounded sets [8 2.1], we have \( f_n(T)Sf_n(T) \rightarrow E_TSE_T \) strongly and hence weakly.

It follows from the inequality
\[
f_n(T)Sf_n(T) \leq \|S\|f_n(T)^2 \leq \|S\|f_n(T) \leq (\|S\|n) T,
\]
that the sequence \( \{f_n(T)Sf_n(T)\} \) lies in \( (T) \). Since \( (T) \) is convex and norm-closed in \( A \) and hence weakly closed [4, p.422], \( E_TSE_T \in (T) \). But \( S = ES = ESSE_S \leq E_TSE_T \in (T) \). Hence \( \{T\}^{+1} \subseteq (T) \).

**Corollary (3.2.2).** Let \( A \) be a C*-algebra and \( T \) a positive element of \( A \). Then the following statements are true:

(i) \( (T) \) is an order ideal.

(ii) If \( S \in (T) \), then \( S^{\frac{1}{2}} \in (T) \).
(iii) If N is a norm-closed order ideal of A and T ∈ N, then T^{1/2} ∈ N.

PROOF. [5, p.393] (i) is clear. To prove (ii), by Theorem (3.2.1), it suffices to show that S^{1/2} ∈ \{T\}^{1,1}. Let p be a positive element of A' with p(T) = 0. Then by Theorem (1.1.1)(iv) p(T^{1/2})^2 ≤ \|p\|p((T^{1/2})^*T^{1/2}) = \|p\|p(T) = 0. Hence p(T^{1/2}) = 0. Consequently, we have f(T^{1/2}) = 0 for all f ∈ \{T^{1/2}\}^{1,1}, i.e., T^{1/2} ∈ \{T\}^{1,1}. This proves (ii).

Finally, if N is a norm-closed order ideal in A such that (T) ⊆ N, then, by (ii), T^{1/2} ∈ (T) ⊆ N. This proves (iii).

LEMMA (3.2.3). Let S, T be positive elements of a W*-algebra A such that S ≤ T. Then there exists a unique element D ∈ A with the following properties:

(i) S^{1/2} = DT^{1/2}.

(ii) The support of D is majorized by the support of T.

PROOF. [2, pp.11-12] For any \( \zeta \in H \), we have

\[ \|S^{1/2} \zeta\|^2 = \langle S \zeta, \zeta \rangle ≤ \langle T \zeta, \zeta \rangle = \|T^{1/2} \zeta\|^2; \]

in particular, T^{1/2} \zeta = 0 implies S^{1/2} \zeta = 0. The mapping T^{1/2} \zeta → S^{1/2} \zeta is a continuous linear operator C which maps T^{1/2}(H) into H. Let D be the unique continuous extension of C to cl(T^{1/2}(H)) = cl(T(H)) into H, where cl(T^{1/2}(H)) and cl(T(H)) denote the closures of T^{1/2}(H) and T(H).

We have S^{1/2} = DT^{1/2}. Next, we observe that ker S ⊇ ker T. Hence the support of S is majorized by the support of T. Finally, let U be a unitary element in
$A^c$, the commutant of $A$ [§2.2] and let $UDU^{-1} = G$. Then

\[ GT^{\frac{1}{2}} = UDU^{-1}T^{\frac{1}{2}} = UDT^{\frac{1}{2}}U^{-1} = US^{\frac{1}{2}}U^{-1} = S^{\frac{1}{2}}U^{-1} = S^{\frac{1}{2}} \]

and hence by uniqueness of $D$, $G = D$. Thus, $UDU^{-1} = D$ which gives $D \in A$.

**Theorem (3.2.4).** If $N$ is a norm-closed left ideal of a $C^*$-algebra $A$, then the positive part $N^+$ is a norm-closed order ideal of $A$.

**Proof.** [5, p.394] It is clear that $N^+$ is norm-closed. Also, $N^+ + N^+ \subseteq N^+$ and $\lambda N^+ \subseteq N^+$ for all positive scalar $\lambda$. Suppose that $T \in N^+$ and $S \in A$ with $0 \leq S \leq T$. We shall show that $S \in N^+$. Since $0 \leq S \leq T$, by Lemma (3.2.3), there exists a unique element $D \in A^*$ (considered as a $W^*$-algebra) such that $S^{\frac{1}{2}} = DT^{\frac{1}{2}}$ and $\|D\| \leq 1$. Then $S = (S^\frac{1}{2})(S^\frac{1}{2})^* = (DT^\frac{1}{2})(DT^\frac{1}{2})^* = DTD^*$. Since $N \cap N^*$ is a 2-sided ideal of $A$, $DTD^* \in N \cap N^*$, i.e. $S \in N \cap N^*$. Since $S \geq 0$, $S \in N^+$. Hence $N^+$ is a norm-closed order ideal of $A$.

**Definition.** Let $A$ be a $C^*$-algebra. A subset $N$ of $A$ is said to be invariant if for all $T \in N$, $S \in A$, $STS^* \in N$.

**Theorem (3.2.5).** Let $A$ be a $C^*$-algebra. Then the map $N \rightarrow N^+$ is a bijection between the norm-closed 2-sided ideals of $A$ and the norm-closed invariant order ideals of $A$.

**Proof.** [5, p.396] If $N$ is a norm-closed 2-sided ideal of $A$, it is clear that $N^+$ is invariant.

Conversely, if $M$ is an invariant norm-closed order ideal of $A$. Let $N = (A^\prime\prime M) \cap A$. Since $A^\prime\prime$ is a $W^*$-algebra [see
the remark to Theorem (2.3.3)], $A^n$ contains the identity and therefore $M \subseteq N^+$. To prove the converse inclusion, let $S \subseteq A^n$ and $P \in M$ with $SP \in A$ and $SP \geq 0$. Then, as
\[
0 \leq (SP)^2 = (SP)^*(SP) = PS^*SP \leq \|S\|^2P^2 \leq (\|S\|^2 \|P\|)P,
\]
we have $(SP)^2 \in M$ since $M$ is an order ideal. Hence, by Corollary (3.2.3), $SP \in M$ and so $N^+ \subseteq M$. Thus $N^+ = M$.

We show next that $N$ is a norm-closed ideal of $A$. Clearly $AN \subseteq N$. To prove that $N$ is closed under addition, let $S, T \in N$. Then $S^*S$ and $T^*T$ are in $N^+ = M$. It follows from the inequality
\[
0 \leq (S + T)^*(S + T) \leq 2(S^*S + T^*T)
\]
that $(S + T)^*(S + T) \in M$ and thus, since $M$ is an order ideal, $P = ((S + T)^*(S + T))^\frac{1}{2} \in M = N^+$. Let $S + T = UP$ be the polar decomposition of $S + T$ with $U \in A^n$.

We see that $S + T \in (A^nM) \cap A = N$.

To show that $N$ is norm-closed, let $\{T_n\}$ be a sequence in $N$ converging in the norm to an element $T$ of $A$. Then the sequence $\{T_n^*T_n\}$ lies in $N^+ = M$ as $N$ is a left ideal and $T_n^*T_n \geq 0$ for each $n$. Since $M$ is norm-closed, $T_n^*T \in M$ and hence $(T_n^*T)^\frac{1}{2} \in M$. From the polar decomposition $T = UT$ of $T$, where $U \in A^n$ and $|T| = (T^*T)^\frac{1}{2}$, we see that $T \in (A^nM) \cap A = N$. Hence $N$ is norm-closed. Thus $N$ is a norm-closed left ideal.

Finally, we shall show that $N$ is also a right ideal of $A$. Let $T \in N$ and let $T = UT$ with $U \in A^n$ be the polar decomposition of $T$. Now $|T| = (T^*T)^\frac{1}{2} \in N^+$ and
A is strongly dense in $A^n$ and $U \in A^n$ (and hence $U^* \in A^n$). Therefore there exists a net \( \{U^*_q\} \) on $A$ converging strongly to $U^*$. The net \( \{U_\alpha T | U^*_q\} \) lies in $M = N^+$ as $M$ is invariant. Also, $U_{\alpha} | T | U_q^*$ converges to $U | T | U^*$ weakly and since $U | T | U^* = |T^*|$, we have $U | T | U^* \in A$. As $M$ is convex and norm-closed, it is relatively closed in $A$ under the weak topology \([4, \text{p.422}]\). Thus $|T^*| \in M = N^+$. Since $T^* = V |T^*|$, for a partial isometry $V \in A^n$, and $N = (A^n M) \cap A$, it follows that $T^* \in N$. This shows that $N$ is self-adjoint, i.e., $N^* = N$, and therefore is a 2-sided ideal of $A$.

From the proof of Theorem (3.2.5), we have the following:

**COROLLARY (3.2.6).** Let $N$ be a norm-closed order ideal of a $C^*$-algebra $A$ and let $A^n N$ be the set of all $SP$ with $S \in A^n$ and $P \in N$. Then $M = (A^n N) \cap A$ is a norm-closed left ideal such that $M^+ = N$.

**THEOREM (3.2.7).** If $N$ is a norm-closed left ideal in a $C^*$-algebra, then $N = (A^n N^+) \cap A$, where $A^n N^+$ denotes the set of all elements of the form $SP$ with $S \in A^n$ and $P \in N^+$. 

**PROOF.** \([5, \text{p.394}]\) Let $T$ be an element of $A$ such that $T = SP$ with $S \in A^n$ and $P \in N^+$. Since $A$ is weakly dense in $A^n$, there exists a net \( \{S_\alpha\} \) in $A$ converging weakly to $S$. Since $N$ is a left ideal of $A$, \( \{S_\alpha P\} \) lies in $N$ and converges weakly to $SP = T$. As $N$ is convex and norm-closed, it is weakly closed in $A$ \([4, \text{p.422}]\) and hence $T \in N$. This proves that $(A^n N^+) \cap A \subseteq N$. 
Conversely, let $T \in N$. Then $|T|^2 = T^*T$ is in the $C^*$-algebra $N \cap N^*$ and hence $|T| \in (N \cap N^*) = N^+$. Since $T = U|T|$ with $U \subset A''$, $T \in (A''N^+) \cap A$. Hence $N \subset (A''N^+) \cap A$ and therefore $N = (A''N^+) \cap A$.

Combining Theorems (3.2.4), (3.2.6) and (3.2.7), we obtain the following:

**THEOREM (3.2.8).** There exists a one-to-one correspondence $N \rightarrow N^+$ between norm-closed left ideals and norm-closed order ideals in a $C^*$ algebra $A$.

**§ 3.3. Polar decomposition for continuous linear functionals.**

**THEOREM (3.3.1).** Let $A$ be a $W^*$-algebra, $E$ a projection in $A$ and $f$ in the predual $A_*$ of $A$. Then we have the following relations:

(i) $\|Ef\|^2 \geq \|Ef\|^2 + \|(I - E)f\|^2$.

(ii) If $\|f\| = \|Ef\|$, then $f = Ef$.

**PROOF.** [3, p.239] (ii) follows from (i). We prove (i).

Let $H$ be the Hilbert space on which $A$ acts, and let $B = B(H)$ be the algebra of all bounded linear operators on $H$ into itself. $B$ is a Banach space under the operator bound norm. Since $A$ is a weakly closed subspace of $B$, it is a norm-closed subspace of $B$ [4, p.422]. If $\overline{A}$ denotes the set of all continuous linear functionals on $B$ which vanish on $A$, and if $g' + \overline{A}$ denotes the class of all elements $g \in B'$ such that $g - g' \in \overline{A}$, the mapping $\Psi: g' + \overline{A} \rightarrow g$, 

where \( g \) is defined by \( g(S) = g'(S) \) for all \( S \in A \), is an isometric isomorphism of \( B'/\mathbb{A} \) onto \( A' \) [15, p.188]. Therefore, if \( f \in A^*_\lambda \), there exists an \( f' + \mathbb{A} \in B'/\mathbb{A} \) such that \( \psi(f' + \mathbb{A}) = f \). Hence \( \|f\| = \|f' + \mathbb{A}\| = \inf \{ \|y\| : y \in f' + \mathbb{A} \} \). If \( \varepsilon > 0 \) is given, there exists \( y \in B' \) such that \( \|y\| + \varepsilon > \|y\| \). If we show that

\[
\|y\|^2 \geq \|Ey\|^2 + \|(I - E)y\|^2,
\]

then we will have

\[
(\|f\| + \varepsilon)^2 \geq \|Ef\|^2 + \|(I - E)f\|^2
\]

from which (i) will follow since \( \varepsilon \) is arbitrary. It remains thus to prove (i) for the case where \( A = B \). Since the set of all weakly continuous functionals on \( A \) is dense in \( A^*_\lambda \) [Theorem (2.1.12)], we may assume that \( f \) is weakly continuous. Then, by Theorem (2.1.5), there exist orthonormal systems \( (e_1, e_2, \ldots, e_n), (e'_1, e'_2, \ldots, e'_n) \) and positive scalars \( \lambda_i, i = 1, 2, \ldots, n \), such that

\[
\|f\| = \lambda_1 + \cdots + \lambda_n \quad \text{and} \quad f(T) = \sum_{i=1}^{n} \lambda_i \langle Te_i, e'_i \rangle
\]

for all \( T \in A \). Hence

\[
(Ef)(T) = \sum_{i=1}^{n} \lambda_i \langle Te_i, Ee'_i \rangle,
\]

\[
(I - E)f(T) = \sum_{i=1}^{n} \lambda_i \langle Te_i, (I - E)e'_i \rangle.
\]

Therefore

\[
\|Ef\|^2 + \|(I - E)f\|^2 \leq \left( \sum_{i=1}^{n} \lambda_i \|Ee'_i\| \right)^2 + \left( \sum_{i=1}^{n} \lambda_i \|(I - E)e'_i\| \right)^2
\]

\[
= \sum_{i=1}^{n} \lambda_i^2 (\|Ee'_i\|^2 + \|(I - E)e'_i\|^2)
\]
\[ + \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \left( \|E_i\| \cdot \|E_j\| \right) + \| (I - E) e_i \| \cdot \| (I - E) e_j \| \right) \\
\leq \sum_{i=1}^{n} \lambda_i^2 + 2 \sum_{1 \leq i < j \leq n} \lambda_i \lambda_j \\
= \left( \sum_{i=1}^{n} \lambda_i \right)^2 = \| f \|^2. \]

**THEOREM (3.3.2).** Suppose that \( A \) is a \( W^\ast \)-algebra and \( f \) is an element of the predual \( A_\prec \) of \( A \). Then

(i) There exists a partial isometry \( U \in A \) such that \( p = U^* f \) positive, \( U p = f \) and \( \| p \| = \| f \| \).

(ii) If \( D \) is any operator of \( A \) with \( \| D \| \leq 1 \), \( D f \) positive, and \( \| D f \| = \| f \| \), then \( D f = p \).

(iii) There exists only one partial isometry \( U \) such that \( U U^* = E_p \), where \( E_p \) is the support of \( p \).

**PROOF.** [3, pp.240-242; 5, pp.298-400] (i) For simplicity, we assume that \( \| f \| = 1 \). Let \( A_1 \) be the unit ball of \( A \) and \( K = \{ T \in A_1 : f(T) = 1 \} \). As \( A_1 \) is weakly compact, it is ultraweakly compact. Since \( f \) is ultraweakly continuous, there exists an element \( T \) of \( A_1 \) such that \( f(T) = 1 \). Multiplying by a scalar, we may assume that \( f(T) = 1 \) so that \( T \in K \) and hence \( K \) is nonempty. Since \( K \) is ultraweakly closed and contained in \( A_1 \), \( K \) is ultraweakly compact. It is clear that \( K \) is also convex. Therefore, by Krein-Milman Theorem, \( K \) has an extremal point \( V \) say. \( V \) is also extremal in \( A_1 \), for if \( V = \frac{1}{2} S + \frac{1}{2} T \), where \( \| S \| \leq 1 \) and \( \| T \| \leq 1 \), then \( \| f(S) \| \leq 1 \),
\(|f(T)| \leq 1\) and \(1 = f(V) = \frac{1}{2} f(S) + \frac{1}{2} f(T)\).

Thus \(f(S) = f(T) = 1\), i.e., \(S\) and \(T\) are in \(K\) and \(V = S = T\). From Kadison's characterization of the extremal points of the unit ball of a C*-algebra \([8, \text{p.} 325]\), \(V\) is also a partial isometry.

Define the linear functional \(p\) on \(A\) by \(p(T) = f(\langle VT\rangle)\) for all \(T \in A\). Then \(p \in A_\pi\), and \(\|p\| \leq 1\). Since \(p(I) = f(V) = 1\), \(p\) is positive by Theorem (1.1.4). Also \(p(V^*V) = f(\langle VV^*\rangle) = f(V) = 1\), since \(V\) is a partial isometry; hence \(V^*V \geq E_p\) where \(E_p\) is the support of \(p\). Let \(U = E_pV^*\). Then \(UU^* = E_p\) and, for all \(S \in A\), we have

\[(U^*f)(S) = f(\langle VE_pS\rangle) = p(E_pS) = p(S)\]

Let \(U^*U = E\). Then as \(\|Ef\| \leq 1\) and

\[(Ef)(U^*) = f(\langle U^*UU^*\rangle) = f(U^*) = f(\langle VE_p\rangle) = p(E_p) = 1\]

we have \(\|Ef\| = 1\). From Theorem (3.3.1), \(Ef = f\), and for all \(S \in A\), we have

\[(Up)(S) = p(US) = f(\langle U^*US\rangle) = (Ef)(S) = f(S),\]

i.e., \(Up = f\).

(ii) Suppose that \(D\) is an operator in \(A_\pi\) such that \(Df\) is positive, \(\|Df\| = 1\) and \(\|D\| \leq 1\). Then \(f(D) = (Df)(I) = \|Df\| = 1\) by Theorem (1.1.4).

If \(L\) and \(\gamma\) are respectively the representation of \(A\) and cyclic vector defined by \(p\), then

\((\gamma, L(D^*U^*)\gamma) = p(UD) = f(D) = 1\) by definition of \(p\).

Since \((\gamma, \gamma) = (L(I)\gamma, \gamma) = p(I) = \|p\| = 1\), we have

\((\gamma, \gamma) = (\gamma, L(D^*U^*)\gamma) \leq \|\gamma\| \|L(D^*U^*)\gamma\| \leq \|\gamma\|^2 \|UD\|\)
\[ \|
\|f\| \leq \|Yf\| \leq \|f\| \]

It follows from the Cauchy-Schwartz's inequality that \( L(D^*U^*) \|f\| = \|f\| \). Thus

\[
(Df)(S) = f(DS) = p(UDS) = (L(UDS) f, f) = (L(S)f, L(D^*U^*)f) = (L(S)f, f) = p(S).
\]

Hence \( Df = p \).

(iii) Let \( Y \) be another partial isometry with \( Y^*f \)
positive \( YY^*f = f \) and \( Y^* = E_p \). Since \( \|f\| = \|Yf\| \)
\[ \leq \|Yf\| \leq \|f\|, \]
we have \( 1 = \|f\| = \|Yf\| \) and, by Theorem \( 3.3.1 \), \( p = Y^*f \). We want to show that \( Y = U \). Let \( X = UY^* \).

Then \( p(X) = p(UY^*) = Up(Y^*) = f(Y^*) = Y^*f(I) = p(I) = 1 \).

Similarly \( p(X^*) = 1 \). Since \( 1 = p(X)^2 \leq p(XX^*) \leq 1 \), we have \( p(XX^*) = 1 \) and \( p((E_p - X)(E_p - X)^*) = 1 - 1 - 1 = 0 \).

Now, since \( U, Y \) are partial isometries such that \( UU^* = E_p \)
and \( YY^* = E_p \), \( U = UU^*U = E_p U \) and \( Y = E_p Y \). Then

\[
X = UY^* = (E_p U)(E_p Y)^* = E_p (UY^*) E_p = E_p X E_p, \text{ i.e.,}
\]

\[
X = E_p X E_p \in E_p A E_p. \]

Since \( p((E_p - X)(E_p - X)) = 0 \) and \( p \)
is faithful on \( E_p A E_p \) [2, p. 61], we have \( (E_p - X)^* (E_p - X) = 0 \)
whence \( E_p - X = 0 \) or \( E_p = X \). Suppose that \( H \) is the underlying Hilbert space for \( A, E_1 \) is the projection of \( H \) onto \( (\ker U)^\perp \), the initial space of \( U \). For each \( \xi \in E_p(H) \), we have \( \|U(Y^* \xi)\| = \|E_p \xi\| = \|\xi\| \) and hence \( \|U(Y^* \xi)\| \geq \|Y^* \xi\| \).

Thus \( \xi \in E_1(H) \). As \( U : E_1(H) \rightarrow E_p(H) \) is an isometry, the equality
\[ UY^* \xi = \xi = UU^* \xi \]
implies that \( Y^* \xi = U^* \xi \); moreover \( U^* \) and \( Y^* \) vanish on \( (E_p(H))^\perp \). Hence \( Y^* = U^* \) or \( Y = U \).
If \( A \) is a \( W^* \)-algebra and \( f \) is an element of \( A^* \), we shall denote the function \( p \) defined in Theorem (3.3.2) by \( |f| \). If \( A \) is a \( C^* \)-algebra and \( f \) belongs to \( A^* \), then, considering \( A^* \) as the predual of \( A^\prime \) [Theorem (2.2.3)], we have that \( |f| \) is defined and is also a member of \( A^* \). The equality \( f = U|f| \), where \(UU^* = E_f \), is called the polar decomposition of \( f \).

**THEOREM (3.3.3).** Suppose that \( A \) is a \( W^* \)-algebra and \( \{f_n\} \) is a sequence in the predual \( A^\times \) of \( A \), converging in the norm topology to a function \( f \). Then there exists a subsequence of the \( \{f_n\} \) converging in the weak topology to \( |f| \).

**PROOF.** [5, p. 401] Let \( f_n = U_n|f_n| \) be the polar decompositions. Taking a subsequence, we may assume that the sequence \( U_n^\times \) converges ultraweakly to some operator \( D \in A \) with \( \|D\| \leq 1 \). For each \( T \in A \), we have

\[
|f_n(T) - Df(T)| \leq |f_n(U_n^\times T) - f(U_n^\times T)| + |f(U_n^\times T) - f(DT)|
\]

\[
\leq \|f_n - f\| \|T\| + |f(U_n^\times T) - f(DT)|.
\]

Since the latter expression tends to 0, the subsequence \( |f_n| \) converges weakly to \( Df \). It follows that \( Df \) is positive, and \( \|f_n\| = |f_n| (I) \) tends to \( \|Df\| = (Df)(I) \).

But \( \|f\| \) is the limit of the sequence \( \|f_n\| \), hence \( \|Df\| = \|f\| \) and we conclude from Theorem (3.3.2) that \( Df = |f| \).

**THEOREM (3.3.4).** If \( f \) and \( g \) are in the predual \( A^\times \) of a \( W^* \)-algebra \( A \), then for all \( T \in A \), we have
\[ |f + g|^2 (T) \leq (|f| + |g|)(|f| (T^*)^T + |g| (T^*)^T) \]

**Proof.** [5, p.401] Let \( f = U |f|, g = V |g| \), and \( f + g = S |f + g| \) be the polar decompositions. Then for all \( T \in A \),

\[
\begin{align*}
|f + g|^2 (T) & = |f| (US^*T) + |g| (VS^*T) |^2 \\
& \leq \left[ |f| (US^*T) + |g| (VS^*T) \right]^2 \\
& \leq \left[ |f| (US^*SU^*)^{1/2} |f| (T^*)^{1/2} \\
& \quad + |g| (VS^*SV^*)^{1/2} |g| (T^*)^{1/2} \right]^2 \\
& \leq \left( |f| + |g| \right)^2 \left( |f| (T^*)^{1/2} + |g| (T^*)^{1/2} \right)^2 \\
& \leq (|f| + |g|)(|f| (T^*)^T + |g| (T^*)^T).
\end{align*}
\]

§ 3.4. Order ideals in the dual of a C*-algebra.

Let \( B \) be a W*-algebra, and \( B_\pi \) its predual. If \( p \) is a positive element of \( B_\pi \), it was shown in [5, p.402] that \( \{p\}^{\pi_\pi} \) is the smallest norm-closed order ideal containing \( p \), i.e., \( \{p\}^{\pi_\pi} \) is the norm-closure of the set of all \( q \in B_\pi \) such that \( 0 \leq q \leq \lambda p \) for some positive scalar \( \lambda \).

**Lemma (3.4.1).** Let \( p \) and \( q \) be positive elements in the predual \( B_\pi \) of a W*-algebra \( B \) such that there exists a constant \( k \) with \( |q(S)|^2 \leq k p(S^*S) \) for all \( S \in B \). Then \( q \in \{p\}^{\pi_\pi} \).

**Proof.** [5, p.403] By Theorem (1.2.3), there exists a representation \( L \) of \( B \) and a cyclic vector \( \psi \) corresponding to \( p \) such that \( p(T) = (L(T) \psi, \psi) \) for all \( T \in B \). To show that \( q \in \{p\}^{\pi_\pi} \), let \( S \geq 0 \) and \( p(S) = 0 \). Then \( L(S) \psi = L(S^{1/2})(L(S^{1/2}) \psi) = 0 \) and \( p(S^*S) = (L(S^*S) \psi, \psi) \).
= (L(S)∗L(S)y, y) = (L(S)y, L(S)y) = 0. Hence $|q(s)|^2 \leq k p(s^*s) = 0$ and $q \in \{p\}^⊥⊥.$

**Theorem (3.4.2).** For each norm-closed order ideal $N$ of the predual $B_*$ of a $W^*$-algebra $B$, there exists a norm-closed left invariant subspace $M$ in $B_*$ such that $M^+ = N.$

**Proof.** [5, p.403] Let $M = BN$, where $BN$ denotes the set of all $Sp$ with $S \in B$ and $p \in N$. We shall show that $M$ is a norm-closed left invariant subspace such that $M^+ = N.$

First of all, we shall show that $M^+ = N.$ Since $B$ is a $W^*$-algebra, it contains the identity operator and hence $M^+ \supseteq N.$ Now, for any $q \in M^+$, $q = Up$ where $U \in B$, $p \in N$. If $C \geq 0$ is in $B$ and $p(C) = 0$, then as shown in the proof of Lemma (3.4.1), $Lp(C)y = 0$ which implies $Lp(Sc)y = 0$ and $0 = (Lp(Sc)y, y) = p(sc) = Sp(c)$ for all $S \in B$. Hence $q(C) = Up(C) = 0$ and $q \in \{p\}^⊥⊥$. Since $N$ is a norm-closed order ideal and $\{p\}^⊥⊥$ is a subset of $N$, we have $q \in N$. Hence $M^+ = N.$ Next, if $f \in M = BN$, then $f = Sp$ for some $S \in B$, $p \in N$. For each $T \in B$, $Tf = T(Sp) = (Ts)p \in BN = M$, hence $M$ is left invariant.

To show that $M$ is closed under addition, let $f$ and $g$ be two elements of $M$ with polar decompositions $f = V|f|$ and $g = U|g|$. Then $|f| = V^*f$ and $|g| = U^*g$ are in $M^+ = N$ since $M$ is left invariant. By Theorem (3.3.4), if $S \in B$, then $\|f + g\|_2 \leq (\|f\| + \|g\|) (\|u| + |g|) (S^*S)$ and hence by Lemma (3.4.1), $f + g \in \{M + \|g\|\}^⊥⊥$, which in turn is a subset of $N$. From its polar decomposition, it
is thus clear that $f + g \in M$. This shows that $M$ is an invariant subspace of $B_x$.

To show that $M$ is norm-closed, let $\{f_n\}$ be a sequence in $M$ converging in the norm to a function $f$, the sequence $\{f_n\}$ lies in $M^* = N$ and, by Lemma (3.3.3), there exists a subsequence converging weakly to $f$. Since $N$ is a norm-closed convex set, it is weakly closed, we have $f \in N$. By making use of the polar decomposition, we have $f \in M$.

**Theorem (3.4.3).** Let $A$ be a $C^*$-algebra and $A'$, $A''$ its dual and bidual respectively. If $N$ is a norm-closed subspace of $A'$, then it is invariant under left multiplication by elements of $A''$.

**Proof.** [1, p.414] Let $p$ be an element of $N$ and $S \in A''$. We want to show that $Sp \in N$. Since $A$ is weakly dense in $A''$, there exists a net $\{S_\alpha\}$ in $A$ converging to $S$ under the topology $\sigma(A'', A')$ on $A''$. Hence $f(S_\alpha) \to f(S)$ for all $f \in A'$; in particular,

$$(fT)S_\alpha \to (fT)S$$

for all $T \in A''$. Since

$$(S_\alpha p)T - (Sp)T = p(S_\alpha T) - p(ST) = (pT)S_\alpha - (pT)S = (pT)(S_\alpha - S),$$

we see that $S_\alpha p \to Sp$ under $\sigma(A', A'')$. Since $N$ is convex and norm-closed, it is weakly closed, and hence $Sp \in N$ as asserted.

**Theorem (3.4.4).** If $N$ is a norm-closed order ideal in the predual $B_x$ of a $W^*$-algebra $B$, then $N^{**} = N$.

**Proof.** [5, p.405] By Theorem (3.4.2), there exists a norm-
closed left invariant subspace $M$ in $B_\star$ such that $M^+ = N$. We claim that $N_\perp = M_\perp$. It is clear that $N_\perp \supseteq M_\perp$. Now, let $S \in N_\perp$ and $f \in M$. Then by Theorem (3.3.2), $f = U^*f$ and hence $|f(S)| = |U^*f| \in M^+ = N$. Thus $|f(S)| = 0$. As shown in the proof of Lemma (3.4.1), we have $|f(S^2)| = |f(S^*S)| = 0$. But $|f(S)|^2 = |f(U^*U)S^*S| = 0$ by Cauchy-Schwartz inequality. Hence $f(S) = 0$ and so $S \in M^+$. Thus $M^+ \supseteq N^+$ and consequently $N^\perp = M^\perp$. On the other hand, we have $M^{\perp\perp} = M_\circ\circ$, where $M_\circ$ denotes the polar of $M$. For clearly, $M^\perp \subseteq M_\circ$ and $M^{\perp\perp} \supseteq M_\circ\circ$. Now, if $p \in M^{\perp\perp}$ and $S \in M_\circ$, then $S^*S \in M_\circ$ since $M$ is left invariant, and from $p(S^*S) = 0$ and the Cauchy-Schwartz inequality, we obtain $p(S) = 0$. Hence $S \in M_\circ^\perp$ and $M^{\perp\perp} \subseteq M_\circ^\perp$. Hence $M^{\perp\perp} = M_\circ^\perp$. Thus, we obtain $N^{\perp\perp} = M^{\perp\perp} = M_\circ^\perp = M_\circ \cap B_\star^\perp$. Since $M$ is a norm-closed subspace of $B_\star$, by the Bipolar Theorem [12, Corollary 1, p.36], we have $M_\circ = M_\circ^\perp = M_\circ^\perp = M_\circ^\perp \cap B_\star^\perp = M_\star$. Hence $M_\circ \cap B_\star^\perp = M^+$ and so $N^{\perp\perp} = M^+ = N$.

**COROLLARY (3.4.5).** If $N$ is a norm-closed order ideal in the dual $A_\star$ of a C*-algebra $A$, then $N = (N^\perp)_{\perp\perp}$.

**PROOF.** [5, p.405]: Since $A_\star$ can be considered as a W*-algebra, and since $A_\star$ can be identified with the predual of $A_\star$ [p.36], the Corollary follows immediately from Theorem (3.4.4).

**THEOREM (3.4.6).** Let $A_\star$ be the dual of a C*-algebra $A$ and $N$ a weak*-closed order ideal of $A_\star$. Then there exists a weak*-closed left invariant subspace $M$ of $A_\star$.
such that $M^+ = N$.

**Proof.** [5, pp. 405-406] Since $N$ is convex and weak*-closed, it is norm-closed, [4, p.422 and § 3.1]. By Theorem (3.4.2), there exists a norm-closed left invariant subspace $M$ of $A'$ such that $M^+ = N$. We shall show that $M$ is also weak*-closed. By Krein-Smulian Theorem [4, p.429], it suffices to show that the intersection of $M$ and the unit ball $A'_1$ of $A'$ is weak*-closed. Let $\{f_\alpha\}$ be a net in $M \cap A'_1$ converging weakly* to $f$, and $f_\alpha = U_\alpha f'$ be the corresponding polar decompositions, $U_\alpha \subseteq A''$. Since $M$ is a left invariant norm-closed subspace of $A'$, $Tf \in M$ for all $T \in A$, $Sf \in M$ for all $S \in A''$ by Theorem (3.4.3). Hence $|f_\alpha| = U_\alpha^* f_\alpha$ lie in $M^+ = N$. Since the unit ball of $A'$ is weak*-compact, we may select a subset $\{f_\beta\}$ converging weakly* to a positive functional $p$, which must lie in $N$. Since $|f_\beta| \subseteq M \cap A'_1$, we have

$$|f_\beta(T)|^2 = |U_\beta| f_\beta(T) |^2 = |f_\beta| (U_\beta T) |^2 \leq |f_\beta| (T^* T) |f_\beta| (U^* U) \leq |f_\beta| (T^* T)$$

for all $T \in A$. Hence $|f(T)|^2 \leq p(T^* T)$. Since $f$ and $p$ are in the predual of $A''$ [p.37], we have $|f(S)|^2 \leq p(S^* S)$ for all $S \in A''$. Letting $f = U |f|$ be the polar decomposition, we have for each $S \in A''$,

$$|f| (S)^2 = |f(U^* S)|^2 \leq p(S^* U^* U S) \leq p(S^* S).$$

from Lemma (3.4.1), $|f| \subseteq \{p\}^\perp$, which in turn is a subset of $N$. Then $f = U |f|$ is in $M$, which shows that $M$
is weak*-closed.

**THEOREM (3.4.7).** If $N$ is a weak*-closed order ideal in the
dual of a $C^*$-algebra $A$, then $N = ( \overline{(N)_{A^*}})$.  

**PROOF.** [5, p.406] By Theorem (3.4.6), there exists a
weak*-closed left invariant subspace $M$ with $M^+ = N$.
Since $A'$ is the predual of $A''$ [p.35], the argument
used in the proof of Theorem (3.4.4) shows that

$$N_{\perp \perp} = (M_{A''})^0 \cap (A')^+.$$  

Hence, since $M$ is weak*-closed, $N_{\perp \perp} = M^+ = N$.  

CHAPTER IV

TWO SIDED IDEALS IN A C*-ALGEBRA

§ 4.1. Invariant faces of a state space.

Let $A$ be a C*-algebra and let $A'$ be the conjugate space of $A$. The state space of $A$ is the set of all states of $A$, topologized by regarding it as a subspace of $A'$ in the weak*-topology, and shall be denoted by $S(A)$. By a face of $S(A)$ we mean a convex subset $F$ of $S(A)$ such that if $f \in F$, $g \in S(A)$ and $\alpha g \leq f$ for some $\alpha > 0$, then $g \in F$. $F$ is an invariant face if $f \in F$ implies the state $S \rightarrow f(T^*T) f(T^*T)^{-1}$ belongs to $F$ whenever $f(T^*T) \neq 0$ and $T \in A$. If $F$ is an invariant face of $S(A)$, let $F^\perp$ denote the set of all elements $T$ such that $f(T) = 0$ for all $f \in F$. Also if $N$ is a subset of $A$, let $N_A^\perp$ denote the set of all $f \in A'$ with $f(T) = 0$ for all $T \in N$, and let $N^\perp$ denote $N_A^\perp \cap S(A)$.

THEOREM (4.1.1). Let $A$ be a C*-algebra. The map $N \rightarrow N^\perp$ is an order inverting bijection between norm-closed 2-sided ideals of $A$ and weak*-closed invariant faces of $S(A)$.

PROOF. If $N$ is a norm-closed 2-sided ideal of $A$, then $N^\perp$ is a weak*-closed invariant face. Moreover, if $N_1$ and $N_2$ are two norm-closed 2-sided ideals of $A$ with $N_1 \subseteq N_2$, then $N_2^\perp \subseteq N_1^\perp$. Hence the mapping is order
inverting. To show that the mapping is also onto, let $F$ be a weak$\ast$-closed invariant face of $S(A)$ and let $G$ be the smallest weak$\ast$-closed order ideal in the conjugate space $A'$ of $A$ generated by $F$; $G$ is an invariant order ideal. By Theorem (3.4.7), $G = (G^\perp)^\perp_{A'}$. Let $N = G^\perp$, we see that $N$ is norm-closed and a 2-sided ideal of $A$ since $G$ is invariant; furthermore $N^\perp_{A'} = (G^\perp)^\perp_{A'}$. Since $G \cap S(A) = F$, $N^\perp = N^\perp_{A'} \cap S(A) = (G^\perp)^\perp_{A'} \cap S(A) = G \cap S(A) = F$. Hence the map $N \rightarrow N^\perp$ is onto. As it is clear that it is also one-to-one, it is an order inverting bijection.

**COROLLARY (4.1.2).** Let $A$ be a C$\ast$-algebra, $N$ a norm-closed 2-sided ideal of $A$ and $F$ a weak$\ast$-closed invariant face of $S(A)$. Then $N^\perp = N$ and $F^\perp = F$.

**PROOF.** Note that if $F$ is a weak$\ast$-closed invariant face of $S(A)$, and $G$ is the weak$\ast$-closed order ideal in $A'$ generated by $F$, then $F^\perp = G^\perp$. It follows that if $f : N \rightarrow N^\perp$ denotes the bijection in the preceding theorem then $f^{-1} : F \rightarrow F^\perp$; in particular, $N = f^{-1}(N^\perp) = N^\perp$. As $F^\perp = G^\perp$ and, by Theorem (3.4.7), $(G^\perp)^\perp_{A'} = G$, we have $F^\perp = (F^\perp)^\perp_{A'} \cap S(A) = (G^\perp)^\perp_{A'} \cap S(A) = G \cap S(A) = F$.

**THEOREM (4.1.3).** Let $N$ and $M$ be closed 2-sided ideals in a C$\ast$-algebra $A$ with identity. Then

(i) $(N + M)^\perp = N^\perp \cap M^\perp$.

(ii) $(N \cap M)^\perp = \text{conv}(N^\perp, M^\perp)$ where $\text{conv}(N^\perp, M^\perp)$ is the convex hull of $N^\perp \cap M^\perp$. 
PROOF. (1) Since $N, M$ are norm-closed 2-sided ideals of the $C^*$-algebra $A$, $N + M$ is again a norm-closed 2-sided ideal of $A$. Since $N \rightarrow N^\perp$ is order inverting and $N, M \subseteq N + M$, $(N + M)^\perp \subseteq N^\perp \cap M^\perp$. Conversely, if $f \in N^\perp \cap M^\perp$, then $f(T) = 0$ and $f(S) = 0$ for all $T \in N$, $S \in M$. Hence $f(P) = 0$ for all $P \in N + M$ which proves that $f \in (N + M)^\perp$. Therefore $N^\perp \cap M^\perp \subseteq (N + M)^\perp$.

(2) Since $N \cap M \subseteq N, M$, we have $(N \cap M)^\perp \supseteq N^\perp, M^\perp$. Now, $(N \cap M)^\perp$ is convex and weak*-closed, and $\text{conv}(N^\perp, M^\perp)$ is the smallest weak*-closed convex subset of $S(A)$ containing $N^\perp$ and $M^\perp$. Hence $(N \cap M)^\perp \supseteq \text{conv}(N^\perp, M^\perp)$. It remains to show that $(N \cap M)^\perp \subseteq \text{conv}(N^\perp, M^\perp)$. By Corollary (4.1.2), this is equivalent to show that $\{\text{conv}(N^\perp, M^\perp)\}^\perp \supseteq (N \cap M)^\perp = N \cap M$. Let $T \in N \cap M$. Then $f(T) = 0$ and $g(T) = 0$ for all $f \in N^\perp$ and $g \in M^\perp$. Hence $h(T) = 0$ for all $h \in \text{conv}(N^\perp, M^\perp)$, and $T \in \{\text{conv}(N^\perp, M^\perp)\}^\perp$. This completes the proof.

DEFINITION. A mapping $\mathcal{g}$ on a convex subset $K$ of a vector space into another vector space is called an affine mapping if $\mathcal{g}(ak + (1 - a)k') = a\mathcal{g}(k) + (1 - a)\mathcal{g}(k')$ for all $k, k' \in K$ and $0 \leq a \leq 1$.

Let $A$ be a $C^*$-algebra with identity. For each $T \in A$, let $\hat{T}(f) = f(T)$ for all $f$ in $A'$, the conjugate space of $A$. Then $T$ is a weak*-continuous linear functional. The restriction of $\hat{T}$ to $S(A)$ is an affine
mapping on \( S(A) \), the state space of \( A \). By definition of \( \hat{T} \),
\[
|\hat{T}(f)| = |f(T)| \leq \|f\| \|T\|, \text{ then } \|\hat{T}\| \leq \|T\|.
\]
Conversely, if \( T \neq 0 \),
then by Hahn-Banach Theorem, there exists an \( f \in A^* \) with
\[
\|f\| = 1 \text{ and } f(T) = \|T\| \quad [4, \text{ p.65}].
\]
Hence \( \|T\| = |f(T)| = |\hat{T}(f)| \leq \|T\| \|f\| = \|T\| \) and so \( \|\hat{T}\| = \|T\| \). Furthermore, \( \hat{T} + \hat{S} = \hat{T} + \hat{S} \) and
\[
\alpha\hat{T} = \alpha\hat{T}.
\]
If \( \hat{T} = \hat{S} \) for some \( T, S \subseteq A \), then \( \hat{T}(f) = \hat{S}(f) \)
for all \( f \in A^* \), i.e., \( f(T) = f(S) \), and \( f(T - S) = 0 \) for all \( f \in A^* \). Hence \( T - S = 0 \) since \( A^* \) is total on \( A \).
Therefore \( S = T \). This shows that the canonical mapping \( T \rightarrow \hat{T} \),
for all \( T \subseteq A \) is an isometric homomorphism on \( A \) into \( A^* \).

**Theorem (4.1.4).** Let \( A \) be a \( C^* \)-algebra with identity.
For each hermitian element \( T \) of \( A \), let \( \hat{T}_S \) be the
restriction of \( \hat{T} \) to \( S(A) \). Then the mapping \( \Psi : T \rightarrow \hat{T}_S \)
is an isometric order-isomorphism of the hermitian part \( A_h \) of
\( A \) onto the Banach space of all weak\(^*\)-continuous real affine
functions on \( S(A) \).

**Proof.** It is clear that each \( \hat{T}_S \) is a real affine function
on \( S(A) \) and is weak\(^*\)-continuous for each hermitian element \( T \) of \( A \). Since \( \omega_\xi \) with \( \|\xi\| = 1 \) is a state and
\[
\sup \{ |\hat{T}_S(\omega_\xi)| : \|\xi\| = 1 \} = \sup \{ |(T_\xi, \xi)| : \|\xi\| = 1 \} = \|T\|,
\]
we see that \( \|\hat{T}\| = \|\hat{T}_S\| \) and \( \Psi \) is an isometry and therefore one-to-one.
It remains to show that it is onto. Let \( \Phi_0 \) be a real affine
weak\(^*\)-continuous function on \( S(A) \). It is well known that each
element of \( A^* \) can be written as the sum of two hermitian linear
functionals on \( A \), and each hermitian functional of \( A^* \) is
the difference of two positive linear functionals on \( A \) \[Theorem
(1.1.4)\]. It follows that \( S(A) \) is total on \( A \), and in
particular $S(A)$ is total on the real Banach space $A_h^*$. Hence $S(A)$ is not contained in any hyperplane of $A_h^*$ and therefore, by [9, Lemma (4.1)], there exists a unique linear functional $\varphi$ on $A_h^*$ into the real field $\mathbb{R}$ and a unique real number $\alpha_0$ in $\mathbb{R}$ such that $\varphi(f) = \varphi(f) + \alpha_0$ for all $f \in S(A)$. Let $\varphi|_{S(A)}$ be the restriction of $\varphi$ to $S(A)$. Since $\varphi_0$ and the constant function $T_{\alpha_0} : f \mapsto \alpha_0$ are weak*-continuous on $S(A)$, it follows that $\varphi|_{S(A)}$ is also weak*-continuous since $\varphi|_{S(A)} = \varphi_0 - T_{\alpha_0}$. Since $S(A)$ is a weak*-compact convex set and the closed unit ball $D$ of $A_h^*$ is given by $D = Q - Q$, where $Q = \{\alpha f : \alpha \in [0, 1], f \in S(A)\}$, we have from [9, Lemma (4.2)] that $\varphi$ is weak*-continuous on $A_h^*$. Therefore, by [15, Theorem (3.81-A)], there exists an element $T \in A_h$ such that $\varphi(f) = f(T)$ for all $f \in A_h^*$. Hence $\varphi$ is onto.

§ 4.2. Two-sided ideals in a C*-algebra.

The purpose of this section is to answer the question asked by J. Dixmier in [3, p.20]. This is done in Theorem (4.2.2). Let $A$ be a C*-algebra with identity, $N$ a norm-closed 2-sided ideal of $A$. If $\Phi$ is the canonical homomorphism of $A$ onto $A/N$, then the map $f \mapsto f \circ \Phi$ is an affine isomorphism of $S(A/N)$ onto $N$. Thus the map $\Phi(T) \mapsto T N$ is an order-isomorphic isometry on the self-adjoint operators in $A/N$. We shall make use of this fact to prove the following

THEOREM (4.2.1). Let $A$ be a C*-algebra with identity $I$, and let $M, N$ be norm-closed 2-sided ideals of $A$. Let $T \in (M+N)^+$, and let $0 \leq \varepsilon < 1$. Then there exist
$B \in M^+$ and $C \in N^+$ such that $0 \leq T - B - C \leq \varepsilon I$.

**Proof.** [14, pp. 255-256] Multiplying $T$ by a scalar, we may assume that $T \leq I$. It follows that $\|T\| \leq 1$. Let $\psi$ be the canonical homomorphism of $A$ onto $A/N$. Then

$\psi(M + N) = \psi(M)$. Now $\psi(T) \geq 0$. Then there exists

$B_1 \in M^+$ such that $\psi(B_1) = \psi(T)$. Then $\hat{B}_1 \big| M^+ \leq 0$ and

$\hat{B}_1 \big| N^+ = \hat{B}_1 \big| N^+$. Since $(M + N)^+ = \text{conv}(M^+, N^+)$ by Theorem (4.1.3), $\hat{B}_1 \big| (M \cap N)^+ \leq \hat{B}_1 \big| (M \cap N)^+$. Let $\overline{f}$ be the canonical homomorphism of $A$ onto $A/M \cap N$. Then $0 \leq \overline{f}(B_1) \leq \overline{f}(T)$. Let $f$ be the real continuous function $f(x) = (3^{-1} \varepsilon)^2$ for $x \leq (3^{-1} \varepsilon)^2$ and $f(x) = x$ for $x > (3^{-1} \varepsilon)^2$. Then

$f(T) = (3^{-1} \varepsilon)^2$ for $T \leq (3^{-1} \varepsilon)^2$ and $f(T) = T$ for $T > (3^{-1} \varepsilon)^2$. Since $T$ is positive, it has a unique positive square root $T^{\frac{1}{2}}$ [7, p. 58]. Since $\|I - T^{\frac{1}{2}}\| < 1$, $T^{\frac{1}{2}}$ has inverse $T^{-\frac{1}{2}}$ [11, p. 12]. With this remark, let

$$S = f(T)^{-\frac{1}{2}} B_1 f(T)^{-\frac{1}{2}}$$

Then $S \geq 0$ and $S \in M^+$ since $M$ is a 2-sided ideal.

Now,

(1) \hspace{1cm} 0 \leq f(S) = f(f(T)^{-\frac{1}{2}} B_1 f(T)^{-\frac{1}{2}}) f(f(T)^{-\frac{1}{2}}) \leq f(f(T)^{-\frac{1}{2}}) f(f(T))^{-\frac{1}{2}} \leq f(I).

Let $g$ be the real continuous function $g(x) = x$ for $x \leq 1$, $g(x) = 1$ for $x > 1$. Since $g(0) = 0$, $g(S)$ is by the Stone-Weierstrass theorem a uniform limit of polynomials in $S$ without constant terms. Since $S \in M^+$, and $M$ is
uniformly closed, \( g(S) \in M^+ \). By (1), we have

\[ f(g(S)) = g(f(S)) = f(S). \]

Let

\[ B = (f(T)^\frac{1}{2} - 3^{-1}\varepsilon I) g(S) (f(T)^\frac{1}{2} - 3^{-1}\varepsilon I). \]

Since \( g(S) \in M^+ \) so is \( B \). Now \( (f(x)^\frac{1}{2} - 3^{-1}\varepsilon)^2 \leq x \) for \( x > 0 \), and \( g(S) \leq I \). Hence \( 0 \leq B \leq T \). By (2)

\[ f(B) = (f(g(T))^\frac{1}{2} - 3^{-1}\varepsilon g(I)) f(g(S)) (f(g(T))^\frac{1}{2} - 3^{-1}\varepsilon g(I)) \]

\[ = f(B_1) - 3^{-1}\varepsilon \left[ f(g(T))^\frac{1}{2} g(S) + g(S) f(g(T))^\frac{1}{2} - 3^{-1}\varepsilon g(S) \right]. \]

Since \( \|f(g(T))^\frac{1}{2}\| \leq 1 \), \( \|g(S)\| \leq 1 \) and \( \varepsilon < 1 \),

\[ \|B_1 |(M \cap N)^\perp - B_1 |(M \cap N)^\perp\| = \|f(B) - f(B_1)\| \leq \varepsilon; \]

in particular,

\[ \|B \|_{N^\perp} = \|T \|_{N^\perp} \leq \|B_1 \|_{N^\perp} \leq \varepsilon. \]

Applying the preceding argument to \( T - B \) instead of \( T \) and to \( N \) instead of \( M \). Choose \( C_1 \in N^+ \) such that \( C_1 \leq T - B \), and

\[ \|\hat{C}_1 \|_{M^\perp} = (\hat{T} - \hat{B}) \leq \|C_1 \|_{M^\perp} \leq \varepsilon. \]

Since \( \hat{C}_1 \|_{N^+} = 0 \), (3) implies

\[ \|\hat{C}_1 \|_{N^\perp} = (\hat{T} - \hat{B}) \|_{N^\perp} \leq \varepsilon. \]

By (4) and (5), we have
||\mathcal{F}(C_1) - \mathcal{F}(T - B)|| = ||\hat{C}_1|_{\text{conv}(M^+, N^+)}|| \leq \varepsilon.

Let \( D = T - (B + C_1) \). Then \( D \geq 0 \), and \( ||\mathcal{F}(D)|| \leq \varepsilon \). Let \( h \) be the real continuous function \( h(x) = 0 \), for \( x \leq \varepsilon \), \( h(x) = x - \varepsilon \) for \( x > \varepsilon \). Then \( \mathcal{F}(h(D)) = h(\mathcal{F}(D)) = 0 \), and \( h(D) \in (M \cap N)^+ \subseteq N^+ \). Furthermore

\[(6)\quad D - \varepsilon I \leq h(D) \leq D.
\]

Let \( C = C_1 + h(D) \). Then \( C \in N^+ \), and by (6), we have
\[
0 \leq B + C \leq B + C_1 + D = T \leq B + C_1 + h(D) + \varepsilon I = B + C + \varepsilon I.
\]

This completes the proof.

**THEOREM (4.2.2).** Let \( N \) and \( M \) be norm-closed 2-sided ideals in a \( C^* \)-algebra \( A \) with identity \( I \). Then

\[
(N + M)^+ = N^+ + M^+.
\]

**PROOF.** [14, pp.256-257] If \( T \in M^+ + N^+ \), then \( T = B + C \) with \( B \in M^+ \) and \( C \in N^+ \). Hence \( T \in (M + N)^+ \). Multiplying \( T \) by a scalar, we may assume that \( 0 \leq T \leq I \). By the foregoing theorem, choose \( B_0 \in M^+ \) and \( C_0 \in N^+ \) such that

\[
0 \leq T - B_0 - C_0 \leq 2^{-1}I.
\]

Then \( ||B_0|| \leq ||T|| \leq 1 \).\( \|C_1\| \leq ||T|| \leq 1 \). Suppose inductively \( B_0, B_1, \ldots, B_{n-1} \) are chosen in \( M^+ \) and \( C_0, C_1, \ldots, C_{n-1} \) are chosen in \( N^+ \) such that \( ||B_i|| \leq 2^{-i} \), and \( ||C_j|| \leq 2^{-j} \),
and

\[ 0 \leq T - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \leq 2^{-n} I. \]

Applying the foregoing theorem to \( T - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j \) and to \( \varepsilon = 2^{-n-1} \). Then there exist \( B_n \in M^+ \) and \( C_n \in N^+ \) such that

\[ 0 \leq T - \sum_{j=0}^{n-1} B_j - \sum_{j=0}^{n-1} C_j - B_n - C_n \leq 2^{-n-1} I \]

or

(7) \[ 0 \leq T - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j \leq 2^{-n-1} I. \]

Moreover, by (7) \( \| B_n \| \leq 2^{-n}, \| C_n \| \leq 2^{-n} \); the induction is complete. Let

\[ B = \sum_{j=0}^{\infty} B_j, \quad C = \sum_{j=0}^{\infty} C_j. \]

Then \( B \in M^+ \) and \( C \in N^+ \), and

\[ \| T - B - C \| = \lim_{n \to \infty} \| A - \sum_{j=0}^{n} B_j - \sum_{j=0}^{n} C_j \| \leq \lim_{n \to \infty} 2^{-n-1} = 0. \]

Thus \( A = B + C \in M^+ + N^+ \), and \( (M+N)^+ \subseteq M^+ + N^+ \).

This completes the proof.
BIBLIOGRAPHY


