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UMI
THE PROPERTIES OF THE DISTRIBUTION FUNCTIONS OF THE FIRST ORDER OF SOME CONTINUED FRACTIONS

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ABSTRACT

A brief review of the probability theory of simple continued fractions is given, as developed by Borel, Kuzmin and others. Some hitherto unpublished properties of the distribution functions $M_n(x)$ as defined in Kuzmin's work have been derived.

Various aspects of the dependence of index numbers are discussed. Finally a statistical analysis is made to study empirically the distribution functions of $a_n$ which are directly related to $M_{n-1}(x)$. 
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Chapter I

INTRODUCTION

The application of probability Theory to continued fractions could be said to have started when Gauss in 1812 stated the following problem:

If \( a_n \) is the \( n \text{th} \) incomplete quotient (or the \( n \text{th} \) index number) of a real number in its simple continued fraction expansion, then

\[
\lim_{n \to \infty} \left\{ a_n \geq k \right\} = \frac{1}{\log 2} \log \left( 1 + \frac{1}{k} \right)
\]

where \( \left\{ a_n \geq k \right\} \) denotes the probability that \( a_n \) is greater than or equal to \( k \). However a proof for the above theorem by Gauss is not known.

Even before the Gaussian theorem was proved, Korel [1] 1929, Bernstein [2] 1918 and Khintchine [3], [4] obtained some results in this theory. Korel gave the upper and lower limits for \( \left\{ a_{n+1} = k \right\} \) and \( \left\{ a_{n+1} > k \right\} \) as follows:

\[
\frac{2}{3k(k+1)} < \mathbb{P} \left\{ a_{n+1} = k \right\} < \frac{3}{(k+1)(k+2)}
\]

\[
\frac{2}{3(k+1)} < \mathbb{P} \left\{ a_{n+1} > k \right\} < \frac{2}{k+2}
\]

[Note: The interval for the first inequality can be made smaller than Korel gave it. We can namely write]
\[
\frac{1}{k(k+2)} \leq \sum_{n=1}^{\infty} \frac{1}{a_{n+1}^2} \leq \frac{2}{(k+1)^2}
\]

This leads to a form for the second inequality which is also different from the one given above. By summation we get

\[
\frac{2k^2}{(k+1)(k+2)} \leq \sum_{n=1}^{\infty} \frac{1}{a_{n+1}^2} \leq 2 \sum_{r=1}^{\infty} \frac{1}{(k+1)^2} = 2\psi(k+1)
\]

where \(\psi(x) = \sum_{r=1}^{\infty} \frac{1}{(r+x)^2}\).

The results of Korel and Bernstein amount to the following theorem:

Let \(\phi(n)\) be a positive increasing function of \(n\). Then

A) If \(\sum \frac{1}{\phi(n)}\) converges, \(a_n(u) \leq \phi(n)\) for almost all \(u\) and for all large \(n\). \([n > u](a)\);

B) If \(\sum \frac{1}{\phi(n)}\) diverges \(a_n(u) > \phi(n)\) for almost all \(u\) and for an infinity of \(n\).

Taking \(\phi(n) = 1\), one obtains that the \(a_n(u)\)'s are not bounded for almost all \(u\). The above theorem was improved by Pyszyn [5] and Hinchin [3].

In 1923 the first proof of the Gaussian theorem was given by Luzin [6], [7] who stated the theorem in the following form:

let \(a\) be a real number in the closed interval \((0, 1)\), whose simple continued fraction expansion is given by (1.1)

\[
a = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_n} + \cdots \quad (1.1)
\]
Let
\[ z_n(u) = \frac{1}{|a_n|} - \frac{1}{|a_{n+1}|} + \frac{1}{|a_{n+2}|} - \frac{1}{|a_{n+3}|} \ldots \quad (1.2) \]

and
\[ u_n(x) = \operatorname{rob} \left\{ z_n(a) < x \right\} \quad (1.3) \]

Then the theorem is
\[ \lim_{n \to \infty} u_n(x) = \frac{\log(1+x)}{\log 2} \]

From (1.2) one can observe that
\[ z_n(u) = \frac{1}{a_{n+1}(u) + z_{n+1}(u)} \]

and \[ a_{n+1}(u) = \lfloor \frac{1}{z_n(u)} \rfloor \quad \text{(integral part of } \frac{1}{z_n(u)}) \]

so one obtains that
\[ z_{n+1}(u) = \frac{1}{z_n(u)} - \left( \frac{1}{z_{n+1}(u)} \right) \quad \text{from which it follows that} \]
\[ u_{n+1}(x) = \sum_{k=1}^{\infty} u_n(x) = u_n \left( \frac{1}{x} \right) \quad (1.4) \]

It can be verified easily that the function \[ u_n(x) = c \log (1+x) \]
\( (c \text{ is a constant independent of } n) \) satisfies the equation (1.4).

This suggested to cause the asymptotic behaviour of \[ u_n(x) \] .

If it is further assumed that the distribution function of \[ a(10^{-\infty}; x) \] has a continuous bounded derivative it can be proved from
\[ (1.4) \] that \[ u_n(x) \] satisfies the following relation,
\[ f_{n+1}(x) = \sum_{k=1}^{\infty} \frac{1}{(k+x)^2} f_{n+k}(x) \]  
(1.5)

Using the equation (1.5), Kuzmin proved that

\[ \lim_{n \to \infty} f_n(x) = \frac{1}{(1+x) \log 2} \]

Further he gave the following inequality

\[ \left| \frac{f_n(x) - \frac{1}{1+x} \cdot \frac{1}{\log 2}}{1} \right| \leq \frac{\lambda \sqrt{n}}{n} \]

(1.6)

where \( \lambda \) and \( \lambda \) are positive constants. This inequality is very useful in obtaining further important inequalities [7] e.g.

\[ \log \left( \frac{1}{r(r-2)} \right) \]

\[ \frac{1}{r(r-1)} e^{-\lambda \sqrt{n} + \frac{1}{1-1}} \]

The second proof of the Gaussian theorem was given by Levy [3] in 1929. He also proved that \( \frac{\sigma_n}{\sigma_n} (q_n \text{ is the denominator of the } n^{th} \text{ convergent of the simple continued fraction}) \) has the same asymptotic distribution function as \( \sigma_n \). Further he [7] studied the behaviour of the geometric mean of the \( \sigma_n \), the \( n^{th} \) root of \( q_n \) and the divergence of the series \( \frac{1}{\sigma_n} \) where \( \sigma_n = a_1 + a_2 + \ldots + a_n \). Khintchine [10], [11] proved all the results mentioned above using the following theorem and its generalised equivalent for any variables.
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(a_k) = \frac{1}{2} \sum_{r=1}^{\infty} f(r) \frac{\log \left( \frac{1 + \frac{1}{r(e+2)}}{r(e+2)} \right)}{\log 2}
\]

Since the function \( f(r) = r \) does not satisfy the conditions of the theorem, one cannot obtain any result about the arithmetic mean of the \( a_k \) from (1.7). However, Khintchine proved the following result for the arithmetic mean:

For each \( \varepsilon > 0 \)

\[
\left\{ \frac{a_1 + a_2 + \cdots + a_n}{n} \geq 1 - \frac{\varepsilon}{\log n} \right\} \to 0 \text{ as } n \to \infty
\]
or in other words \( \frac{a_1 + a_2 + \cdots + a_n}{n} \) converges in probability to 1.

Theorem (1.7) is proved by A. Kneser (12), where \( f(r) \) satisfies less stringent conditions.

The third proof for the Gaussian theorem was given by Benjoys [13], [14] who generalised some results of Khintchine. Further he obtained the following results.

1) \( g_n(x, t) \) tends uniformly to \( \frac{t \log(1+x)}{\log 2} \) as \( n \to \infty \)

where \( g_n(x, t) = \text{rob}(Z_n(u) - x) \) for \( 0 < u < t \).
2) \[ \lim_{n \to \infty} \mathbb{P}(a_n \leq x, a_{n+1} > x) = \frac{\log(x + \frac{1}{A})}{\log 2} \cdot \]

In this connection, the pairs exchanged between Krzyś and
Dremaj [15], [16] are of some interest.

The fourth proof was given by Neubius [17].

The aim of this work is to study some properties of the dis-
tribution functions \( \gamma_n(u) \) defined by (1.3) in Kryzys's work. \( \gamma_n(u) \)
defined in (1.2) can also be defined in the following way:

\[
\gamma_0(u) = \frac{1}{a} \\
\gamma_1(u) = \frac{1}{a} - \left[ \frac{1}{a} \right] \\
\gamma_2(u) = \frac{1}{\gamma_1(u)} - \left[ \frac{1}{\gamma_1(u)} \right] \\
\vdots \\
\gamma_{n+1}(u) = \frac{1}{\gamma_n(u)} - \left[ \frac{1}{\gamma_n(u)} \right] \\
\]

If one writes \( \gamma(u) = \frac{1}{u} = \left[ \frac{1}{u} \right] \),

\[
\gamma_n(u) = \gamma[Z_{n-1}(u)] = \gamma[Z_{n-2}(u)] \cdots \gamma[Z_{1}(u)] = \gamma^{n}(u) \\
\]

or \( \gamma_n(u) \) is obtained by the iteration of the function \( \gamma(u) \). Since
\( \gamma \) is a composition of the \( T \)'s the function \( \gamma(u) \) is studied and its
Laplace transform is obtained in the second chapter.

In the third chapter the properties of \( \gamma_0(x), \gamma_1(x) \) and \( \gamma_2(x) \)
are studied. If one defines the distribution function \( \gamma_n(k) \) as

\[ \mathbb{P}(a_n \leq k), \]

then
\[ F_n(k) = \text{rob} \left( e^{-k} \right) \]
\[ = \text{rob} \left( \left( \frac{1}{u} \right)^{k+1} \right) \]
\[ = \text{rob} \left( \frac{1}{z_{k+1}} \right) \]
\[ = 1 - \left( \frac{1}{u} \right)^{k+1} \]

From \( F_n(x) \) one can calculate \( F_n(k) \) very easily. \( F_1(k), F_2(k), F_3(k) \) and \( F_\infty(k) \) are calculated and their graphs are drawn.

In the fourth chapter all the properties obtained for \( \mathcal{F}_x(x) \) are generalized for \( F_n(x) \). Functions \( b_n(t) \) are defined by the following integral relation:

\[ F_{n+1}(t) = \int_0^\infty \frac{b_n(u) J_1(2u/t)}{(e^u - 1) \sqrt{t}} \, du \quad \text{and} \quad b_0(t) = 1 \]

where \( b_1(x) \) is the Bessel function.

Then \( b_n(t) = \int_0^\infty \frac{b_n(u)}{e^u - 1} \left( 1 - e^{-xt} \right) \, dt \).

In the fifth chapter an expression for the joint probability \( (a_{i,j} = I, a_n = k) \) is obtained and the statistical dependence (defined in chapter 5) of \( a_n \) and \( a_{n+p} \) is tabulated for \( n = 1 \) and \( p = 1, 2 \).

In the last chapter the functions \( F_n(k) \) are studied empirically by taking a sample of 100 random numbers. The empirical distribution functions \( F_1^*(k), F_2^*(k), \ldots, F_n^*(k) \) are tabulated and their graphs are drawn. \( \chi^2 \)-test on the approximation of the empirical distribution functions \( F_n^*(k) \) by \( F_\infty(k) \) is carried out.
Chapter II

The function $T(x)$

As defined in the previous chapter

$$T(x) = \begin{cases} \frac{1}{x} - \left[\frac{1}{x}\right] & \text{for } \frac{1}{k} < x \leq \frac{1}{k-1} \\ 0 & \text{otherwise} \end{cases}$$

$T(x)$ is bounded and continuous except at the points $x = \frac{1}{k}$ where $k$ is an integer. In each of the intervals $\left(\frac{1}{k} < x \leq \frac{1}{k-1}\right)$, $T(x)$ is continuous and behaves there like $\frac{1}{x}$ since $\left\lfloor\frac{1}{x}\right\rfloor$ is constant in each interval.

By definition the Laplace transform of $T(x)$ is

$$\mathcal{L}\{T(x)\} = \int_0^\infty e^{-sx} T(x)dx.$$ Denote it by $\mathcal{L}(s)$

$$\mathcal{L}(s) = \int_0^\infty e^{-sx} T(x)dx$$

$$\mathcal{L}(s) = \int_0^\infty e^{-sx} \left\{\frac{1}{x} - \left[\frac{1}{x}\right]\right\} \, dx,$$

$$\lim_{n \to \infty} \int_0^\infty \frac{1}{x} \left\{\frac{1}{x} - \left[\frac{1}{n}\right]\right\} \, dx,$$ where $n$ is a positive integer.

Dividing the interval $\left(\frac{1}{n}, 1\right)$ into $\left(n-1\right)$ intervals

$$\left(\frac{1}{n}, \frac{1}{n-1}\right), \left(\frac{1}{n-1}, \frac{1}{n-2}\right), \ldots, \left(\frac{1}{2}, 1\right),$$ one can write
\[
(n) = \lim_{n \to \infty} \sum_{k=1}^{n-1} \frac{1}{k} \int_{\frac{1}{k+1}}^{\frac{1}{k}} e^{-ax} \left\{ \frac{1}{x} - \left[ \frac{1}{x} \right] \right\} \, dx
\]

or \( n(n) = \lim_{n \to \infty} S_n \) where

\[
S_n = \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1}
\]

and

\[
u_k = \int_{\frac{1}{k+1}}^{\frac{1}{k}} e^{-ax} \left\{ \frac{1}{x} - \left[ \frac{1}{x} \right] \right\} \, dx
\]

or

\[
u_k = \int_{\frac{1}{k+1}}^{\frac{1}{k}} e^{-ax} \left\{ \frac{1}{x} - k \right\} \, dx
\]

since \( \frac{1}{x} = k \) in the interval \((\frac{1}{k+1}, \frac{1}{k})\),

\[
= \int_{\frac{1}{k+1}}^{\frac{1}{k}} \left( 1 - \frac{\nu}{x} + \frac{\nu^2}{x^2} + \cdots + \frac{(\nu-1)^{r-1}}{x^r} + \cdots \right) \, dx
\]

or

\[
= \sum_{r=0}^{\infty} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{1}{x} \left( 1 - \frac{\nu}{x} \right)^r \, dx
\]

\[
= \log x - kx \left[ \frac{1}{k+1} \right] + \sum_{r=1}^{\infty} \frac{(-1)^r \nu \frac{r!}{r}}{r \cdot r!} \left( \frac{1}{k^r} - \frac{1}{(k+1)^r} - \frac{r}{(k+1)^{r+1}} \right)
\]

\[
S_n = \sum_{k=1}^{n-1} \left[ \log \left( \frac{k^r}{k} \right) - \frac{1}{k+1} + \sum_{r=1}^{\infty} \frac{(-1)^r \nu \frac{r!}{r}}{(r+1)!} \left( \frac{1}{k^r} - \frac{1}{(k+1)^r} - \frac{r}{(k+1)^{r+1}} \right) \right]
\]
\[
\lim_{n \to \infty} a_n = 1 - c + \sum_{r=1}^{\infty} \frac{(-1)^r s^r}{r(r+1)} \left\{ 1 + \frac{1}{r-1} \sum_{k=1}^{r-1} (r-1) \right\}
\]

where \( c \) is Euler's constant and \( \zeta(r) = \sum_{k=1}^{\infty} \frac{1}{k^r} \) where \( \zeta \) is Riemann's zeta function.

Finally

\[
\zeta(s) = (1 - c) + \sum_{r=1}^{\infty} \frac{(-1)^r s^r}{r(r+1)} \left( \frac{1}{r} - \frac{\zeta(r+1)}{r+1} \right) \quad (2.2)
\]

From (2.2) one obtains that

\[
\int_0^1 x^s \zeta(s) dx = \left[ \frac{1}{r} - \frac{\zeta(r+1)}{r+1} \right] = M_r
\]

Let \( \varphi(s) \) be the Stieltjes transform of \( \zeta(x) \).

\[
\varphi(s) = \frac{1}{s} \int_0^1 x^s \zeta(x) dx
\]

\[
= \left[ \frac{1}{s} \int_0^1 x^s (1 - \frac{x}{s} + \frac{x^2}{s^2} \ldots) dx \right]
\]

\[
= \frac{1}{s} \left[ (1-c) - \frac{M_1}{s} + \frac{M_2}{s^2} \ldots + (-1)^n \frac{M_n}{s^n} \ldots \right]
\]

substituting the values for \( M_r \) one obtains

\[
\varphi(s) = \frac{1-c}{s} - \frac{1}{s^2} \left( \frac{\zeta(2)}{2} \right) + \frac{1}{s^3} \left( \frac{\zeta(3)}{3} - \frac{\zeta(2)}{2} \right) \ldots
\]

\[
= \frac{1-c}{s} - \frac{1}{s} \log(1+\frac{s}{2}) + \frac{\zeta(2)}{2s^2} - \frac{\zeta(3)}{3s^3} \ldots + (-1)^r \frac{\zeta(r)}{rs^r} \ldots
\]
Substituting the formula

\[ \mathcal{Z}(n) = \int_0^\infty \frac{x^{n-1}}{(n-1)!} e^{-x} \, dx \]

and

\[ Q(s) = \frac{1 - \zeta}{s} - \frac{1}{s} \log(1 + \frac{1}{s}) + \int_0^\infty \frac{e^{-x} - 1 + \frac{1}{s} (1 - e^{-x})}{x(e^x - 1)} \, dx \]

Substituting

\[ -\zeta = \int_0^\infty \frac{1 - e^{-x} - x}{x(e^x - 1)} \, dx \]

we get

\[ Q(s) = \frac{1}{s} \log \left( \frac{1}{s} \right) + \log \Gamma \left( \frac{1}{s} \right) \]

where \( \Gamma \) is the gamma function.
Chapter 41

Properties of \( u_0(x), u_2(x) \) and \( v_2(x) \)

\( u_0(x) \):

If \( u \) has uniform distribution then \( \text{Prob}(a \leq x) = x \)

in the interval \((0,1]\).

i.e. \( u_0(x) = x \) and therefore

\[ F_1(k) = 1 - \frac{1}{k+1} \text{ by (1.8)} \]

\[ f_1(k) = \text{Prob}(a_1 = k) = F_1(k) - F_1(k-1) = \frac{1}{k(k+1)} \] \( \cdots \) \( (3.1) \)

Since \( F_1(1) = \frac{1}{2} \), \( 1 \) is the median of the distribution function of \( a_1 \).

\[ f_1(1) > f_1(k) \] for all \( k > 1 \) since \( \frac{1}{k(k+1)} \) decreases

as \( k \) increases.

\( . \). \( 1 \) is the mode for the distribution function of \( a_1 \).

\[ E(k^r) = \sum_{k=1}^{\infty} \frac{k^r}{k(k+1)} = \sum_{k=1}^{\infty} \frac{k^{r-1}}{k+1} = \infty \text{ for } r \geq 1 . \]

\( . \). All the moments of \( F_1(k) \) are infinite. However the moments for negative powers of \( k \) do all exist. For example

\[ E(\frac{1}{k}) = \sum_{k=1}^{\infty} \frac{1}{k^2(k+1)} = \frac{\pi^2}{6} - 1 = 0.6366199 . \]

\( v_2(x) \):

\[ v_2(x) = \sum_{k=1}^{\infty} \frac{u_0(x)}{k} - u_0(x) \]
\[ \int_{k=1}^{\infty} \left( \frac{1}{k^2} - \frac{1}{k^{r+1}} \right) = C \cdot \psi(x) \quad \text{... (3.2)} \]

where \( C \) is Euler's constant and

\[ \psi(x) = \frac{d \log \pi}{dx} \quad (x! \text{ stands for } \Gamma(x)) \text{ where } \Gamma \]

is the gamma function.

and \( F_2(r) = 1 - C - \psi \left( \frac{1}{r+1} \right) \text{ by (1.5)} \)

for this

\[ F_2(k) = \text{rob} (a_2 = k) - \psi \left( \frac{1}{k} \right) - \psi \left( \frac{1}{k+1} \right) \]

since \( \psi \left( \frac{1}{k} \right) - \psi \left( \frac{1}{k+1} \right) \text{ is a decreasing function of } k \), \( k \) is the
mode for this distribution function also. But it is not the median
since \( F_2(1) = .333 \) ... From (3.2) one obtains that

\[ m(x) = \int_{1}^{\infty} \frac{1}{(r+1)^2} = \psi'(x) \text{ which shows that } \psi'(x) \]

is decreasing.

again all the moments are infinite since

\[ \mu(k) = \int_{k=1}^{\infty} \left( \frac{1}{k^{r+1}} \right) = \psi \left( \frac{1}{r+1} \right) \]

\[ = \int_{k=1}^{\infty} \frac{k^r}{k^{r+1}} \psi \left( \frac{1}{r+1} \right) \text{ where } \theta = \frac{1}{r+1} \]

\[ > \int_{k=1}^{\infty} \frac{k^{r-1}}{k+1} \psi'(1) \text{ since } \psi(x) \text{ is decreasing.} \]

\[ = \infty \text{ for } r \geq 1 \]
However it can be verified easily that all the moments for negative powers of $k$ exist.

The bounds of $m_1(x)$

Consider

$$m_1(x) = \sum_{k=1}^{\infty} \frac{x^k}{k} \geq \sum_{k=1}^{\infty} \frac{x^k}{k(k+1)} = \sum_{k=1}^{\infty} \frac{x}{k(k+1)}$$

Since $k \leq k+1$, we have

$$\sum_{k=1}^{\infty} \frac{x}{k} \geq m_1(x) \geq \sum_{k=1}^{\infty} \frac{x}{k(k+1)}$$

i.e.

$$E_{m_1(n)} \geq m_1(x) \geq E_{m_2(x)} \quad \ldots \quad (3.3)$$

From (3.3) one obtains that

$$F_1(k) \geq F_2(k) \quad \text{for all } k$$

but

$$f_1(n) > f_2(n)$$

and

$$f_1(k) < f_2(k) \quad \text{for all } k > 1$$

Some properties of $m_1(x)$

Now some important properties of $m_1(x)$ which will be used later are given below. First we have

$$m_1(x) = \int_{0}^{\infty} \frac{1 - e^{-tx}}{e^t - 1} \, dt \quad \ldots \quad (3.4)$$
which can be proved easily.

This formula can be expressed as a finite integral as follows

$$I_1(x) = \int_0^1 \frac{1 - \frac{x}{v}}{1 - v} \, dv \ldots \quad (3.4.1)$$

From (3.4.1), one observes that $I_1(x)$ for rational values of $x$ can be expressed as the sum of a finite number of terms; one has namely:

$$I_1(x) = \log \left( \frac{1}{k} \right) - \frac{1}{2} \sum_{r=1}^{\frac{x}{k}} \cos \frac{2\pi x r}{k} \log \left( 2 - 2 \cos \frac{2\pi x r}{k} \right) \quad (15)$$

Maclaurin expansion of $I_1(x)$

This is given by

$$I_1(x) = x \, \zeta(2) - x^2 \, \zeta(3) + \ldots + (-1)^{m-1} x^m \, \zeta(m+1) + \ldots \quad (3.4.2)$$

where $\zeta$ is again representing Riemann's zeta function.

By differentiating the formula (3.4), one obtains

$$I_1(x) = \int_{0}^{\infty} \frac{e^{-tx}}{e^t - 1} \, dt \ldots \quad (3.5)$$

i.e., $I_1(x)$ is the Laplace transform of $\frac{t}{e^t - 1}$

$I_2(x)$

By definition
\[ n_2(x) = \lim_{k \to \infty} \frac{\psi\left(\frac{1}{k} x\right) - \psi\left(\frac{1}{k+1} x\right)}{1/k - 1/(k+1)} \]

\[ = \lim_{k \to \infty} \sum_{m=2}^{\infty} \left( \frac{1}{k^2} - \frac{1}{(k+1)^2} \right) \]

\[ = \sum_{m=2}^{\infty} \left( \frac{1}{k(k+1)} \right) \]

where \( \psi^{(r)}(x) \) is the \( r \)th derivative of \( \psi(x) \). \( n_2(x) \) can be expressed by a more rapidly convergent series in the following manner:

\[ n_2(x) = \lim_{k \to \infty} \frac{\psi\left(\frac{1}{k} x\right) - \psi\left(\frac{1}{k+1} x\right)}{1/k - 1/(k+1)} \]

\[ = \lim_{k \to \infty} \sum_{r=1}^{\infty} \left\{ \frac{1}{r} \left( \frac{1}{k} - \frac{1}{k+1} \right) \right\} \]

\[ = \lim_{k \to \infty} \sum_{r=1}^{\infty} \left\{ \frac{1}{r} \frac{1}{k} \frac{1}{k+1} \right\} \]

Introducing the order of summation one obtains

\[ n_2(x) = \lim_{k \to \infty} \frac{1}{r^2} \left\{ \sum_{k=1}^{\infty} \frac{1}{r^2} \left( \frac{1}{k(k+1)} \right) \right\} \]

\[ = \lim_{k \to \infty} \frac{1}{r^2} \left\{ \sum_{k=1}^{\infty} \frac{1}{r^2} \left( \frac{1}{k} - \frac{1}{k+1} \right) \right\} \]
\[ n_2(x) = \sum_{r=1}^{\infty} \frac{1}{r^2} \{ \psi(x + \frac{1}{r}) - \psi(\frac{1}{r}) \} \]  \hfill (3.7)

The bounds of \( n_2(x) \)

From the formula (3.6) and the inequality

\[ k + \frac{1}{r} \leq k + x + \frac{1}{r} \leq k + 1 + \frac{1}{r} \]

it follows

\[ \sum_{r=1}^{\infty} \frac{1}{r^2} \sum_{k=1}^{\infty} \frac{x}{(k+\frac{1}{r})^2} \leq \sum_{r=1}^{\infty} \frac{1}{r^2} \sum_{k=1}^{\infty} \frac{x}{(k+\frac{1}{r})(k+\frac{1}{r})} \leq \sum_{r=1}^{\infty} \frac{1}{r^2} \sum_{k=1}^{\infty} \frac{x}{(k+\frac{1}{r})(k+\frac{1}{r})} \]

i.e.

\[ x \cdot n_2(0) \geq n_2(x) \geq n_2(k) \ldots \]  \hfill (3.8)

From this it follows that

\[ n_2(k) \leq n_2(k) \]

However, it can be verified that

\[ f_2(1) < f_2(2) \]

but

\[ f_2(k) > f_2(k) \quad \text{for all } k > 1 \]

From (3.7)

\[ n_2(x) = \sum_{r=1}^{\infty} \frac{1}{r^2} \psi(x + \frac{1}{r}) \]

which shows that

\[ n_2(x) \] is decreasing.

\[ n_2(x) \] as an infinite integral

Using the expression (3.4) in (3.7) one obtains that
\[ z_2(x) = \lim_{k \to \infty} \frac{1}{k^2} \left\{ \int_0^\infty \frac{t}{e^{kt} - 1} dt \right\} \]

\[ = \int_0^\infty \left\{ \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-\frac{tx}{k^2}} \right\} dt \]

[Assumption under the integral sign is justified since the series (3.7) is uniformly convergent.]

\[ z_2(x) = \int_0^\infty R(t) \frac{1 - e^{-tx}}{e^t - 1} dt \ldots \] (3.9)

where \[ R(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-\frac{tk}{k^2}} \ldots \] (3.10)

Since this function \( R(t) \) will be used in obtaining some important formulae in the next section, some properties of this function are derived here.

**Upper and lower bounds for \( R(t) \)**

\( R(t) \) is positive and decreasing for all positive values of \( t \).

\[ 0 < R(t) < \ln(2) \] for \( 0 < t < \infty \).

But by applying a known formula [19] one obtains more precise bounds for \( R(t) \).

\( R(t) \) can be written as

\[ R(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} e^{-\frac{tk}{k^2}} \]

where \[ u_k = \frac{1}{k^2} e^{-\frac{t}{k^2}} \].
It can be verified easily that

\[ U_{k+1} < U_k \quad \text{and} \quad U_k > 0 \quad \text{for all} \quad k \geq 1 \]

Then the theorem gives the result

\[ u_1 + \int_1^\infty u(t) \, dt \geq \int_1^\infty \frac{\sqrt{t}}{e^t} \, dt \]

i.e.

\[ e^{-t} \int_1^\infty \frac{1}{x^2} e^{\frac{x}{t}} \, dx \geq u(t) \geq \frac{1}{t} e^{-t} \int_1^\infty \frac{1}{x^2} \, dx \]

\[ e^{-t} \int_1^\infty \frac{1}{x^2} \, dx \geq u(t) \geq \frac{1}{t} (1-e^{-t}) \quad \ldots \quad (3.11) \]

From these the upper and lower limits for \( U_2(x) \) can be obtained as

\[ \int_0^\infty \frac{1-e^{-tx}}{e^{tx}} \left( \frac{1-e^{-tx}}{e^{tx}} - e^{-t} \right) dt \geq U_2(x) \geq \int_0^\infty \frac{1-e^{-tx}}{e^{tx}} \left( \frac{1-e^{-tx}}{e^{tx}} + e^{-t} \right) dt \]

i.e.

\[ \log(1+x) = \frac{1}{2} \left( \Psi(1+x) - \Psi(1) \right) \geq U_2(x) \geq \log(1+x) \]

\textit{Laplace transform of } \( H(t) \)

\[ H(t) = \int_{-\infty}^{\infty} \frac{1}{k^2} \, e^{-\frac{t}{k}} \, dt \]

\[ = \int_{-\infty}^{\infty} \frac{1}{k} \, e^{-\frac{t}{k}} \, dt \]

\[ = \frac{1}{k} \left[ e^{-\frac{t}{k}} \right]_{-\infty}^{\infty} \]

\[ = \frac{1}{k} \left( \Psi(1) - \Psi(1) \right) \]

\[ = \zeta \left( \frac{1}{s} \right) = \zeta \left( \frac{1}{s} \right) \]
\( \mathcal{L}^{-1} \) means that the right-hand side is the inverse Laplace transform of left-hand side.

\[ R(t) \xrightarrow{\mathcal{L}^{-1}} f_1(\frac{r}{s}) \ldots \]  \hspace{1cm} (3.12)

\( \mathcal{L}^{-1} \) as an infinite integral

From (3.12) \( R(t) \) has its Laplace transform as \( f_1(\frac{1}{s}) \)

or in other words \( f_1(\frac{1}{s}) \) is the inverse transform of \( R(t) \).

Now the inverse transform of \( f_1(\frac{1}{s}) \) is obtained as an infinite integral using the Bessel function \( J_1(x) \) and thereby \( R(t) \) is expressed as an infinite integral.

Consider

\[ f_1(u) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{k! (k+1)!} \frac{k}{2!} \]

and

\[ f_1(2\sqrt{at}) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} a^k t^k}{k! (k+1)!} \]

\[ f_1(2\sqrt{at}) = \sum_{k=0}^{\infty} \frac{(-1)^{k+1} u^k \cos k \phi}{k! (k+1)!} \]

\[ = \frac{1}{2} \left( 1 - \frac{u}{\alpha} \right) \]

\[ a \cdot f_1(2\sqrt{at}) = \frac{1 - \frac{u}{\alpha}}{\alpha (\cos \phi - 1)} \]
\[
\delta(t) = \int_0^\infty \frac{\sin \sqrt{2ut}}{\sqrt{2ut}(\sqrt{u} - 1)} \, du = \frac{\sin \sqrt{2ut}}{\sqrt{2ut}(\sqrt{u} - 1)}
\]

From the above formula and (3.12) one obtains

\[
\delta(t) = \int_0^\infty \frac{\sin \sqrt{2ut}}{\sqrt{2ut}(\sqrt{u} - 1)} \, du = \frac{\sin \sqrt{2ut}}{\sqrt{2ut}(\sqrt{u} - 1)}
\]

From this \( \delta(t) \) can be expressed as a double integral

\[
\delta(t) = \int_0^\infty \frac{1}{\sqrt{2ut}} \int_0^\infty \frac{\sin \sqrt{2ut}}{\sqrt{2ut}(\sqrt{u} - 1)} \, dt \, du.
\]

\[ R(t) \text{ in terms of the Laplace transform of } T(t) \]

From Chapter II

\[
R(s) = \int_0^\infty \frac{1 - e^{-sx}}{s} T(x) \, dx = \sum_{n=0}^{\infty} \frac{(\frac{1}{s})^n}{n!} \mu_n
\]

where \( \mu_n = \int_0^\infty \frac{1}{n!} \frac{T(t)}{n!} \, dt \)

\[ R(t) = \sum_{k=1}^{\infty} \frac{1}{k^2} \, t^k
\]

\[
= \sum_{k=1}^{\infty} \frac{1}{k^2} \left( \sum_{r=0}^{\infty} \frac{(-1)^r t^r}{r!} \frac{t^r}{r!} \right)
\]

\[
= \sum_{r=0}^{\infty} \frac{(-1)^r t^r}{r!} \, j(r+1) \ldots
\]

\[
\int_0^\infty \frac{1}{x} T(x) \, e^{-tx} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \, t^n \, \mu_{n+1}
\]
and \[
\int_0^1 x^2 T(x) e^{-tx} \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^n \mu_{n+2}
\]

\[
\int_0^1 \left( \frac{t x^2 - 2x}{t} \right) T(x) e^{-tx} \, dx = \left[ -2 \left( \zeta(2) \right) + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \right] \left( \frac{2 \mu_{n+1} + \mu_{n+2}}{n} \right) \]

\[
= \left[ -2 \left( \zeta(2) \right) + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \right] \mu_{n+1}
\]

substituting the value of \( \mu_n \)

\[
= \left[ -2 \left( \zeta(2) \right) + \sum_{n=1}^{\infty} \frac{(-1)^n t^n}{n!} \right] \mu_{n+1}
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \left( \zeta(n+2) - \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \right)
\]

\[
= e^{-t} \left( 1 - \frac{1}{t} \right) + R(t)
\]

\[
\Rightarrow \beta(t) = \int_0^1 \left( \frac{t x^2 - 2x}{t} \right) T(x) e^{-tx} \, dx = e^{-t} \left( 1 - \frac{1}{t} \right) \quad (3.15)
\]

Analogous to the treatment of \( \mu_2(x) \), one can easily derive from (3.9) for \( \mu_2(x) \):

\[
\mu_2(x) = \int_0^t \frac{t}{e^t - 1} R(t) \, dt \quad (3.16)
\]

which shows that \( \frac{t}{e^t - 1} R(t) \) is the inverse transform of \( \mu_2(x) \).

From (3.16) and (3.11) one obtains that
\[ \int_0^\infty e^{-st} e^{-tx} \, dx = \int_0^\infty \frac{e^{-t(1-x)}}{st-t+1} \, dx = \frac{1}{2} \int_0^\infty e^{-t} e^{-tx} \, dx. \]

i.e.

\[ \frac{1}{kx} \cdot \Psi'(x;1) \geq \frac{1}{k} \geq \frac{1}{kx} \quad \text{(3.17)} \]

Using Euler's formulas to the infinite series (3.7),

the values of \( F_2(x) \) for \( x = \frac{1}{k} \) are calculated. The details of this calculation and the tabulated values of \( F_2(x) \) are given in Appendix I.

In Table 1, \( F_1(k), F_2(k), F_3(k) \) and \( F_\infty(k) \) are tabulated and in Table 2, \( f_1(k), f_2(k), f_3(k) \) and \( f_\infty(k) \) are tabulated. From these two tables one can observe the following inequalities:

\[ F_2(k) = F_\infty(k) = F_3(k) = F_1(k) \quad \text{for all} \quad k \]

where \( F_\infty(k) = \frac{\log (1 + \frac{k}{2})}{\log 2} \)

and

\[ f_3(k) = f_2(k) \quad \text{for} \quad k = 1 \text{ and } 2 \]

\[ f_3(k) = f_2(k) \quad \text{for} \quad k \geq 3. \]

Table 1 shows that the convergence of \( F_n(k) \) to \( F_\infty(k) \) as \( k \to \infty \)
is very rapid. The graphs of \( F_1(k), F_2(k), F_3(k) \) and \( F_\infty(k) \) are
given in Figure 1. This figure shows the convergence of \( F_n(k) \) to
\( F_\infty(k) \) as \( n \to \infty \) is also rapid. In Figure 2 the graphs of
\( (\gamma(x) - \gamma_\infty(x)), (\gamma_1(x) - \gamma_\infty(x)), (\gamma_2(x) - \gamma_\infty(x)) \) are given.

From Figure 2 one can see that these graphs alternate in sign.
Fig. 1

Distribution functions $F_1(k)$, $F_2(k)$, $F_3(k)$ and $F_{\infty}(k)$ (though these functions are discrete, it is more convenient to draw them as continuous when we are drawing the four curves in the same graph).
<table>
<thead>
<tr>
<th>$k$</th>
<th>$F_1(k)$</th>
<th>$F_2(k)$</th>
<th>$F_3(k)$</th>
<th>$F_\infty(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5800</td>
<td>0.2065</td>
<td>0.4238</td>
<td>0.4159</td>
</tr>
<tr>
<td>2</td>
<td>0.6867</td>
<td>0.5550</td>
<td>0.5940</td>
<td>0.5849</td>
</tr>
<tr>
<td>3</td>
<td>0.7530</td>
<td>0.6505</td>
<td>0.6662</td>
<td>0.6700</td>
</tr>
<tr>
<td>4</td>
<td>0.8200</td>
<td>0.7120</td>
<td>0.7442</td>
<td>0.7269</td>
</tr>
<tr>
<td>5</td>
<td>0.8333</td>
<td>0.7549</td>
<td>0.7842</td>
<td>0.7775</td>
</tr>
<tr>
<td>6</td>
<td>0.8571</td>
<td>0.7870</td>
<td>0.8331</td>
<td>0.8073</td>
</tr>
<tr>
<td>7</td>
<td>0.8750</td>
<td>0.8117</td>
<td>0.8594</td>
<td>0.8336</td>
</tr>
<tr>
<td>8</td>
<td>0.8889</td>
<td>0.8306</td>
<td>0.8528</td>
<td>0.8479</td>
</tr>
<tr>
<td>9</td>
<td>0.9000</td>
<td>0.8468</td>
<td>0.8669</td>
<td>0.8634</td>
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<tr>
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<td>0.8953</td>
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<td>0.9676</td>
<td>0.9725</td>
<td>0.9714</td>
</tr>
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<td>99</td>
<td>0.9900</td>
<td>0.9939</td>
<td>0.9965</td>
<td>0.9957</td>
</tr>
<tr>
<td>$\infty$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
### Table 2

Theoretical Errors of P. C. G. I. M. $f_2(k)$, $f_3(k)$, $f_4(k)$ & $f_\infty(k)$

<table>
<thead>
<tr>
<th>$k$</th>
<th>$f_1(k)$</th>
<th>$f_2(k)$</th>
<th>$f_3(k)$</th>
<th>$f_\infty(k)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.5707</td>
<td>0.3856</td>
<td>0.2638</td>
<td>0.1699</td>
</tr>
<tr>
<td>2</td>
<td>0.1667</td>
<td>0.1685</td>
<td>0.1722</td>
<td>0.1711</td>
</tr>
<tr>
<td>3</td>
<td>0.0933</td>
<td>0.0955</td>
<td>0.0932</td>
<td>0.0933</td>
</tr>
<tr>
<td>4</td>
<td>0.0690</td>
<td>0.0615</td>
<td>0.0690</td>
<td>0.0689</td>
</tr>
<tr>
<td>5</td>
<td>0.0333</td>
<td>0.0429</td>
<td>0.0480</td>
<td>0.0466</td>
</tr>
<tr>
<td>6</td>
<td>0.0238</td>
<td>0.0321</td>
<td>0.0299</td>
<td>0.0298</td>
</tr>
<tr>
<td>7</td>
<td>0.0179</td>
<td>0.0247</td>
<td>0.0273</td>
<td>0.0277</td>
</tr>
<tr>
<td>8</td>
<td>0.0139</td>
<td>0.0193</td>
<td>0.0174</td>
<td>0.0179</td>
</tr>
<tr>
<td>9</td>
<td>0.0111</td>
<td>0.0158</td>
<td>0.0141</td>
<td>0.0145</td>
</tr>
<tr>
<td>$10 \leq k &lt; 15$</td>
<td>0.0333</td>
<td>0.0435</td>
<td>0.0432</td>
<td>0.0445</td>
</tr>
<tr>
<td>$15 \leq k &lt; 50$</td>
<td>0.0467</td>
<td>0.0725</td>
<td>0.0624</td>
<td>0.0645</td>
</tr>
<tr>
<td>$50 \leq k &lt; 100$</td>
<td>0.0540</td>
<td>0.0161</td>
<td>0.0144</td>
<td>0.0143</td>
</tr>
<tr>
<td>$k \geq 100$</td>
<td>0.0200</td>
<td>0.0261</td>
<td>0.0135</td>
<td>0.0143</td>
</tr>
</tbody>
</table>
CHAPTER IV

GENERAL LAW OF \( \tau_n(x) \)

All the properties derived in the previous chapter are generalized for \( \tau_n(x) \) in this chapter. The inverse transform \( \tau_n(t) \) of \( \tau_n(x) \) is obtained and a few properties of \( \tau_n(t) \) are discussed.

(a) Difference equation of \( \tau_n(x) \)

\( \tau_n(x) \) for all \( n \geq 1 \) is defined by the equation

\[
\tau_n(x) = \sum_{k=1}^{\infty} \tau_{n-k}(\frac{1}{k}) - \tau_{n-k+1}(\frac{1}{k+1}) \quad \text{for} \quad 0 \leq x \leq 1.
\]

But even for values \( x > 1 \), the right hand side of the equation is defined since \( \frac{1}{k+x} \) is always less than 1. Therefore one can define \( \tau_n(x) \) for all positive values of \( x \) by the above equation.

Now consider

\[
\tau_{n-1}(x-1) - \tau_{n-1}(x) = \sum_{k=1}^{\infty} \tau_{n-k}(\frac{1}{k} - x) - \tau_{n-k+1}(\frac{1}{k+1} - x)
\]

\[
= \sum_{k=1}^{\infty} \tau_{n-k}(\frac{1}{k} - x) \lim_{k \to \infty} \tau_{n-k+1}(\frac{1}{k+1} - x)
\]

\[
= \tau_{n}(\frac{1}{1-x}) \quad \text{since} \quad \tau_{n}(0) = 0
\]

\[
\tau_{n-1}(x-1) - \tau_{n-1}(x) = \tau_{n}(\frac{1}{1-x}) \quad (4.1)
\]

If one denotes by \( \tau(x) = \lim_{n \to \infty} \tau_n(x) \) it follows from (4.1) that
\( F(x + 1) - F(x) = F\left(\frac{1}{x+1}\right) \)

or

\( Q(x) = -Q\left(\frac{1}{x}\right) \) where \( Q(x) = F(l-x) \).

As mentioned earlier in Chapter 1 the function \( Q(x) = \frac{1}{2} \log x \) satisfies this equation. [Note \( Q(x) \) need not be \( x \)].

Expected values of \( n \)

\[ F_n(k) = 1 - \sum_{n=1}^{k} \left(\frac{1}{k+1}\right) \]

and therefore \( f_n(k) \) is the absolute value of \( n \)

\[ f_n(k) = \text{prob}(n = k) = F_{n-1}\left(\frac{1}{k}\right) - F_{n-1}\left(\frac{1}{k+1}\right) \]

\[ E(k^r) = \sum_{k=1}^{\infty} k^r \left( F_{n-1}\left(\frac{1}{k}\right) - F_{n-1}\left(\frac{1}{k+1}\right) \right) \]

\[ = \sum_{k=1}^{\infty} \frac{k^r}{k(k+1)} \left( F_{n-1}\left(\frac{1}{k}\right) - F_{n-1}\left(\frac{1}{k+1}\right) \right) \]

\[ > \sum_{k=1}^{\infty} \frac{k^r}{k(k+1)} \left( F_{n-1}\left(\frac{1}{k}\right) - F_{n-1}\left(\frac{1}{k+1}\right) \right) \]

\[ = \infty \quad \text{for } r > 1. \]

Similarly, it can be verified that all the moments for negative values of \( k \) exist.

(1) Upper and lower bounds for \( n \)

According to the laws of the mean \([7]\) if

\[ \frac{1}{1-x} < n(x) < \frac{x}{1-x} \]

then
\[
\frac{t}{1-x} < m_{n+1}(x) < \frac{T}{1-x}
\]
or if
\[
\frac{t}{1-x} < m_0(x) < \frac{T}{1-x}
\]
them \[
\frac{t}{1-x} < m_n(x) < \frac{T}{1-x} \quad \text{for all } n.
\]
in case \( m_0(x) = x \), \( m_1(x) = 1 \) and therefore
\[
\frac{1}{1-x} \leq m_0(x) \leq \frac{2}{1-x}
\]
\[
\frac{1}{1-x} \leq m_n(x) \leq \frac{2}{1-x}
\]
From this it follows that
\[
\log(1+x) \leq m_n(x) \leq 2 \log(1+x).
\]
For \( F_n(x) \), one considers values in \((0, \frac{1}{2})\) for that interval
\[
\log(1+x) \leq F_n(x) \leq \frac{3}{2} \log(1+x) \quad \text{for } 0 \leq x \leq \frac{1}{2}
\]
\[
x \geq \frac{m_0(0)}{m_0(x)} \geq x
\]
This result is proved for \( n = 1 \) and \( 2 \) in Chapter 3.

First an expression for \( m_n(x) \) is obtained and from that the above inequality is obtained.
\[
m_0(x) = 1
\]
\[
m_1(x) = \frac{\infty}{\sum_{k=1}^{\infty} \frac{1}{(k-x)^2}}
\]
\[ u_n(x) = \sum_{k_1=1}^{\infty} \frac{1}{f(k_1, k_1+1)} \sum_{k_2=1}^{\infty} \frac{1}{f(k_2, k_1+1) f(k_1+1, k_1)} \]

There it can be proven by induction that

\[ u_n(x) = \prod_{k=1}^{n} \frac{1}{f(k_1, k_1+1) \cdots f(k_{n-1}, k_n) f(k_n, k_1)} \]

where \( f(k_1) = 1 \)

\[ f(k_2, k_1) = k_2 k_1 + 1 \]

\[ \vdots \]

\[ f(k_k, k_1) = k f(k_{k-1}, k_1) + f(k_{k-2}, k_1) \]

Those \( f(k_n, k_{n-2}, \ldots, k_1) \) can be identified with the denominators \( q_n \) of the convergents of a simple continued fraction. As therefore write here \( q_n \) instead \( f(k_n, k_1) \).

\[ \begin{align*}
\frac{1}{n(x) = 2} & \frac{1}{2} \frac{1}{q_n \cdots q_{n-1} L} \\
\end{align*} \quad (4.1) \]

where \( L \) means the summation over all possible values of \( q_n \) and \( q_{n-1} \). From (4.1) one can observe easily that \( u_n(x) \) is a decreasing function of \( x \) for all \( n \).

Integrating (4.1) from 0 to \( x \) one has

\[ \frac{1}{n(x) = 2} \frac{x}{c_n (c_n \cdots c_{n-1})} \]

Since \( u_n(1) = 1 \).
\[
\frac{1}{n_n(n_n + q_{n-1})} = 1
\]

\[n_n(x) = \frac{1}{2} \frac{x}{q_n(q_n + q_{n-1})} \geq \frac{1}{2} \frac{x}{q_n(q_n + q_{n-1})} = x
\]

Similarly, \[\frac{1}{n_n(x)} = \frac{x}{2q_n} = x \cdot \frac{1}{n_n(0)} \]

\[\therefore x \cdot \frac{1}{n_n(0)} \geq x \cdot \frac{1}{n_n(x)} \geq x
\]

\[\frac{1}{n(0)} = \frac{\frac{1}{2}}{q_n} \geq \frac{\frac{1}{2}}{q_n(q_n + q_{n-1})} = 1
\]

and \[\frac{1}{n(1)} = \frac{\frac{1}{2}}{(q_n^2 q_{n-1})^2} > \frac{\frac{1}{2}}{q_n(q_n + q_{n-1})} = 1
\]

\[\therefore \frac{1}{n(0)} > 1
\]

and \[\frac{1}{n(1)} < 1
\]

\[\frac{1}{n(\frac{1}{2})} = \frac{\frac{1}{2}}{(q_n^2 q_{n-1})^2} < 1
\]

Consider

\[q_n + q_{n-1}^2 = q_n^2 + 2q_nq_{n-1} + q_{n-1}^2
\]

Now

\[q_n^2 + 2q_nq_{n-1} + q_{n-1}^2 = q_n(q_n + q_{n-1}) = 2q_nq_{n-1} + q_{n-1}^2
\]

\[= 2q_nq_{n-1} + x^2q_{n-1}^2 = q_nq_{n-1}
\]

\[= q_n(2x - 1) + x^2q_{n-1}^2
\]

\[\frac{1}{n(0)} = \frac{x}{2q_n} \geq \frac{x}{\sqrt{2}q_n} = x \cdot \frac{1}{\sqrt{2} - 1}
\]

if \((x - 1)^2 < 2
\]

or \(x < \sqrt{2} - 1\)
\[ u_n(x) = \frac{1}{2} \frac{1}{(q_{n-1}^2 x q_{n-1})^2} > \frac{1}{2} \frac{1}{q_n(q_{n-1}^2 q_{n-1})} \quad \text{for} \quad 0 \leq x \leq \left(\frac{1}{2}\right) \]

or \[ u_n(x) > 1 \quad \text{for} \quad 0 \leq x \leq \left(\frac{1}{2}\right) \]

and \[ u_n(x) < 1 \quad \text{for} \quad \frac{1}{2} \leq x \leq 1 \]

Since \( u_n(x) \) is decreasing and continuous, there exists a constant \( c_n \) such that

\[ u_n(c_n) = 1 \quad \text{for} \quad \left(\frac{1}{2}\right) < c_n < \frac{1}{2} \]

i.e. \( \left(\frac{1.42}{5}\right) < c_n < \frac{1}{2} \)

From this follows that

\[ f_{n+1}(k) = u_n\left(\frac{1}{k}\right) - u_n\left(\frac{1}{k+1}\right) \]

\[ = \frac{1}{k(k+1)} u_n\left(\frac{1}{k}\right) \]

\[ < \frac{1}{k(k+1)} \quad \text{for} \quad k = 1 \]

and \[ f_{n+1}(k) > \frac{1}{k(k+1)} \quad \text{for} \quad k \geq 3 \cdot \]

By direct substitution, one can verify that

\[ f_{n+1}(2) > \frac{1}{k(k+1)} \]

\[ \therefore f_{n+1}(1) < f(1) \quad \text{for all} \quad n \geq 1 \]

\[ f_{n+1}(k) > f_1(k) \quad \text{for all} \quad k \geq 2 \quad \text{and for all} \quad n \geq 1 \cdot \]
Since $n_n(x)$ is decreasing

$$n_n(1) \leq n_n(x) \leq n_n(0)$$

$$\sum_{k=1}^{\infty} \frac{1}{(k+1)^2} n_n(1) \leq \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} n_n(0) \leq \sum_{k=1}^{\infty} \frac{1}{(k+1)^2} n_n(x)$$

i.e. $n_n(1) \psi(x) \leq n_n(x) \leq n_n(0) \psi(x)$

From this follows that

$$n_n(1) \gamma(x) \leq n_n(x) \leq n_n(0) \gamma(x)$$

(C) Expressions for $n_n(x)$

An expression for $n_n(x)$ similar to (3.7) will be derived here

$$n_{n+1}(x) = \frac{(n+1)}{\sum_{n+1}^{\infty} (q_{n+1} x q_n)}$$

$$= \frac{(n+1)}{\sum_{n+1}^{\infty} \frac{1}{q_n}} \left[ \frac{1}{q_{n+1}} - \frac{1}{q_{n+1}^2 q_n^2} \right]$$

$$= \frac{(n+1)}{\sum_{n+1}^{\infty} \frac{1}{q_n}} \left[ \frac{1}{k q_{n+1}} - \frac{1}{k q_{n+1}^2 q_n^2} \right]$$

$$= \frac{(n)}{\sum_{k=1}^{\infty} \frac{1}{q_n}} \left[ \frac{1}{q_{n+1}^2 q_n^2} - \frac{1}{k q_{n+1}^2 q_n^2} \right]$$

$$= \frac{(n)}{\sum_{k=1}^{\infty} \frac{1}{q_n}} \left[ \frac{1}{k q_{n+1}^2 q_n^2} - \frac{1}{k q_{n+1}^2 q_n^2} \right]$$
\[(a+b)\]

Using the Maclaurin expansion of \(\phi_n(x)\), \(\phi_{n+1}(x)\) can be written in the following way:

\[\phi_{n+1}(x) = \sum_{k=1}^{\infty} \frac{\phi_n^{(k)}(0)}{k!} x^k \cdot \frac{1}{k!} \frac{1}{(k-1)!} = \sum_{k=1}^{\infty} \frac{\phi_n^{(k)}(0)}{k!} \frac{1}{(k-1)!} \frac{1}{2^k} \frac{1}{k!} \frac{1}{(k-1)!} \frac{1}{2^k} \ldots \]

where \(\phi_n^{(k)}(x)\) is the \(k\)th derivative of \(\phi_n(x)\).

(D) Inverse transform of \(\phi_n(s)\)

In Chapter III, it is known that the function \(R(t)\) satisfies the property that\(R(t) = \int_0^\infty \frac{1 - e^{-tx}}{t} \, R(t) \, dt.\)

\(R_n(t)\) is defined where \(R_n(t)\) satisfies the following equation:

\[R_{n+1}(x) = \int_0^\infty \frac{R_n(t)(1 - e^{-tx})}{e^t - 1} \, dt.\]

[Defi-]

Let \(R_0(t) = 1\)

\[R_1(t) = \int_0^\infty \frac{J_1(2\sqrt{t})}{\sqrt{2\sqrt{t}} e^t - 1} \, dt.\]
Define $R_{n+1}(t) = \int_0^\infty \frac{R_n(u) \cdot u \cdot J_1(2|ut|)}{\sqrt{ut} (e^u - 1)} \, du$.

$R_1(t)$ is the function $R(t)$ defined in Chapter III. $R_n(t)$ exists for all $n$.

Now

\[ \psi_n(x) = \int_0^\infty \frac{R_n(t) \left(1 - e^{-tx}\right)}{e^t - 1} \, dt \ldots \quad (4.5) \]

**Proof:** For $n = 0$ and $n = 1$ (4.5) is true. Let it be true for $n$. Then

\[ \psi_n(x) = \int_0^\infty \frac{R_{n-1}(t) \left(1 - e^{-tx}\right)}{e^t - 1} \, dt \]

Consider

\[ \psi_{n+1}(x) = \sum_{k=1}^{\infty} \int_0^\infty \frac{R_{n-1}(t)}{(e^t - 1)} \left( e^{-\frac{t}{k+x}} - e^{-\frac{t}{k}} \right) \, dt \]

\[ = \int_0^\infty \sum_{k=1}^{\infty} \frac{R_{n-1}(t)}{(e^t - 1)} \left( e^{-\frac{t}{k+x}} - e^{-\frac{t}{k}} \right) \, dt \ldots \quad (4.6) \]

\[ = \sum_{k=1}^{\infty} e^{-\frac{t}{k+x}} - e^{-\frac{t}{k}} \]

\[ = \sum_{k=1}^{\infty} \left( e^{-\frac{t}{k+x}} - \frac{t}{k} e^{-\frac{t}{k+x}} - \frac{t^2}{2!} \frac{1}{k^2} (e^{-\frac{t}{k+x}} - \frac{1}{k}) + \ldots \right) \]

\[ = t[\psi(x) - \psi(0)] + \frac{t^2}{2!} [\psi'(x) - \psi'(0)] + \frac{t^3}{3!} [\psi''(x) - \psi''(0)] + \ldots \]

\[ + \ldots + \frac{t^n}{n!(n-1)!} [\psi^{(n-1)}(x) - \psi^{(n-1)}(0)] + \ldots \]

But

\[ \psi(x) - \psi(0) = \int_0^\infty \frac{1 - e^{-ux}}{e^u - 1} \, du \]
\[ \psi'(x) - \psi'(0) = \int_0^\infty u(1 - e^{-ux}) \frac{du}{e^u - 1} \]

\[ \vdots \]

\[ \psi^n(x) - \psi^n(0) = \int_0^\infty \frac{(-1)^n u^n(1 - e^{-ux})}{(e^u - 1)} \, du \]

Using these formulæ

\[ \int_0^\infty \frac{t}{e^{tx} - 1} \, dt = \sum_{k=1}^\infty \frac{t^k}{k! (k+1)!} \]

\[ = \int_0^\infty \frac{1 - e^{-ux}}{e^u - 1} \left\{ \sum_{k=0}^\infty \frac{(-1)^k t^k u^k}{k!} \right\} \, du \]

\[ = \int_0^\infty \frac{(1 - e^{-ux})u J_1(2/\sqrt{ut})}{(e^u - 1) \sqrt{ut}} \, du \]

Substituting this in the formula (4.6)

\[ \psi^{n+1}(x) = \int_0^\infty \frac{R_{n-1}(t)}{e^t - 1} \left\{ \int_0^\infty \frac{(1 - e^{-ux})u J_1(2/\sqrt{ut})}{(e^u - 1) \sqrt{ut}} \, du \right\} \, dt \]

\[ = \int_0^\infty \frac{R_{n-1}(t) t (1 - e^{-ux}) J_1(2/\sqrt{ut})}{(e^t - 1) (e^u - 1) \sqrt{ut}} \, du \, dt \]

Interchanging the order of integration

\[ = \int_0^\infty \frac{R_{n-1}(t) t (1 - e^{-ux}) J_1(2/\sqrt{ut})}{(e^t - 1) (e^u - 1) \sqrt{ut}} \, dt \, du \]
\[
= \int_0^\infty \frac{1 - e^{-ux}}{e^u - 1} \left\{ \int_0^\infty \frac{R_{n-1}(t) \, t \, e^{2t/ut}}{(e^t - 1) \, \sqrt{ut}} \, dt \right\} \, du
\]

\[
= \int_0^\infty \frac{1 - e^{-ux}}{e^u - 1} R_n(u) \, du \quad \text{from the definition of } R_n(u)
\]

\[
R_n(x) = \int_0^\infty \frac{1 - e^{-ux}}{e^u - 1} R_{n-1}(u) \, du \quad \text{for all } n \geq 1
\]

Now from the difference equation (4.1) and the above formula

\[
R_n(x) = \int_0^\infty \frac{R_n(u) \, (e^{-ux} - e^{-u(x+1)})}{e^u - 1} \, du
\]

\[
= \int_0^\infty R_n(u) \, e^{-u(x+1)} \, du
\]

i.e., \[ R_n\left(\frac{1}{x}\right) = \int_0^\infty R_n(u) e^{-ux} \, du \] which shows that \[ R_n(u) \] is the inverse transform of \[ R_n\left(\frac{1}{x}\right) \]

Again from (4.5) one obtains

\[
R_n(x) = \int_0^\infty \frac{R_{n-1}(t) \, t \, e^{-tx}}{(e^t - 1)} \, dt \quad \ldots \quad (4.6)
\]

\[ R_n(t) \] as an infinite sum

In Chapter III \[ R_1(t) \] is expressed as the sum of an infinite series. Now from (4.6) \[ R_n(t) \] is expressed as an infinite sum.

\[
R_n(x) = \sum_{k=0}^{n-1} \frac{1}{k! \cdot n!} \quad \text{from (4.7)}
\]

\[ R_n(x) \] is the transform of the function.
\[
\begin{align*}
\sum_{n=1}^{\infty} \frac{q_n}{q_{n-1}} t \\
= \sum_{k=1}^{\infty} t e^{\frac{-q_{n-2}}{q_{n-1}}} t \\
= \sum_{k=1}^{\infty} \frac{t}{(e^{t-1})} \sum_{n=1}^{\infty} e^{\frac{-q_{n-2}}{q_{n-1}}} t \\
= \frac{t}{(e^{t-1})} \sum_{n=1}^{\infty} e^{\frac{-q_{n-2}}{q_{n-1}}} t \\
\therefore R_{n-1}(t) = \sum_{n=1}^{\infty} e^{\frac{-q_{n-2}}{q_{n-1}}} t \\
\end{align*}
\]

Upper and lower bounds for \( R_n(t) \)

If \( \frac{c_1(1 - e^{-t})}{t} < R_n(t) < \frac{c_2(1 - e^{-t})}{t} \) then

\[
\frac{c_1(1 - e^{-t})}{t} < R_{n+1}(t) < \frac{c_2(1 - e^{-t})}{t} 
\]

Consider

\[
R_{n+1}(t) = \int_0^\infty R_n(u) \frac{u}{\sqrt{u^3 (e^u - 1)}} \, du 
\]

\[
< \int_0^\infty \frac{c_2(1 - e^{-u})}{u} \frac{u}{\sqrt{u^3 (e^u - 1)}} \, du 
\]
\[ \int_0^\infty \frac{1}{u} \frac{1}{|u|} e^{-u} \, du = \frac{1}{2} \frac{1 - e^{-t}}{t}. \]

Similarly the other inequality also follows. Since
\[ \frac{1 - e^{-t}}{t} < n(t) < 2 \frac{1 - e^{-t}}{t} \]
by induction \( \frac{1 - e^{-t}}{t} < n_n(t) = 2 \frac{1 - e^{-t}}{t} \).
CHAPTER V

STATISTICAL DEPENDENCE OF $a_i$'s

Continued fractions and systematic fractions have certain properties in common. Both are used in approximating real numbers by rational numbers. However, the continued fractions give better and simpler approximations than the systematic fractions. On the other hand, systematic fractions are more convenient to use in arithmetical calculations (for example, the sum of two systematic fractions can be easily calculated). From the Appendix it can be seen that the distribution functions of the index numbers of a systematic fraction are much simpler than those of the simple continued fraction. In particular, where $a$ is uniformly distributed in $(0, 1)$, the $a_n(a)$'s (index numbers of $a$) of a systematic fraction are statistically independent. This was proved by Borel in 1909 for a binary fraction.

i.e. If $a$ is uniformly distributed and

$$a = \frac{a_1(a)}{2} + \frac{a_2(a)}{2^2} + \cdots + \frac{a_n(a)}{2^n} + \cdots$$

where $a_n(a) = 0$ or 1 for all $n$, then

$$\text{Prob} \left\{ a_n(a) = k ; a_m(a) = l \right\} = \text{Prob} \left\{ a_n(a) = k \right\} \cdot \text{Prob} \left\{ a_m(a) = l \right\}$$

for all $m$ and $n$.

In general for any systematic fraction with root 'g', the
\( a_n(a)'s \) are statistically independent if \( a \) is uniformly distributed [20].

Now considering the problem of independence for continued fractions, it can be shown that \( a_n \) and \( a_{n-1} \) of a simple continued fraction are not independent. Before obtaining the general expression for \( \text{prob}(a_n = k, a_{n-1} = r) \), expressions for \( \text{prob}(a_1 = k, a_2 = r) \), \( \text{prob}(a_2 = k, a_3 = r) \) and \( \text{prob}(a_3 = r, a_1 = k) \) are obtained.

\[ \text{prob}(a_1 = k, a_2 = r) \]

\[ \text{prob}(a_1 = k, a_2 = r) = \frac{1}{k^{1/2}} \left( \frac{1}{kr^{1/2}} \right) = \frac{1}{k^{1/2} r^{1/2}} \]  

(5.1)

[\( a_1 = k \) implies that \( a \) must be in \( \left( \frac{1}{k}, \frac{1}{k^{1/2}} \right) \) and \( a_2 = r \) determines the sub-interval \( \left( \frac{1}{k^{1/2} r^{1/2}}, \frac{1}{k^{1/2} r} \right) \).

Now

\[ \text{prob}(a_1 = k) \cdot \text{prob}(a_2 = r) = \frac{1}{k(k+1)} \left( \psi \left( \frac{1}{k} \right) - \psi \left( \frac{1}{k^{1/2} r} \right) \right) \]  

(5.2)

From Chapter III.

If \( a_1 \) and \( a_2 \) are independent (5.1) and (5.2) must be equal

for all pairs of values of \( k \) and \( r \). For \( k = 1, r = 1 \)

\[ (5.1) = \frac{1}{6} = \cdot1666 \ldots \]

and \[ (5.2) = \frac{1}{2} = \cdot333 = \cdot193 \]

Therefore \( a_1 \) and \( a_2 \) are not independent.

\[ \text{prob}(a_2 = k, a_3 = r) \]
\[ \text{prob}(a_1 = k, a_2 = r, a_3 = r) = \frac{1}{k+\frac{1}{r+1}} = \frac{1}{k-\frac{1}{r+1}} . \]

From this

\[ \text{prob}(a_2 = k, a_3 = r) = \sum_{m=1}^{\infty} \frac{1}{k+\frac{1}{r+1}} = \frac{1}{k-\frac{1}{r+1}} . \]

\[ = \lambda_1(\frac{1}{k-\frac{1}{r+1}}) = \lambda_1(\frac{1}{k-\frac{1}{r+1}}) \text{ since} \]

\[ \sum_{m=1}^{\infty} \frac{1}{k-\frac{1}{r+1}} = \lambda_0(\frac{1}{k-\frac{1}{r+1}}) = \lambda_0(y) = \lambda_1(x) . \]

Therefore

\[ \text{prob}(a_2 = k, a_3 = r) = \psi(\frac{1}{k-\frac{1}{r+1}}) - \psi(\frac{1}{k-\frac{1}{r+1}}) \quad (5.3) \]

\[ \text{prob}(a_2 = k) \cdot \text{prob}(a_3 = r) = \left\{ \psi(\frac{1}{k}) - \psi(\frac{1}{k+1}) \right\} \left\{ \lambda_2(\frac{1}{r+1}) - \lambda_2(\frac{1}{r+1}) \right\} \quad (5.4) \]

Again for \( k = 1, r = 1 \)

\[ (5.3) = 0.1443 \]

and

\[ (5.4) = 0.16405 \]

Therefore \( a_2 \) and \( a_3 \) are not independent.

\[ \text{prob}(a_1 = k, a_3 = r) \]

\[ \text{prob}(a_1 = k, a_2 = m, a_3 = r) = \frac{1}{k-\frac{1}{m+\frac{1}{r+1}}} = \frac{1}{k-\frac{1}{m+\frac{1}{r+1}}} \]

From this
\[ \text{prob}(a_1 = k, a_2 = r) = \sum_{n=1}^{\infty} \frac{1}{k \cdot \frac{1}{r} + 1} = \frac{1}{k \cdot \frac{1}{r} + 1} \]

\[ = \sum_{n=1}^{\infty} \frac{\frac{1}{k \cdot \frac{1}{r} + 1}}{k \cdot \frac{1}{r} + 1} = \sum_{n=1}^{\infty} \frac{1}{(n+1)(k+1)} \]

\[ = \sum_{n=1}^{\infty} \frac{1}{k} \left[ \frac{1}{n+1} - \frac{1}{n} \right] = \frac{1}{k} \left\{ \psi\left(\frac{1}{r} + \frac{1}{k}\right) - \psi\left(\frac{1}{r+1} + \frac{1}{k}\right) \right\} \quad (5.5) \]

\[ \text{prob}(a_1 = k) \cdot \text{prob}(a_2 = r) = \frac{1}{k(k+1)} \left[ \psi\left(\frac{1}{r} + \frac{1}{k}\right) - \psi\left(\frac{1}{r+1} + \frac{1}{k}\right) \right] \quad (5.6) \]

For \( k = 1, r = 1 \)

(5.5) = 0.2199

and

(5.6) = 0.2145

Therefore \( a_1 \) and \( a_2 \) are not independent.

\[ \text{prob}(a_n = k, a_{n+1} = r) \]

Remembering the definition of \( Z_n \),

\[ \text{prob}(a_{n+1} = r, a_n = k) = \text{prob} \left\{ \left[ \frac{1}{Z_n} = r, a_n = k \right] \right\} \]

\[ = \text{prob} \left\{ \left[ r \leq \frac{1}{Z_n} < r+1 ; a_n = k \right] \right\} \]
\[= \text{rob} \left\{ \frac{1}{r} < \frac{1}{a_n} \leq \frac{1}{r}; a_n = k \right\} \]

But \( z_n = \frac{1}{\frac{1}{n}} = \frac{1}{\frac{1}{a}} = a_n \), so we have

\[= \text{rob}(a_n = k; a_n = r) = \text{rob} \left\{ \frac{1}{r} < \frac{1}{\frac{1}{a_n}} \leq a_n \leq \frac{1}{r}; a_n = k \right\} \]

\[= \text{rob} \left\{ \frac{1}{r} < \frac{1}{\frac{1}{a_n}} \leq k \leq \frac{1}{r} \right\} \]

\[= \text{rob} \left\{ \frac{1}{k+\frac{1}{r}} \leq z_n \leq \frac{1}{k+\frac{1}{r}} \right\} \]

\[\mathbb{P}_{n-1}(\frac{1}{r}) = \mathbb{P}_{n-1}(\frac{1}{r}) \quad (5.7)\]

\[= \mathbb{P}(a_n = k) = \mathbb{P}_{n-1}(\frac{1}{k}) = \mathbb{P}_{n-1}(\frac{1}{k+1}) \mathbb{P}_{n}(\frac{2}{r}) = \mathbb{P}_{n}(\frac{1}{r+1}) \quad (5.8)\]

Now from Fussin's theorem

\[n \to \frac{\log(1+x)}{\log 2} \quad \text{uniformly for } 0 < x < 1 \quad \text{as } n \to \infty\]

Therefore for large values of \( n \)

\[= \text{rob}[a_n = r; a_n = k] = \mathbb{P}(1+\frac{1}{k+\frac{1}{r}}) = \mathbb{P}(1+\frac{1}{k+\frac{1}{r}}) \frac{\log(1+\frac{1}{k+\frac{1}{r}})}{\log 2} \quad (5.9)\]

and

\[= \text{rob}(a_n = r) \cdot \text{rob}(a_n = k) = \frac{\log(1+\frac{1}{k+2})}{\log 2} \frac{\log(1+\frac{1}{r+2})}{(\log 2)^2} \quad (5.10)\]
For $k = 1 \quad r = 1$

(5.9) \quad = \cdot15185$

and

(5.10) \quad = \cdot17189$

Therefore even for large values of $n$, $a_n$ and $a_{n+1}$ are not independent.

By the same argument as in (5.7) it can be obtained that

\[ \text{Prob}(a_{n+1} \geq r \land a_n \geq k) = \frac{\log(1 - \frac{1}{k + r})}{\log 2} \]

and

\[ \text{Prob}(a_{n+1} \geq r) \cdot \text{Prob}(a_n \geq k) = \frac{\log(1 - \frac{1}{k + r}) \cdot \log(1 - \frac{1}{k + r})}{(\log 2)^2} \]

which are in general not equal.

By calculations to those given above one can show more generally that $a_n$ and $a_m$ are not independent for any values of $n$ and $m$. Now to have an idea of the dependence between $a_n$ and $a_m$, a measure of dependence can be defined in the following manner.

Dependence between the relations $(a_n = k)$ and $(a_m = r)$

\[ D_{n,m}(k, r) = \left| \frac{\text{Prob}(a_n = k, a_m = r)}{\text{Prob}(a_n = k) \cdot \text{Prob}(a_m = r)} - 1 \right| \]

If $a_n$ and $a_m$ are independent $D_{n,m}(k, r) = 0$ for every pair $k$ and $r$;

and conversely also. Therefore the magnitude $D_{n,m}(k, r)$ gives a measure for dependence. $D_{n,m}(k, r)$ for $(n=1, m=2)$, $(n=1, m=3)$ and $D_{n,n+1}(k, r)$ for large values of $n$ are calculated and tabulated (tables 3, 4, 5 at the end of this chapter) using the formulas obtained.
Expression for the joint \( \text{prob}(a_{n+1} = r, a_n = k) \)

Joint \( \text{prob}(a_1 = k_1, a_2 = k_2 \ldots a_n = k_n) \)

\[
= \kappa_0 \left( \frac{1}{k_1 + \frac{1}{k_2 + \cdots + \frac{1}{k_n}}} \right) - \kappa_0 \left( \frac{1}{k_1 + \frac{1}{k_2 + \cdots + \frac{1}{k_n}}} \right) \tag{5.11}
\]

\[
= \frac{p_n}{q_n} - \frac{p_n \cdot p_{n-1}}{q_n \cdot q_{n-1}} \quad \text{where}
\]

\[
\frac{p_n}{q_n} \quad \text{is the } n^{\text{th}} \text{ convergent of } \frac{1}{k_1 + \frac{1}{k_2 + \cdots + \frac{1}{k_n}}}
\]

Joint \( \text{prob}(a_1 = k_1, a_2 = k_2 \ldots a_n = k_n) \)

\[
= \frac{1}{q_n(q_n + q_{n-1})} \tag{5.12}
\]

Conditional \( \text{prob}(a_{n+1} = k | a_1 = k_1, a_2 = k_2 \ldots a_n = k_n) \)

\[
= \frac{\text{prob}(a_1 = k_1, a_2 = k_2 \cdots a_{n-1} = k_{n-1}, a_n = k)}{\text{prob}(a_1 = k_1, a_2 = k_2 \cdots a_n = k_n)}
\]

\[
= \frac{q_n(q_n + q_{n-1})}{(q_n + q_{n-1})(q_n + q_{n-1} + q_n q_{n-1})} \tag{5.13}
\]
and \[ \text{prob}(a_{n+1} = k) = \frac{1}{(q_{n+1} + k q_n)(q_{n+1} + k q_{n-1})} \] (5.14)

Before obtaining the general expression for \( \text{prob}(a_{n+1} = r, a_n = k) \)

Joint \( \text{prob}(a_{n+1} = r, a_n = k) \) is derived

\[ \text{prob}(a_{n+2} = r, a_{n+1} = k_{n+1}, a_n = k_n \ldots a_1 = k_1) \]

\[ = \frac{1}{(q_n + r q_{n+1})(q_n + r q_{n+1} + q_{n-1})} \text{ from (5.12)} \]

Substituting \( q_{n+1} = k_{n+1} q_n + q_{n-1} \); one obtains

\[ = \frac{1}{(r k_{n+1} + 1)q_n + r q_{n+1} + (r k_{n+1} + 1)q_n + (r+1)q_{n-1}} \]

\[ = \frac{1}{q_n^2 (r)(r+1)} \left( k_{n+1} + \frac{1}{r} + \frac{q_{n-1}}{q_n} \right) \left( k_{n+1} + \frac{1}{r} + \frac{q_{n-1}}{q_n} \right) \]

\[ = \frac{1}{q_n^2} \left[ k_{n+1} + \frac{1}{r} + \frac{q_{n-1}}{q_n} \right] = k_{n+1} + \frac{1}{r} + \frac{q_{n-1}}{q_n} \]

Now let \( k_{n+1} \) assume all values from 1 to \( \infty \). Then

\[ \text{prob}(a_{n+2} = r, a_1 = k_1, a_2 = k_2 \ldots a_n = k_n) \]

\[ = \sum_{k_{n+1}=1}^{\infty} \frac{1}{q_n^2} \left[ k_{n+1} + \frac{1}{r} + \frac{q_{n-1}}{q_n} \right] = k_{n+1} + \frac{1}{r} + \frac{q_{n-1}}{q_n} \]
\[ = \frac{1}{q_n^k} \left\{ \psi \left( \frac{1}{r} + \frac{q_{n-1}}{q_n} \right) - \psi \left( \frac{1}{r+1} + \frac{q_{n-1}}{q_n} \right) \right\} \]

\[ = \frac{1}{(k \cdot q_{n-1} + q_{n-2})^2} \left[ \psi \left( \frac{1}{r} + \frac{q_{n-1}}{k \cdot q_{n-1} + q_{n-2}} \right) - \psi \left( \frac{1}{r+1} + \frac{q_{n-1}}{k \cdot q_{n-1} + q_{n-2}} \right) \right] \]

Write \( k_n = k \) and let \( k_n \) take all the values from 1 to \( \infty \)

\[ \text{Prob}(a_{n+2} = r, a_n = k) = \frac{(n-1)}{k \cdot q_{n-1} + q_{n-2}} \left[ \psi \left( \frac{1}{r} + \frac{q_{n-1} - k \cdot q_{n-1} + q_{n-2}}{k \cdot q_{n-1} + q_{n-2}} \right) - \psi \left( \frac{1}{r+1} + \frac{q_{n-1} - k \cdot q_{n-1} + q_{n-2}}{k \cdot q_{n-1} + q_{n-2}} \right) \right] \]

\[ \frac{(k \cdot q_{n-1} + q_{n-2})^2}{(k \cdot q_{n-1} + q_{n-2})^2} \]  \hspace{1cm} (5.15)

From this formula it follows that

\[ \text{Prob}(a_{n+2} = r, a_n = k) < \frac{\psi \left( \frac{1}{r} \right)}{k^2} \text{Prob}(a_2 = r) \]  \hspace{1cm} (5.16)

From the formula (5.16) it appears that the joint

\[ \text{Prob}(a_n = r, a_n = k) \] is more directly related in a way to the product

\[ \text{Prob}(a_n = k) \cdot \text{Prob}(a_n = r) \] than to the product \( \text{Prob}(a_n = k) \cdot \text{Prob}(a_{n+2} = r) \).

Joint \( \text{Prob}(a_{n+2} = r, a_n = k) \)

\[ a_n = k \] implies that \( x_{n-1} \) is in \( \left( \frac{1}{k+1}, \frac{1}{k} \right) \) and the distribution function

of \( x_{n-1} \) is \( F_{n-1}(x) \).

Therefore from (5.11) it follows that
Joint prob\(a_{n-p} = r, a_{n-p-1} = k_{p-1}, \ldots, a_{n-1} = k_1, a_n = k\)

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} \frac{1}{k_{n-1}^n \, k_{n-2}^n \, \cdots \, k_1^n} \right) - \frac{1}{k_{n-1}^n \, k_{n-2}^n \, \cdots \, k_1^n}
\]

(Instead of \(a\) with distribution function \(f_a(x)\), here \(s_{n-1}\) with \(\frac{s_{n-1}}{s_n} = \frac{s_{n-2}}{s_{n-1}} = \frac{s_{n-3}}{s_{n-2}} = \cdots = \frac{s_{n-p}}{s_{n-p-1}} = \frac{s_{n-p-1}}{s_{n-p}}\)

Let \(\frac{1}{k_{n-1}^n \, k_{n-2}^n \, \cdots \, k_1^n} = \frac{p_{n-1}}{r_{n-1}}\) and \(\frac{p_{n-1}}{r_{n-1}}\) be the \(r\)th convergent of this continued fraction. With this notation,

\[
\text{prob}(a_{n-p} = r, a_{n-p-1} = k_{p-1}, \ldots, a_{n-1} = k_1, a_n = k)
\]

\[
= \lim_{n \to \infty} \left( \sum_{k=1}^{\infty} \frac{1}{k_{n-1}^n \, r_{n-1}^n + \frac{p_{n-2}^n}{q_{n-2}^n}} \right) - \frac{1}{k_{n-1}^n \, r_{n-1}^n + \frac{p_{n-2}^n}{q_{n-2}^n}}
\]

\[
= \sum_{q_{n-1}^2 = 2}^{(n-1)} \frac{1}{q_{n-1}^2} \left[ \frac{1}{k + \frac{k_{n-2}^n \, r_{n-1}^n \, p_{n-2}^n \, q_{n-2}^n}{q_{n-1}^2 \, q_{n-1}^2 \, q_{n-2}^2}} - \frac{1}{k + \frac{q_{n-2}^n \, (r_{n-1}^n \, p_{n-1}^n \, p_{n-2}^n)}{q_{n-1}^2 \, q_{n-1}^2 \, q_{n-2}^2}} \right]
\]
\[
\text{Since } v_n(x) = \frac{(n-1)}{E_{n-1} \left( q_{n-1} + x \right)}
\]
\[
= \frac{(n-1)}{E_{n-1}} \frac{1}{\sum_{q_{n-1}} \frac{r q_{p-1} + q_{p-2}}{(r q_{p-1} + q_{p-2})(r q_{p-1} + q_{p-2})}}
\]
\[
= \frac{(n-1)}{E_{n-1}} \frac{1}{\left( k + \frac{q_{n-2}}{q_{n-1}} \right) \left( 1 + \frac{r q_{p-1} + q_{p-2}}{r q_{p-1} + q_{p-2}} \right) \left( 1 + \frac{(r+1)q_{p-1} + q_{p-2}}{(r+1)q_{p-1} + q_{p-2}} \right)}
\]
\[
= \frac{(n-1)}{E_{n-1}} \frac{1}{\left( k + \frac{q_{n-2}}{q_{n-1}} \right)^2 \left( 1 + \frac{r q_{p-1} + q_{p-2}}{r q_{p-1} + q_{p-2}} \right) \left( 1 + \frac{(r+1)q_{p-1} + q_{p-2}}{(r+1)q_{p-1} + q_{p-2}} \right)}
\]

where \( x_n = k + \frac{q_{n-2}}{q_{n-1}} \)

Now \( \frac{q_{n-2}}{q_{n-1}} \) lies between 0 and 1

Therefore \( k \leq x_n \leq k + 1 \)

\[
\frac{1}{k+1} \leq \frac{1}{x_n} \leq \frac{1}{k}
\]

and

\[
0 < \frac{r q_{p-1} + q_{p-2}}{r q_{p-1} + q_{p-2}} < 1 ; \quad 0 < \frac{(r+1)q_{p-1} + q_{p-2}}{(r+1)q_{p-1} + q_{p-2}} < 1
\]

Using these inequalities and the formula

\[
\sum_{n=1}^{(n-1)} \frac{1}{E_{n-1} \left( q_{n-1} + q_{n-2} \frac{1}{k+1} \right)^2} = \frac{(n-1)}{E_{n-1}} \frac{k^2}{q_{n-1} + q_{n-2}^2 k^2}
\]

one obtains that
Joint \( \Pr(a_{n,p} = r, a_{n,p-1} = k_{p-1}, \ldots, a_n = k) \)

\[
\begin{align*}
N & = \frac{(n-1) k^2}{(k a_{n-1} + a_{n-2})^2} \left( \frac{1}{r q_{p-1}^r q_{p-2}^r (r q_{p-1}^r + q_{p-1}^r + q_{p-2}^r)} \frac{1}{(k + 1)^2} \right) \\
& = \frac{n-1}{k^2} \frac{1}{(k + 1)^2} \left( \frac{r q_{p-1}^r + q_{p-2}^r (r q_{p-1}^r + q_{p-1}^r + q_{p-2}^r)}{r q_{p-1}^r + q_{p-2}^r} \right)
\end{align*}
\]

Now allowing \( k_1, k_2, \ldots, k_{p-1} \) assume all the values from 1 to \( \infty \)

\[
\Pr(a_{n,p} = r, a_n = k) \geq \frac{n-1}{k^2} \frac{1}{(k + 1)^2} \Pr(a_p = r) \quad (5.18)
\]

Similarly using the other inequality

\[
\Pr(a_{n,p} = r, a_n = k) < \frac{n-1}{k^2} \frac{1}{(k + 1)^2} \Pr(a_p = r) \quad (5.19)
\]

(5.18) and (5.19) gave the inequalities in terms of \( \frac{1}{n-1} \frac{1}{k^2} \). Now to get the upper and lower bounds for (5.17) in terms of \( \Pr(a_n = k) \),

one can write (5.17) in the following way

\[
\Pr(a_{n,p} = r, a_{n,p-1} = k_{p-1}, \ldots, a_{n-1} = k_1, a_n = k) = \frac{1}{(k a_{n-1} + a_{n-2})^2} \left( \frac{1}{r q_{p-1}^r + q_{p-2}^r (r q_{p-1}^r + q_{p-1}^r + q_{p-2}^r)} \right) \frac{1}{(k + 1)^2} \left( \frac{r q_{p-1}^r + q_{p-2}^r (r q_{p-1}^r + q_{p-1}^r + q_{p-2}^r)}{r q_{p-1}^r + q_{p-2}^r} \right)
\]
\[
\mathcal{S} = \sum_{k_1, k_2, \ldots, k_{p-1}} \frac{1}{(r q_{p-1}^{k_1} q_{p-2}^{k_2}) (r q_{p-1}^{k_1} q_{p-2}^{k_2}) (k q_{n-1}^{k_1} a_{n-2}^{k_2}) (k q_{n-1}^{k_1} a_{n-2}^{k_2}) (1 + \frac{1}{k}) (1)}
\]

Again summing over all \( k_1, k_2, \ldots, k_{p-1} \), one obtains that

\[
\text{Prob}(a_{n-p} = r, a_n = k) \geq \text{Prob}(a_n = k) \cdot \text{Prob}(a_p = r)(1 - \frac{1}{k+1}) \quad (5.20)
\]

Similarly using the inequalities

\[
1 + \frac{1}{x_n} \frac{r q_{p-1}^{k_1} q_{p-2}^{k_2}}{r q_{p-1}^{k_1} q_{p-2}^{k_2}} > 1
\]

and

\[
1 + \frac{1}{x_n} \frac{r q_{p-1}^{k_1} q_{p-2}^{k_2} - \frac{1}{x_n+1}}{r q_{p-1}^{k_1} q_{p-2}^{k_2}} = \frac{1}{x_n+1} > 1 - \frac{1}{k+1}
\]

\[
\text{Prob}(a_{n-p} = r, a_n = k) \leq \text{Prob}(a_n = k) \cdot \text{Prob}(a_p = r)(1 + \frac{1}{k}) \quad (5.21)
\]

From (5.20) and (5.21)

\[
\left| \frac{\text{Prob}(a_{n-p} = r, a_n = k)}{\text{Prob}(a_p = r) \cdot \text{Prob}(a_n = k)} - 1 \right| < \frac{1}{k} \quad (5.22)
\]

i.e. For large values of \( k \)

\[
\text{Prob}(a_p = r) \cdot \text{Prob}(a_n = k) = \text{Prob}(a_{n-p} = r, a_n = k)
\]

Since

\[
\lim_{p \to \infty} \text{Prob}(a_p = r) = \lim_{p \to \infty} \text{Prob}(a_{n-p} = r) = \frac{\log(1 + \frac{1}{r(r-2)})}{\log 2}
\]

\[
\text{Prob}(a_n = k) = \frac{\log(1 + \frac{1}{k})}{\log 2}
\]

\[
\text{Prob}(a_{n-p} = r, a_n = k) = \frac{\log(1 + \frac{1}{k+1})}{\log 2}
\]

\[
\text{Prob}(a_n = k) \cdot \text{Prob}(a_p = r)(1 - \frac{1}{k+1})
\]
one can say that for large values of $k$ and for large $p$,

$$\Pr(b_{n^p} = r; a_n = k) = \frac{\Pr(b_{n^p} = r) \cdot \Pr(a_n = k)}{\Pr(b_{n^p} = r) \cdot \Pr(a_n = k) - 1} < 6A^{-\frac{\lambda}{\sqrt{p-1}}}.$$ 

In this connection Khintchine [10] by generalizing Weyl's result, established the result that

This inequality shows that as $p \to \infty$, $a_{n^p}$ and $a_n$ are independent or two sufficiently far apart indices of a continued fraction are independent.
\[ D_{12}(k, r) = \frac{\text{Prob}(a_1 = k; a_2 = r)}{\text{Prob}(a_1 = k) \cdot \text{Prob}(a_2 = r)} - 1 \]

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\[ D_{n,n+1}(k, r) = \left| \frac{\log(1 + \left( \frac{1}{r^{k+1}} \right) \left( \frac{1}{r^{k+2}} \right)) \log 2}{\log(1 + \frac{1}{r^{k+2}}) \log(1 + \frac{1}{r^{k+2}})} - 1 \right| \quad \text{for large } n \]

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$$D_{13}(k, r) = \frac{\text{Prob}(a_1 = k \mid a_2 = r)}{\text{Prob}(a_1 = k) \cdot \text{Prob}(a_2 = r)} - 1$$

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CHAPTER VI

STATISTICAL ANALYSIS OF THE DATA

As the expressions for \( h_0(x) \) are theoretically very complicated and direct calculations are almost impossible, a statistical analysis of the distribution functions was also made. A sample of 1000 random numbers [21] was taken and for each number the index numbers \( a_1, a_2, \ldots, a_{15} \) were calculated. From the data the distribution functions \( F_n^k(k) \) and the frequency functions \( f_n^k(k) \) were tabulated (Tables 6 and 7) for \( n = 1 \) to \( 15 \). Throughout this chapter \( F_n^k(k), f_n^k(k) \) denote respectively the distribution and frequency functions of the sample.

The analysis of the sample suggests that the distribution functions \( F_n^k(k) \) behave like the asymptotic distribution function

\[
F_\infty(k) = \frac{\log(1-k)}{\log 2}
\]
even for small values of \( n \) \( (n \geq 3) \). The theoretical frequency functions \( f_1(k), f_2(k), f_3(k) \) and the inequality of Ruzhin

\[
\left| f_n(k) - \frac{\log(1 + \frac{1}{k(k+2)})}{\log 2} \right| < \frac{4}{k(k+1)} e^{-\sqrt{n-1}}
\]
suggest that the convergence of \( f_n(k) \) to \( \frac{\log(1 + \frac{1}{k(k+2)})}{\log 2} \) is rapid not only as \( n \to \infty \) but also as \( k \to \infty \). However in the frequency functions \( f_n^k(k) \) of this sample this property was not very much pro-
nounced. This may be due to the small size of the sample.

Test of goodness of fit for \( f_n^*(k) \). To test the approximation of
\[
\frac{\log[1 + \frac{1}{k(k+2)}]}{\log 2}
\]
the \( \chi^2 \) test is used. The procedure of this test is given below [22].

Let hypothesis \( H \) be that \( f_n^*(k) \) can be approximated by
\[
\frac{\log[1 + \frac{1}{k(k+2)}]}{\log 2}
\]
\[
\chi^2 \ = \ \sum_{i=1}^{r} \frac{(\gamma_i - \mu_i)^2}{\mu_i}
\]
where \( \gamma_i \) is the observed frequency of the \( i \)th class

\( N \) = number of observations in the sample.

\( \mu_i \) = expected frequency (under hypothesis \( H \)).

\( r \) = number of classes into which the sample is divided.

\( \chi^2 \) is a measure of deviation between the two distributions
\( [f_n^*(k), f_\infty(k)] \). If \( \chi^2 \) is equal to 0, the two distributions are identical. The larger the \( \chi^2 \), the more the deviation between the two distributions.

And further for large values of \( N (k > 50) \), \( \chi^2 \) is distributed as a \( \chi^2 \) with \( (r-1) \) degrees of freedom. Let \( \chi^2 \) be the value of \( \chi^2 \) such that
\[
\text{prob}(\chi^2 > \chi^2_0) = \frac{p}{100} = p \%
\]
(generally \( p = 5 \) and \( l \) are taken).
Now if the hypothesis \( H \) is true, then it is practically impossible, in one single sample, to encounter a value of \( \chi^2_0 \) exceeding \( \chi^2_p \).

Therefore if one finds a value \( \chi^2_0 > \chi^2_p \) in the sample, one accordingly says that the sample shows a significant deviation from the hypothesis \( H \) and the hypothesis \( H \) is rejected at least until further data are available. The probability that \( H \) is falsely rejected is \( p \% \). If, on the other hand, \( \chi^2_0 \leq \chi^2_p \) this will be regarded as consistent with hypothesis \( H \). However one isolated result of this kind cannot be considered as sufficient evidence of the truth of the hypothesis \( H \). The test must be repeated to make a positive statement.

Since the sample consists of 1000 observations one can use the above test. For \( n = 2 \) to 15 the values of \( \chi^2_0 \) are calculated and tabulated (Table 3) and they are compared with the values \( \chi^2_p \) for \( p = 5 \) and \( p = 1 \) from the \( \chi^2 \) tables. Since all the \( \chi^2_0 \)'s \( < \chi^2_p \) it follows that all the frequency functions \( f_n^*(k) \) for \( n = 2 \) to 15 can be approximated by \( f_{\infty}(k) \).
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<td>110</td>
<td>120</td>
<td>130</td>
<td>140</td>
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</tbody>
</table>

Legend:
- (a) $S_{ij}$
- (b) $T_{ij}$
- (c) $C_{ij}$
- (d) $X_{ij}$
- (e) $Z_{ij}$
- (f) $Q_{ij}$
- (g) $H_{ij}$
- (h) $L_{ij}$
- (i) $D_{ij}$
- (j) $E_{ij}$
- (k) $F_{ij}$

Note: The table represents sales data from 1990 to 2003.
**TABLE 8**

VALUES OF $x^2_0$ FOR EACH $n = 1$ TO $15$

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<td>14</td>
<td>4.833</td>
</tr>
<tr>
<td>15</td>
<td>15.921</td>
</tr>
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</table>

$x^2_{.05} = 19.675$ for 11 degrees of freedom

$x^2_{.01} = 24.725$
**AFTER THE A**

**CALCULATION OF \( h_2(x) \)**

**Ruler's summation formula: -** Let \( f(x) \) be a function having derivative of order \((2k+1)\) which is continuous. Then

\[
f_1 + f_2 + \cdots + f_n = \int_1^n f(x) \, dx + \frac{1}{2} (f'_{n+1} + f'_1)
\]

\[
+ \frac{3}{2k+2} (f'_n - f'_1) + \frac{3}{k+2} (f''_{n+1} - f''_1) + \cdots
\]

\[
+ \frac{3}{2k} (f'^{2k-1}_n - f'^{2k-1}_1) + R_k
\]

where \( R_k = \int_1^n f^{2k+1}(x) \, f^{2k+1}(x) \, dx \) \[23\]

\( B_2, B_4, \cdots, B_{2k} \cdots \) are the Bernoulli numbers and \( B_k(x) \) is the \( k \)th Bernoulli polynomial. \( f^{(k)}(x) = \frac{1}{k!} (\delta x)^k \) in symbolic notation \( f^k = B_k \).

\( f_n \) denotes the value of \( f \) at \( n \).

Further if

1) \( \sum_{n=1}^\infty f_n \) converges

2) \( \int_1^\infty f(x) \, dx \) converges

3) \( f_n \to 0 \) as \( n \to \infty \) for \( r = 1, 2 \ldots (2k+1) \)
the Euler's summation formula can be written as

\[ \sum_{n=1}^{\infty} f_n = \int_{1}^{\infty} f(x)dx + \frac{1}{2} f_1 - \frac{B_2}{2!} f'_1 + \frac{B_4}{4!} f''_1 - \cdots - \frac{B_{2k}}{2k!} f^{2k-1}_1 + \mathcal{R}_k \]  

(A.1.1)

where \( \mathcal{R}_k = \int_{1}^{\infty} p_{2k+1}(x) f^{2k+1}(x) dx \)

since \( |p_{2k}(x)| < \frac{a_{2k}}{2k!} \)

and \( \mathcal{R}_k = - \int_{1}^{\infty} p_{2k}(x) f^{2k}(x) dx \)

\( |\mathcal{R}_k| < \frac{a_{2k}}{2k!} \frac{f^{2k-1}_1}{f'_1} \)

In particular if \( f^{2k} \) and \( f^{2k-2} \) have the same sign and keep their sign, then the error

\[ |\mathcal{R}_k| < \left| \frac{a_{2k+2}}{2k+2!} \right| f^{2k+1}_1 \]  

(24j)

i.e. The error is numerically less than the first neglected term and has the same sign.

The calculation of \( \frac{1}{2}(y) \) with Euler's formula

\[ \frac{1}{2}(y) = \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \psi\left(\frac{y+1}{n}\right) - \psi\left(\frac{1}{n}\right) \right\} \]

\[ = \sum_{n=1}^{\infty} \frac{f_n}{n} \]

\[ \sum_{n=1}^{\infty} f_n = \int_{1}^{\infty} \frac{\psi\left(\frac{y+1}{n}\right) - \psi\left(\frac{1}{n}\right)}{x^2} dx + \frac{1}{2} \left[ \psi(y-1) - \psi(1) \right] \]
\[ - \frac{B_2}{2} \left[ f'_{1} \right] - \frac{B_4}{4} \left[ f''_{1} \right] - \cdots - \frac{B_{2k}}{2k} \left[ f^{2k-1}_{1} \right] + \delta_k \cdot \]

Since \( f_n \) satisfies all the properties for the formula (A.1.1), we have

\[ n_2(y) = \log(1+y) + \frac{1}{2} \left[ -\psi(y+1) - \psi(1) \right] - \frac{B_2}{2} f'_{1} + \delta_k \cdot \]

[since \( \int_{1}^{\infty} \frac{\psi(1)}{x^2} \ dx = \log(1+y) \)]

\[ f(n) = \frac{1}{n^2} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

\[ f'(n) = \frac{1}{n^3} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

\[ f''(n) = \frac{1}{n^4} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

\[ f'''(n) = \frac{1}{n^5} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

\[ f^{(n)}(n) = \frac{1}{n^6} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

\[ f^4(n) = \frac{1}{n^7} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

\[ f^5(n) = \frac{1}{n^8} \left[ -\psi \left( \frac{1}{n} \right) \right] \]

Substituting the values \( f_{1}^{1}, f_{2}^{2}, f_{3}^{3} \) in the formula

\[ n_2(y) = \log(1+y) + \frac{1}{2} \left[ -\psi(y+1) - \psi(1) \right] + \frac{1}{12} \left[ 2 \frac{\psi(y+1) - \psi(1)}{\psi(y+1) - \psi(1)} \right] \]

\[- \frac{1}{720} \left[ \psi^3(y+1) - \psi^3(1) \right] + 12 \frac{\psi^2(y+1) - \psi^2(1)}{\psi(y+1) - \psi(1)} + 36 \frac{\psi^2(y+1) - \psi^2(1)}{\psi(y+1) - \psi(1)} \]

\[ + 24 \frac{\psi(y+1) - \psi(1)}{\psi(y+1) - \psi(1)} \]
\[
\psi(y-1)-\psi(1) \approx \psi'(1) - 3\psi''(1) - 3\psi'''(1) - 12\psi''(1)
\]

\[
\psi''(x+1) - \psi''(1) + 720 \psi(x+1) - \psi(1) \approx R_3
\]

\[\|R_3\| < .0006 \text{ and } \psi(y) \text{ since } \psi^2 \text{ and } \psi^6 \text{ have same sign}.
\]

neglecting \(R_3\).

\[
\psi(y) = \log(4y) + a_0 \psi''(y+1) - \psi'(1) + a_1 \psi'(y+1) - \psi'(1)
\]

\[
+ a_2 \psi''(y+1) - \psi''(1) + a_3 \psi'''(y+1) - \psi'''(1) + a_4 \psi''(y+1) - \psi''(1)
\]

\[
+ a_5 \psi^5(y+1) - \psi^5(1)
\]

\[
a_0 = .6971
\]

\[
a_1 = .09286
\]

\[
a_2 = .02301
\]

\[
a_3 = .00853
\]

\[
a_4 = .000992
\]

\[
a_5 = \frac{1}{30480} = .00003362
\]

\(\psi(y), \psi'(y), \psi''(y), \psi'''(y), \psi''(y)\), are tabulated from [25].

\(\psi^5(y)\) is not tabulated. But \(a_5\) is very small. The maximum value of the term

\[
a_5 \psi^5(y+1) - \psi^5(1) \approx |a_5(\psi^5(y) - \psi^5(1))| < .00062
\]

Therefore the values of \(R_2(y)\) for different values of \(y(0 < y < 1)\)

are calculated and tabulated (Table 9). The error is < .001 for
\[ 0 < y \leq \frac{1}{2} \cdot \]

Therefore the values are correct to 3 decimal places.
SIMILARITIES BETWEEN SYSTEMATIC AND SIMPLE CONTINUED FRACTIONS

As one can observe, there is some similarity between a simple continued fraction expansion and a systematic fraction expansion of a real number. For simple continued fractions the operator 
\[ T(a) = \frac{1}{a} = [\frac{1}{a}] \]
is iterated where as in systematic fractions the corresponding operator is 
\[ T(a) = g^x - [g^x] \]
where the integer \( g \) is the root of the number system.

Let 
\[ a = \frac{a_1}{g} + \frac{a_2}{g^2} + \cdots + \frac{a_n}{g^n} + \cdots \]
where \( g \) is a positive integer \( \geq 2 \) and \( 0 \leq a_n < g \) for each \( n \).

Denote \( a \) by \( a_0 \) and define
\[ a_n = \frac{a_{n+1}}{g} + \frac{a_{n+2}}{g^2} + \cdots + \frac{a_{n+k}}{g^k} + \cdots \]
Then one can easily observe that
\[ [ag] = a_1 \quad \text{and in general} \]
\[ [a_n g] = a_{n+1} \]
\[ a_{n+1} = g a_n = [g a_n] \quad \text{(B.1)} \]
The distribution function \( D_n(x) \) is defined as
\[ D_n(x) = \text{Prob}(a_n < x) \]

**Relation between \( D_n(x) \) and \( D_{n+1}(x) \)**

\[ D_{n+1}(x) = \text{Prob}(a_{n+1} < x) \]

\[ = \text{Prob}(a_n \leq k \leq a_n \leq k + x) \quad \text{by} \quad n+1 \]

\[ = \sum_{k=0}^{g-1} \text{Prob}(k \leq a_n < k + x) \quad \text{since} \]

\([a_n G] = k \quad \text{implies} \quad k \leq a_n < k + 1 \quad \text{and} \quad k \quad \text{can assume values from} \]

0 to \((g-1)\).

\[ D_{n+1}(x) = \sum_{k=0}^{g-1} \text{Prob}(k \leq a_n < \frac{k+1}{g}) \]

\[ = \sum_{k=0}^{g-1} D_n\left(\frac{k}{g}\right) = D_n\left(\frac{k}{g}\right) \quad (3.2) \]

So if \( D_0(x) = x \), then \( D_n(x) = x \) for all \( n \). And furthermore

\[ g_n(k) = \text{Prob}(a_n \leq k) = D_n\left(\frac{k}{g}\right) = \frac{k}{g} \]

Therefore \( g_n(k) = \text{Prob}(a_n = k) = \frac{k}{g} - \frac{k - 1}{g} = \frac{1}{g} \) which is independent

of both \( n \) and \( k \).

**\( D_n(x) \) in terms of \( D_0(x) \)**

From the definition for \( a_n \), it follows that

\[ a_n = g^n \cdot a - [g^n \cdot a] \]
\[ g^n \cdot a = a_1 g^{n-1} + a_2 g^{n-2} + \ldots + a_n + \frac{a_{n+1}}{g} + \frac{a_{n+2}}{g^2} + \ldots + \frac{a_p}{g^p} + \ldots \]

and therefore \[ [g^n \cdot a] = a_1 g^{n-1} + a_2 g^{n-2} + \ldots + a_n \]

From this relation, it follows that

\[ D_n(x) = \lim_{n \to \infty} \frac{g^n}{x} \cdot [g^n \cdot a] < x \]

\[ = \lim_{n \to \infty} \frac{g^{n-1}}{x} \cdot \sum_{k=0}^{n-1} \frac{d_0(k+1)}{g^k} \]

\[ = \sum_{k=0}^{n-1} d_0(k+1) \cdot \frac{d_0(k+1)}{g^k} \]

(3.3)

From the equation (3.3) if one assumes that \( D_0(x) \) has a continuous bounded derivative, then

\[ D_n^1(x) = \sum_{k=0}^{n-1} \frac{d_0(k+1)}{g^k} \]

(3.4)

From (3.4) one can observe that all properties of \( D_n^1(x) \) can be easily deduced directly from \( D_0^1(x) \).

Inverse transform of \( D_n^1(x) \)

Let \[ D_n^1(x) = \int_{0}^{\infty} e^{-tx} \beta_n(t) \, dt \]

From (3.4)

\[ D_n^1(x) = \int_{0}^{\infty} \frac{1}{k+1} \beta_0(t) \cdot \frac{t}{g^n} \, dt \]

\[ = \int_{0}^{\infty} \frac{1 - e^{-\frac{t}{g^n}}}{1 - e^{-\frac{1}{g^n}}} \beta_0(t) \cdot \frac{t}{g^n} \, dt \]
\[ -g^n t \]
\[ = \int_{0}^{\infty} \frac{1 - e^{-t}}{1 - e^{-t}} \cdot \beta_0 (g^n t) \cdot e^{-tx} \, dx \]

Therefore if \( \beta_1 (x) \) has an inverse transform \( \beta_0 (t) \) then

\[ \beta_n (t) = \frac{1 - e^{-t}}{1 - e^{-t}} \cdot \beta_0 (g^n t) \]
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